Fluid Mechanics (Spring 2016)

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Contents

1 Fundamentals of continuum mechanics

Continuum mechanics is concerned with the dynamics of a continuous body such as a fluid (gas, liquid) or an elastic body. Our goals here are:

1. to clearly elaborate a mathematical description of a continuum,
2. to describe the kinematics of a continuum,
3. to derive the equations of motion that govern the dynamics of a continuum.

1.1 Kinematics

To describe a continuum we assume the following.

1. At the reference time $t_0 = 0$ (others could be used, but $t_0 = 0$ is no loss of generality) the continuum occupies an open set $\Omega_0 \subseteq \mathbb{R}^3$. We will typically assume that $\partial\Omega_0$ is at least Lipschitz (locally the graph of a Lipschitz function) whenever $\partial\Omega_0 \neq \varnothing$. However, in deriving the equations of motion it is often convenient to assume $\partial\Omega_0$ is more regular. Throughout section 1 we will be a bit vague about the precise regulartiy assumptions, but this has no real impact on the results. The set Ω_0 is often called a reference configuration or a material configuration in the material, i.e. $y \in \Omega_0$ corresponds to a material point/particle.

2. For times $t \geq 0$ the continuum occupies a set $\Omega(t) \subseteq \mathbb{R}^3$ that is given as a deformation of the reference configuration Ω_0 by a map $\eta(\cdot, t) : \Omega_0 \to \mathbb{R}^3$. We assume that for each $t \geq 0$ the continuum does not self-penetrate, which means that we assume that $\eta(\cdot, t)$ is injective. Thus $\eta(\cdot, t) : \Omega_0 \to \eta(\Omega_0, t) = \Omega(t) \subseteq \mathbb{R}^3$ is a bijection for all $t \geq 0$. Then the map $\mathbb{R}^+ \ni t \mapsto \eta(y, t) \in \mathbb{R}^3$ gives the trajectory of the material point $y \in \Omega_0$. We will call η the flow map.

3. We will assume that actually η is differentiable and that for all $t \geq 0$, $\eta(\cdot, t) : \Omega_0 \to \Omega(t)$ is a C^1 diffeomorphism that preserves orientation.

We now turn to one of the most important examples of flow maps.

Definition. *Let $z : \mathbb{R}^+ \to \mathbb{R}^3, R : \mathbb{R}^+ \to SO(3) = \{M \in \mathbb{R}^{3\times 3} \mid M^\top = M^{-1}, \det M = 1\}$ be maps with $z(0) = 0, R(0) = I$. We say that the map $\eta : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$ given by $\eta(y, t) = z(t) + R(t)y$ is a rigid motion.*

Clearly if η is a rigid motion, then $|\eta(y_1, t) - \eta(y_2, t)| = |R(t)(y_1 - y_2)| = |y_1 - y_2|$ for all $t \in \mathbb{R}^+, y_1, y_2 \in \mathbb{R}^3$. This is what justifies the name. In fact, this characterizes these maps.

Proposition 1.1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. Then the following are equivalent:*

1. *$|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$*
2. *There exists $z \in \mathbb{R}^n, R \in O(n)$ such that $f(x) = z + Rx$ for all $x \in \mathbb{R}^n$.*

Proof. The direction from item 2 to item 1 is trivial. To prove the other direction, we will first prove a modified result: if $f(0) = 0$ and $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$, then $f(x) = Rx$ for some $R \in SO(n)$. Indeed, in this case we know

$$x \cdot y = \frac{|x|^2 + |y|^2 - |x-y|^2}{2} = \frac{|x-0|^2 + |y-0|^2 - |x-y|^2}{2}$$
$$= \frac{|f(x) - f(0)|^2 + |f(y) - f(0)|^2 - |f(x) - f(y)|^2}{2} = f(x) \cdot f(y)$$

for all $x, y \in \mathbb{R}^n$. Since f is a bijection we can choose $y_1, \dots, y_n \in \mathbb{R}^n$ such that $f(y_i) = e_i$ for $i = 1, \dots, n$. Then for $x \in \mathbb{R}^n$,

$$f(x) = \sum_{i=1}^{n} (f(x) \cdot e_i) e_i = \sum_{i=1}^{n} (f(x) \cdot f(y_i)) e_i = \sum_{i=1}^{n} (x \cdot y_i) e_i = Rx$$

$R \in \mathbb{R}^{n \times n}$ is the matrix with y_i in the ith row. Thus f is linear and $x \cdot y = Rx \cdot Ry$ for all $x, y \in \mathbb{R}^n$ and thus $R^\top R = I \implies R \in O(n)$. This proves the modified result.

Now, in the general case we set $z = f(0)$ and consider the map $g : \mathbb{R}^n \to \mathbb{R}^n$ given by $g(x) = f(x) - z$. This is clearly a bijection and satisfies $g(0) = 0$ and $|g(x) - g(y)| = |x - y|$ for all x, y. Thus there exists $R \in O(n)$ such that $g(x) = Rx$, and so $f(x) = z + Rx$. $\square$

Remark 1.2. *In general we can't show $R \in SO(3)$ without postulating that η preserves orientation.*

Definition. *The velocity of a material particle $y \in \Omega_0$ at time $t \in \mathbb{R}^+$ is given by $\partial_t \eta(y,t) \in \mathbb{R}^3$. We define the velocity as $v : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^3$ given by $v(y,t) = \partial_t \eta(y,t)$. We similarly define the acceleration $a : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^3$ via $a(y,t) = \partial_t^2 \eta(y,t) = \partial_t v(y,t) \in \mathbb{R}^3$.*

Those definitions are consistent with the usual meaning in the context of the kinematics of particles. Indeed if we set $x(t) = \eta(y,t) \in \mathbb{R}^3$ for some fixed $y \in \Omega_0$, then $v(y,t) = \dot{x}(t), a(y,t) = \ddot{x}(t)$.

The description of the velocity and acceleration in Ω_0 is called the Lagrangian description and the coordinates $(y,t) \in \Omega_0 \times \mathbb{R}^+$ are called Lagrangian coordinates. It turns out that it is often more convenient to work in Eulerian (or sometimes laboratory) coordinates, which are given by $x = \eta(y,t) \in \Omega(t)$. In other words, $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ are coordinates relative to a fixed frame (the laboratory) through which the continuum moves.

Let's examine the velocity and acceleration in Eulerian coordinates. We write $u(\cdot, t) : \Omega(t) \to \mathbb{R}^3$ via

$$v(y,t) = u(\eta(y,t),t) = u \circ \eta \qquad \text{or} \qquad u(x,t) = v(\eta^{-1}(x,t),t) = v \circ \eta^{-1}.$$

Next we compute:

$$\begin{aligned} a(y,t) = \partial_t v(y,t) &= \partial_t u(\eta(y,t),t) + Du(\eta(y,t),t)\partial_t \eta(y,t) \\ &= \partial_t u(\eta(y,t),t) + Du(\eta(y,t),t) v(y,t) \\ &= \partial_t u(\eta(y,t),t) + Du(\eta(y,t),t) u(\eta(y,t),t). \end{aligned}$$

Now, $(Du \cdot u)_i = \sum_{j=1}^3 (Du)_{ij} u_j = \sum_{j=1}^3 \partial_j u_i u_j = u \cdot \nabla u_i$, which leads us to define $u \cdot \nabla u \in \mathbb{R}^3$ via $(u \cdot \nabla u)_i = u_j \partial_j u_i$. Thus

$$a(y,t) = \partial_t u(\eta(y,t),t) + (u \cdot \nabla u)(\eta(y,t),t),$$

and so the Eulerian description of the acceleration is the field $\partial_t u(x,t)+u(x,t)\cdot\nabla u(x,t)$ for $x \in \Omega(t)$.

Given η or v we can compute u and $\Omega(t)$, but we can also go the other way! Say we know $\Omega(t)$ and $u(\cdot,t) : \Omega(t) \to \mathbb{R}^3$ for $t \geq 0$, i.e. we know the Eulerian velocity. We find η by solving

$$\begin{cases} \partial_t \eta(y,t) = u(\eta(y,t),t) \\ \eta(y,0) = y \in \Omega(0) = \Omega_0 \end{cases}.$$

Assuming u is sufficiently regular, there exists a unique η solving the ODE, and the basic theory of ODE tells us that $\{\eta(\cdot,t)\}_{t\geq 0}$ is a 1-parameter family of diffeomorphisms. Moreover,

$$\begin{cases} \partial_t \det D\eta(y,t) = \operatorname{div} u(\eta(y,t),t) \det D\eta(y,t) \\ \det D\eta(y,0) = \det I = 1 \end{cases}, \tag{1}$$

so

$$\det D\eta(y,t) = \exp\left(\int_0^t \operatorname{div} u(\eta(y,s))\ ds\right).$$

This has an important consequence: $\det D\eta > 0$ for all $t \geq 0$, i.e. η is orientation-preserving. Thus we guarantee that η is a flow map.

The formula (1) has other important consequences.

Definition. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $f : \Omega \to f(\Omega) \subseteq \mathbb{R}^n$ be a C^1 diffeomorphism. We say f is locally volume-preserving if $|U| = |f(U)|$ for every measurable set $U \subseteq \Omega$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure.*

Theorem 1.3. *Let $\eta : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^3$ be a flow map. Then the following are equivalent:*

1. *For all $t \in \mathbb{R}^+$ the map $\eta(\cdot,t) : \Omega_0 \to \Omega(t)$ is locally volume-preserving.*
2. $\det D\eta(y,t) = 1$ *for all $y \in \Omega_0, t \in \mathbb{R}^+$*
3. *If $u(\cdot,t) : \Omega(t) \to \mathbb{R}^3$ is the Eulerian velocity associated to η, then* $\operatorname{div} u(x,t) = 0$ *for all $x \in \Omega(t)$ for all $t \in \mathbb{R}^+$.*

Proof. We first show the first and second items are equivalent. We know from measure theory that if $U \subseteq \Omega_0$ is measurable, then $\eta(U,t)$ is measurable, and

$$|\eta(U,t)| = \int_U \det D\eta(y,t)\ dy.$$

If $|U| = |\eta(U,t)|$ for all U, then

$$\int_U [\det D\eta(y,t) - 1]\ dy = 0$$

for all $U \subseteq \Omega_0$ measurable, and hence $\det D\eta(y,t) = 1$ for all $y \in \Omega_0, t \in \mathbb{R}^+$. The converse is trivial.

Next, suppose the second item. Since $\det D\eta(y,t) = 1$ we have that

$$0 = \partial_t \det D\eta(y,t) = \operatorname{div} u(\eta(y,t),t) \det D\eta(y,t) = \operatorname{div} u(\eta(y,t),t)$$

for all $y \in \Omega_0, t \in \mathbb{R}^+$. Since $\eta(\cdot,t) : \Omega_0 \to \Omega(t)$ is a diffeomorphism, we deduce that $\operatorname{div} u(x,t) = 0$ for all $t \in \mathbb{R}^+, x \in \Omega(t)$.

Finally, suppose the third item. We know

$$\begin{cases} \partial_t \det D\eta(y,t) = \operatorname{div} u(\eta(y,t),t) \det D\eta(y,t) = 0 \\ \det D\eta(y,0) = 1 \end{cases}$$

and thus $\det D\eta(y,t) = 1$ for all y, t. □

This suggests some notation.

Definition. *The Eulerian velocity u is called incompressible if* $\operatorname{div} u(\cdot,t) = 0$ *for all* $t \in \mathbb{R}^+$.

The theorem then says: $\eta(\cdot,t)$ is locally volume-preserving for all t if and only if u is incompressible. Note, though that this <u>does not mean</u> that $\operatorname{div}_y v(y,t) = 0$.

Now we use the evolution of $\det D\eta$ to construct an essential tool.

Theorem 1.4 (Transport theorem)**.** *Let $U_0 \subseteq \Omega_0$ be open and set $U(t) = \eta(U_0,t) \subseteq \Omega(t)$ for $t \in \mathbb{R}^+$. Let $f(\cdot,t) : \Omega(t) \to \mathbb{R}$ for $t \geq 0$. Then*

$$\frac{d}{dt}\int_{U(t)} f(x,t)\ dx = \int_{U(t)} \partial_t f(x,t) + \operatorname{div}(f(x,t)u(x,t))\ dx.$$

Proof. We have that $\eta(\cdot,t) : U_0 \to U(t) \subseteq \Omega(t)$ is a diffeomorphism, so

$$\int_{U(t)} f(x,t)\ dx = \int_{U_0} f(\eta(y,t),t) \det D\eta(y,t)\ dy$$

for $t \in \mathbb{R}^+$. Thus

$$\begin{aligned}
\frac{d}{dt}\int_{U(t)} f(x,t)\ dx &= \frac{d}{dt}\int_{U_0} f(\eta(y,t),t)\det D\eta(y,t)\ dy \\
&= \int_{U_0} [\partial_t f(\eta(y,t),t) + \nabla f(\eta(y,t),t)\cdot\partial_t\eta(y,t)]\det D\eta(y,t) \\
&\qquad + f(\eta(y,t),t)\partial_t \det D\eta(y,t)\ dy \\
&= \int_{U_0} [\partial_t f(\eta(y,t),t) + \nabla f(\eta(y,t),t)\cdot u(\eta(y,t),t) \\
&\qquad + f(\eta(y,t),t)\operatorname{div} u(\eta(y,t),t)]\det D\eta(y,t)\ dy \\
&= \int_{U(t)} [\partial_t f(x,t) + \nabla f(x,t)\cdot u(x,t) + f(x,t)\operatorname{div} u(x,t)]\ dx \\
&= \int_{U(t)} \partial_t f + \operatorname{div}(fu).
\end{aligned}$$

□

Remark 1.5. *The theorem trivially extends to $f(\cdot,t) : \Omega(t) \to \mathbb{R}^m$ for $m \geq 2$. In this case*

$$\frac{d}{dt}\int_{U(t)} f = \int_{U(t)} \partial_t f + \operatorname{div}(f \otimes u)$$

where $f \otimes u \in \mathbb{R}^{m\times 3}$ is given by $(f\otimes u)_{ij} = f_i u_j$ for $i = 1,\ldots,m$ and $j = 1,\ldots,3$ and if $M : U(t) \to \mathbb{R}^{m\times n}$ then $\operatorname{div} M \in \mathbb{R}^m$ is given by $(\operatorname{div} M)_i = \sum_{j=1}^n \partial_j M_{ij}$. In other words, divergence acts on matrix functions along each row.

Remark 1.6. *If $\partial U(t)$ is Lipschitz for $t \in \mathbb{R}^+$, then*

$$\frac{d}{dt} \int_{U(t)} f = \int_{U(t)} \partial_t f + \int_{\partial U(t)} f \otimes u \cdot \nu,$$

and the latter term is interpreted as the "flux of f" across the surface $\partial U(t)$. Note: $f \otimes u \cdot \nu = f(u \cdot \nu)$.

We will apply the transport theorem very often, so we introduce some notation.

Definition. *We say $\{U(t)\}_{t \in \mathbb{R}^+}$ is a flow if $U_0 = U(0) \subseteq \Omega_0$ is open with ∂U_0 either $\varnothing$ or else Lipschitz, and $U(t) = \eta(U_0, t)$ for $t > 0$. We say a flow is interior if $U(t) \Subset \Omega(t)$ for all $t \geq 0$.*

Remark 1.7. *Here we have that $U(t)$ is open with $\partial U(t)$ either empty or Lipschitz. We do this so that $U(t)$ is always measurable and $\partial U(t)$ has a $\mathcal{H}^2$-a.e. defined unit normal ν (if $\partial U(t) \neq \varnothing$) and the divergence theorem holds! This could possibly be improved, but we will settle for this, as it already allows for a very large class of flows.*

1.2 Mass and the continuity equation

Continuum mass assumptions:

1. We assume that one of the defining characteristics of a continuum is that it has mass. This is analogous to the concept of mass for point particles, except that we assume that the mass is distributed throughout $\Omega(t)$ with a C^1 density distribution $\rho(\cdot, t) : \Omega(t) \to (0, \infty)$.

2. For any $U \subseteq \Omega(t)$ measurable we say that the mass contained in U is

   $$\mathcal{M}(U) = \int_U \rho(x, t) \; dx.$$

   Note that if U is open then $\mathcal{M}(U) > 0$, i.e. all open subsets of $\Omega(t)$ carry mass.

   With this notion in hand we can state the third mass assumption.

3. We assume the principle of mass conservation: if $\{U(t)\}$ is a flow then $\mathcal{M}(U(t)) = \mathcal{M}(U_0)$ for all $t \in \mathbb{R}^+$, i.e. the mass is constant along a flow. In particular this means that $\mathcal{M}(\Omega(t)) = \mathcal{M}(\Omega_0)$, i.e. the total mass of the flow does not change.

Let's now combine these assumptions with the transport theorem.

Theorem 1.8 (Continuity equation derivation)**.** *The density function $\rho(\cdot, t) : \Omega(t) \to (0, \infty)$ satisfies the PDE $\partial_t \rho + \operatorname{div}(\rho u) = 0$ in $\Omega(t)$ for all $t \geq 0$.*

Proof. Let $\{U(t)\}$ be a flow. Since $\mathcal{M}(U(t)) = \mathcal{M}(U_0)$ for all $t \geq 0$ we have that $d/dt\mathcal{M}(U(t)) = 0$ for all $t \geq 0$. Thus

$$0 = \frac{d}{dt}\mathcal{M}(U(t)) = \frac{d}{dt} \int_{U(t)} \rho = \int_{U(t)} \partial_t \rho + \operatorname{div}(\rho u)$$

for all $t \geq 0$. Since this holds for every flow $\{U(t)\}$ we deduce that $\partial_t \rho + \operatorname{div}(\rho u) = 0$ in $\Omega(t)$ for all $t \geq 0$. □

Definition. *The equation $\partial_t \rho + \operatorname{div}(\rho u) = 0$ is called the continuity equation or the conservation of mass equation.*

Remark 1.9. *Two remarks on the equation $\partial_t \rho + \operatorname{div}(\rho u) = 0$:*

1. *The PDE is first order and linear in ρ if we know u.*
2. *Transport:*

$$\underbrace{\partial_t \rho + u \cdot \nabla \rho}_{\text{transport by } u} + \underbrace{\rho \operatorname{div} u}_{\operatorname{div} u \text{ acts as a source}}$$

$$\begin{cases} \operatorname{div} u > 0 \implies \text{density decreases along characteristic curve} \\ \operatorname{div} u < 0 \implies \text{density increases along characteristic curve} \end{cases}$$

It turns out that something interesting happens to the density in Lagrangian coordinates. To see this, we make a few definitions.

Definition. *The Lagrangian mass density is $\alpha : \Omega_0 \times \mathbb{R}^+ \to (0, \infty)$ given by $\alpha(y,t) = \rho(\eta(y,t),t) \iff \rho(x,t) = \alpha(\eta^{-1}(x,t),t)$. The Lagrangian Jacobian is $J : \Omega_0 \times \mathbb{R}^+ \to (0,\infty)$ given by $J = \det D\eta$.*

Theorem 1.10. $\partial_t(\alpha J) = 0$ *in* $\Omega_0 \times \mathbb{R}^+$.

Proof. We know $\partial_t J(y,t) = \operatorname{div} u(\eta(y,t),t) J(y,t)$ and

$$\begin{aligned} \partial_t \alpha(y,t) = \frac{d}{dt}\rho(\eta(y,t),t) &= \partial_t \rho(\eta(y,t),t) + \nabla \rho(\eta(y,t),t) \cdot \partial_t \eta(y,t) \\ &= \partial_t \rho(\eta(y,t),t) + \nabla \rho(\eta(y,t),t) \cdot u(\eta(y,t),t) \\ &= -\operatorname{div} u(\eta(y,t),t)\rho(\eta(y,t),t) \\ &= -\operatorname{div} u(\eta(y,t),t)\alpha(y,t). \end{aligned}$$

Thus

$$\partial_t(\alpha J) = \partial_t \alpha J + \alpha \partial_t J = (\operatorname{div} u \circ \eta - \operatorname{div} u \circ \eta)\, \alpha J = 0.$$

$\square$

Corollary 1.11.

$$\rho(\eta(y,t),t)J(y,t) = \alpha(y,t)J(y,t) = \alpha(y,0) = \rho(y,0) \implies \rho(\eta(y,t),t) = \frac{\rho(y,0)}{J(y,t)}.$$

In particular, the flow is locally volume-preserving if and only if $\rho(\eta(y,t),t) = \rho(y,0)$ for all $t \geq 0$.

The upshot of this is that the local volume distortion and mass are inversely proportional.

1.3 Momentum, force, torque, and momentum balance

1.3.1 Newton's laws – 1687

Let's begin recalling Newton's laws:

1. An object in motion remains in the state of motion until acted upon by a force.
2. The rate of change of an object's momentum equals the force acting on it.
3. The force one object exerts on another is equal to the opposite of the force the other exerts on it.

Math translation: the first and second items state that $\dot{p} = F$ where $p = mv = m\dot{x}$, and the third items states that $F_{12} = -F_{21}$. Note: the angular momentum measure with respect to $x_0 \in \mathbb{R}^3$ is $L = (x - x_0) \times p$, and $\dot{L} = (x - x_0) \times F$ is the torque.

Newton's laws are formulated for particles and so don't quite work for rigid bodies or continua. Newtonian mechanics was extended to rigid bodies by Euler.

1.3.2 Euler's laws – 1750

A rigid body is given by $\Omega(t) = \eta(\Omega_0, t)$, where η is a rigid motion. We define the linear and angular momentum by

$$p(t) = \int_{\Omega(t)} \rho u, \qquad L(t) = \int_{\Omega(t)} x \times \rho u.$$

1. The total force acting on $\Omega(t)$ is
$$\frac{d}{dt} p(t) = \mathcal{F}(t)$$

2. The total torque acting on $\Omega(t)$ is
$$\frac{d}{dt} L(t) = \mathcal{T}(t)$$

We delay a specification of $\mathcal{F}, \mathcal{T}$ for a moment. Rigid body mechanics is more complicated than particle mechanics because of L and the inertia (which we don't define) related to the distribution of mass through a body. Euler was able to deduce his laws by idealizing the rigid body as a collection of point particles and passing to the limit. Actually, we can go the other way and derive particle mechanics from Euler by considering shrinking rigid bodies and assuming something called "Euler's cut principle". So in some sense the Newtonian and Eulerian laws are the same.

To specify the dynamics of a (non-rigid) continuum we will employ a version of Euler's laws expanded upon by Cauchy.

1.3.3 Cauchy-Euler Laws – 1822

Let $\{U(t)\}_t$ be a flow. The linear/angular momenta are

$$\mathcal{P}(U(t)) = \int_{U(t)} \rho u, \qquad \mathcal{L}(U(t)) = \int_{U(t)} x \times \rho u.$$

Note: we could also define $\mathcal{L}_{x_0}(U(t)) = \int_{U(t)} (x - x_0) \times \rho u$.

We will also assume that to each flow we can specify the force and torque acting on $U(t)$ by $\mathcal{F}(U(t))$ and $\mathcal{T}(U(t))$, respectively. Furthermore, we assume that there are two types of forces/torques:

1. Body/bulk: these are long-range forces/torques that act on the interior of $U(t)$, e.g. gravity, electromagnetism. These are given by a bulk force density $f(\cdot, t) : \Omega(t) \to \mathbb{R}^3$.

2. Surface: these are short-range forces/torques that act on $\partial U(t)$. They are of two subtypes:

   (a) Contact: these occur only on $\partial U(t) \setminus \partial\Omega(t)$ and are caused by the contact between $U(t)$ and $U(t)^c = \Omega(t) \setminus U(t)$.

   (b) Boundary: these occur only on $\partial\Omega(t) \cap \partial U(t)$ and are due to things like surface tension. This is given by a density $\psi(\circ, t) : \partial\Omega(t) \to \mathbb{R}^3$.

Then the Cauchy-Euler laws are as follows:

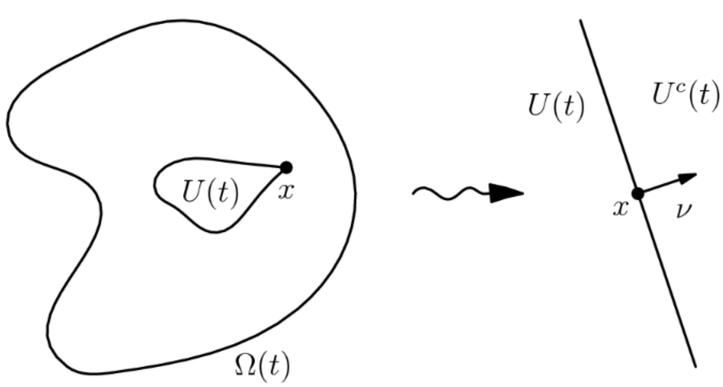
1. For any flow $\{U(t)\}_t$ we have that
$$\frac{d}{dt} \mathcal{P}(U(t)) = \mathcal{F}(U(t)).$$

2.
$$\frac{d}{dt}\mathcal{L}(U(t)) = \mathcal{T}(U(t)).$$

For these to be useful we must specify $\mathcal{F}$ and $\mathcal{T}$. This is done in the third law.

3. (Cauchy's hypothesis) Let $\{U(t)\}$ be a flow. Then we have that
$$\mathcal{F}(U(t)) = \mathcal{F}_b(U(t)) + \mathcal{F}_s(U(t))$$
$$\mathcal{T}(U(t)) = \underbrace{\mathcal{T}_b(U(t))}_{\text{bulk}} + \underbrace{\mathcal{T}_s(U(t))}_{\text{surface}}$$
where
$$\mathcal{F}_b(U(t)) = \int_{U(t)} f(x,t)\ dx, \qquad \mathcal{T}_b(U(t)) = \int_{U(t)} x \times f(x,t)\ dx$$
and
$$\mathcal{F}_s(U(t)) = \int_{\partial U(t)\cap\partial\Omega(t)} \psi(x,t)\ dx + \int_{\partial U(t)\setminus\partial\Omega(t)} T(\nu,x,t)\ dx$$
$$\mathcal{T}_s(U(t)) = \int_{\partial U(t)\cap\partial\Omega(t)} x \times \psi(x,t)\ dx + \int_{\partial U(t)\setminus\partial\Omega(t)} x \times T(\nu,x,t)\ dx$$
for ν the outward unit normal and $T(\cdot,\cdot,t) : \mathbb{S}^2 \times \Omega(t) \to \mathbb{R}^3$ a map called the Cauchy traction. We assume T is at least C^1.

Where does the traction come from? Heuristics: suppose $\{U(t)\}$ is an interior flow. Then
$$\mathcal{F}_s(U(t)) = \int_{\partial U(t)} \Psi$$
where Ψ is the contact force on $\partial U(t)$. Since Ψ is supposed to be short-range, it's reasonable to assume $S = T(\nu,x,t)$. Indeed by zooming in on x, we see that Ψ should on depend on x, t, and the local geometry of $\partial U(t)$ near x.



Since $U(t)$ is just an abstract splitting of $\Omega(t)$ we further expect Ψ to only depend on the tangent space, since all reasonable ways of probing Ψ at x have the same tangent space. Since the tangent is determined by ν, we're led to Cauchy's hypothesis: $S = T(\nu,x,t)$.

Let's now explore the implications of our assumptions.

Lemma 1.12 (Balance of momentum, version 1)**.** *Let $\{U(t)\}_{t\in\mathbb{R}^+}$ be an interior flow. Then*

$$\int_{U(t)} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \int_{U(t)} f + \int_{\partial U(t)} T(\nu, \cdot, \cdot).$$

Proof. We simply combine the transport theorem and Cauchy-Euler 1 and Cauchy-Euler 3:

$$\int_{U(t)} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \frac{d}{dt}\int_{U(t)} \rho u = \frac{d}{dt}\mathcal{P}(U(t)) = \mathcal{F}(U(t)) = \int_{U(t)} f + \int_{\partial U(t)} T(\nu, \cdot, \cdot).$$

□

Now we derive an important property of T.

Lemma 1.13 (Cauchy's lemma / Newton's 3rd for contact forces)**.** *We have that $T(\nu, x, t) = -T(-\nu, x, t)$ for all $t \in \mathbb{R}^+, x \in \Omega(t), \nu \in \mathbb{S}^2$.*

Proof. Fix $t_0 \in \mathbb{R}^+$ and $x_0 \in \Omega(t_0)$, and consider $\nu \in \mathbb{S}^2$. Let $\Gamma \in \mathbb{R}^2$ be a hyperplane such that $x \in \Gamma$ with ν normal to Γ. Let $W \Subset \Omega(t_0)$ be open with ∂W Lipschitz. Set $U = W \cap H^+$, $V = W \cap H^-$ where H^+, H^- are the open half-spaces related to Γ, with ν pointing into H^+. Then $\partial U, \partial V$ are Lipschitz, $x \in \partial U \cap \partial V$, and $\Sigma = \partial U \cap \partial V \subset \Gamma$ is relatively open in Γ. Assume ν points into U. Set $U_0 = \eta^{-1}(U, t_0)$, $V_0 = \eta^{-1}(V, t_0)$, $W_0 = \eta^{-1}(W, t_0)$ which are open with Lipschitz boundaries. Then for the flows $\{U(t)\}, \{V(t)\}, \{W(t)\}$, we know

$$\frac{d}{dt}\mathcal{P}(U(t)) = \mathcal{F}(U(t)), \quad \frac{d}{dt}\mathcal{P}(V(t)) = \mathcal{F}(V(t)), \quad \frac{d}{dt}\mathcal{P}(W(t)) = \mathcal{F}(W(t)).$$

Clearly, $\mathcal{P}(U(t)) + \mathcal{P}(V(t)) = \mathcal{P}(W(t))$, so $\mathcal{F}(U(t)) + \mathcal{F}(V(t)) = \mathcal{F}(W(t))$, and hence

$$\begin{aligned}
&\int_{U(t_0)} f + \int_{V(t_0)} f + \int_{\partial U(t_0)} T + \int_{\partial V(t_0)} T = \int_{W(t_0)} f + \int_{\partial W(t_0)} T \\
\implies &\int_{W(t_0)} f + \int_{\partial W(t_0)} T + \int_{\Sigma} T(\nu, \cdot, t_0) + T(-\nu, \cdot, t_0) = \int_{W(t_0)} f + \int_{\partial W(t_0)} T \\
\implies &\int_{\Sigma} T(\nu, \cdot, t_0) + T(-\nu, \cdot, t_0) = 0.
\end{aligned}$$

The above construction can be carried out such that $\Sigma = B(x_0, r) \cap \Gamma$ for any $r > 0$, and so

$$\frac{1}{\pi r^2}\int_{B(x_0,r)\cap\Gamma} T(\nu, \cdot, t_0) + T(-\nu, \cdot, t_0) = 0$$

for all $r > 0$ and thus

$$T(\nu, x_0, t_0) + T(-\nu, x_0, t_0) = \lim_{r\to 0}\frac{1}{\pi r^2}\int_{B(x_0,r)\cap\Gamma} T(\nu, \cdot, t_0) + T(-\nu, \cdot, t_0) = 0.$$

□

With the previous two lemmas in hand we can now prove an extremely important result.

Theorem 1.14 (Cauchy's stress tensor theorem)**.** *For $t \in \mathbb{R}^+$ there exists a tensor field $S(\cdot, t) : \Omega(t) \to \mathbb{R}^{3\times 3}$ such that $T(\nu, x, t) = -S(x, t)\nu$ for all $\nu \in \mathbb{S}^2, x \in \Omega(t)$. S is called the (Cauchy) stress tensor. Moreover, $S_j(x, t) = -T(e_j, x, t) \cdot e_j$ and so S is as regular as T.*

Proof.

- **Step 1**

  We claim that if $t \in \mathbb{R}^+, x \in \Omega(t)$, $\{z_i\}_{i=1}^3$ is an orthonormal basis of $\mathbb{R}^3$ and $\nu \in \mathbb{S}^2$ is such that $z_i \cdot \nu > 0$ for $i = 1, 2, 3$, then

  $$T(\nu, x, t) = (\nu \cdot z_i) T(z_i, x, t).$$

  Let $\varepsilon > 0$ and consider the tetrahedron $W_\varepsilon \subset \Omega(t)$ such that

  1. x is a vertex of W_ε
  2. Three sides have normal vectors $-z_i$, $i = 1, 2, 3 : \Gamma_\varepsilon^i$
  3. The fourth side has normal vector $\nu : \Gamma_\varepsilon^\nu$
  4. We have that $x + \varepsilon\nu \in \Gamma_\varepsilon^\nu$.

  Now, by balance of momentum version 1, we know that

  $$\int_{W_\varepsilon} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - f = \int_{\partial W_\varepsilon} T.$$

  We leave as an exercise for the reader to show that

  $$\int_{\partial W_\varepsilon} T = \left[T(-z_i, x, t)(z_i \cdot \nu) + T(\nu, x, t)\right] \mathcal{H}^2(\Gamma_\varepsilon^\nu) + o(\varepsilon^2).$$

  Then since $\mathcal{H}^2(\Gamma_\varepsilon^\nu) = \Theta(\varepsilon^2)$ we have that

  $$o(1) = \frac{1}{\mathcal{H}^2(\Gamma_\varepsilon^\nu)} \int_{W_\varepsilon} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = T(\nu, x, t) + T(-z_i, x, t)(z_i \cdot \nu) + o(1)$$

  and thus by sending $\varepsilon \to 0$, we find that

  $$T(\nu, x, t) = -T(-z_i, x, t)(z_i \cdot \nu) = T(z_i, x, t)(z_i \cdot \nu)$$

  by Cauchy's lemma. This proves the claim.

- **Step 2**

  Let $t \in \mathbb{R}^+, x \in \Omega(t), \nu \in \mathbb{S}^2$ and choose $\{z_i\}_{i=1}^3 \subseteq \mathbb{R}^3$ an orthonormal basis such that $\nu \cdot z_i \neq 0$ for $i = 1, 2, 3$. Set $w_i = \operatorname{sgn}(\nu \cdot z_i) z_i$, and note that $\{w_i\}$ is still an orthonormal basis. Then step 1 implies that

  $$T(\nu, x, t) = T(w_i, x, t)(w_i \cdot \nu) = T(\operatorname{sgn}(\nu \cdot z_i) z_i, x, t) \operatorname{sgn}(\nu \cdot z_i)(z_i \cdot \nu) = T(z_i, x, t)(z_i \cdot \nu)$$

  by Cauchy's lemma. Define $S(\cdot, t) : \Omega(t) \to \mathbb{R}^{3\times 3}$ by

  $$S_{ij}(x, t) = -T(e_j, x, t) \cdot e_i$$

  and consider $\mathbb{S}_*^2 = \mathbb{S}^2 \setminus \{\pm e_1, \pm e_2, \pm e_3\}$. The above analysis shows that if $\nu \in \mathbb{S}_*^2$ then

  $$T(\nu, x, t) = T(e_i, x, t)(e_i \cdot \nu) = (T(e_i, x, t) \cdot e_j)(e_i \cdot \nu) e_j = -S_{ji}(x, t)\nu_i e_j = -S(x, t)\nu.$$

  Since T is continuous and $\mathbb{S}_*^2 \subseteq \mathbb{S}^2$ is dense, we deduce that in fact $T(\nu, x, t) = -S(x, t)\nu$ for all $\nu \in \mathbb{S}^2$.

□

We can now use the stress tensor to refine balance of momentum.

Theorem 1.15 (Balance of momentum, version 2). *We have that*

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \operatorname{div} S = f$$

in $\Omega(t)$ *for* $t \in \mathbb{R}^+$.

Proof. Let $\{U(t)\}_t$ be an interior flow. Then balance of mass version 1 says

$$\int_{U(t)} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \int_{U(t)} f + \int_{\partial U(t)} T(\nu, \cdot, t) = \int_{U(t)} f - \int_{\partial U(t)} S(\cdot, t)\nu = \int_{U(t)} f - \operatorname{div} S,$$

and hence

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \operatorname{div} S = f$$

in $\Omega(t)$. □

Remark 1.16. *We compute*

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = [\partial_t \rho + \operatorname{div}(\rho u)]\, u + \rho(\partial_t u + u \cdot \nabla u),$$

so

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \operatorname{div} S = f \end{cases} \iff \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \operatorname{div} S = f \end{cases}.$$

Thus far we have not used Cauchy-Euler 2. Let's use it now.

Theorem 1.17. *For* $t \in \mathbb{R}^+$ *and* $x \in \Omega(t)$ *we have that* $S(x,t) = S^\top(x,t)$, *i.e.* S *is symmetric.*

Proof. Let $\{U(t)\}_t$ be an interior flow. Then Cauchy-Euler 2 says that

$$\frac{d}{dt}\int_{U(t)} x \times \rho u = \int_{U(t)} x \times f + \int_{\partial U(t)} x \times T = \int_{U(t)} x \times f - \int_{\partial U(t)} x \times S\nu.$$

We leave as an exercise to show that

$$\frac{d}{dt}\int_{U(t)} x \times \rho u = \int_{U(t)} x \times [\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u)] + u \times \rho u = \int_{U(t)} x \times [\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u)]$$

and

$$\int_{\partial U(t)} x \times S(x,t)\nu = \int_{U(t)} x \times \operatorname{div} S(x,t) + \int_{U(t)} \varepsilon_{ijk} S_{jk}(x,t) e_i$$

where

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) \text{ is an even permutation} \\ -1 & \text{if } (ijk) \text{ is an odd permutation} \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\int_{U(t)} x \times [\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u)] = \int_{U(t)} x \times (f - \operatorname{div} S) - \int_{U(t)} \varepsilon_{ijk} S_{jk} e_i \implies \int_{U(t)} \varepsilon_{ijk} S_{jk}(x,t) e_i \, dx = 0.$$

Since $\{U(t)\}$ was arbitrary, we deduce that $\varepsilon_{ijk} S_{jk} e_i = 0$ in $\Omega(t)$ for $t \in \mathbb{R}^+$. Then $\varepsilon_{ijk} S_{jk} = 0$ for $i = 1, 2, 3$ and thus $S_{23} = S_{32}, S_{13} = S_{31}, S_{12} = S_{21} \implies S = S^\top$. □

Remark 1.18. *A modification of the above argument shows that the conditions Cauchy-Euler 1, Cauchy-Euler 3, and* $S = S^\top$, *are equivalent to Cauchy-Euler 1, Cauchy-Euler 2, and Cauchy-Euler 3. Thus, Cauchy-Euler 2 does not contribute a new PDE (balance of angular momentum) but instead yields the* <u>*structural result*</u> *that* $S = S^\top$.

1.4 Energy and dissipation

In this course we will only consider continua evolving through purely mechanical processes. This means that we will neglect temperature and heat dependent effects and thereby avoid the need to develop thermodynamics.

We assume the following:

1. The continuum possesses mechanical energy that is distributed via $E(\cdot,t) : \Omega(t) \to \mathbb{R}$. Moreover, we have that $E = \rho |u|^2/2 + \rho\varepsilon$, where $\rho |u|^2/2$ is the kinetic energy and $\rho\varepsilon$ is the "free energy density" given with $\varepsilon(\cdot,t) : \Omega(t) \to \mathbb{R}$.

2. If $\{U(t)\}_t$ is a flow we define
$$\mathcal{E}(U(t)) = \int_{U(t)} E(x,t)\;dx$$
to be the energy,
$$\mathcal{W}(U(t)) = \int_{U(t)} f \cdot u - \int_{\partial U(t) \setminus \partial\Omega(t)} S\nu \cdot u + \int_{\partial U(t) \cap \partial\Omega(t)} \psi \cdot u$$
to be the external power.

3. Dissipation hypothesis: if $\{U(t)\}$ is a flow then
$$\frac{d}{dt}\mathcal{E}(U(t)) \leq \mathcal{W}(U(t)).$$
This means that not all of the power goes to changing the energy: some is dissipated. We could set $\mathcal{D}(U(t)) = \mathcal{W}(U(t)) - \dot{\mathcal{E}}(U(t)) \geq 0$.

Theorem 1.19 (Energy dissipation inequality)**.** *We have that for $t \in \mathbb{R}^+$*
$$\partial_t E + \operatorname{div}(Eu) \leq f \cdot u - \operatorname{div}(Su).$$

Proof. Let $\{U(t)\}_t$ be an interior flow. Then
$$\begin{aligned}\int_{U(t)} \partial_t E + \operatorname{div}(Eu) = \frac{d}{dt}\int_{U(t)} E &\leq \int_{U(t)} f \cdot u - \int_{\partial U(t)} S\nu \cdot u \\ &= \int_{U(t)} f \cdot u - \int_{\partial U(t)} Su \cdot \nu = \int_{U(t)} f \cdot u - \operatorname{div}(Su).\end{aligned}$$
This holds for all flows, so $\partial_t E + \operatorname{div}(Eu) \leq f \cdot u - \operatorname{div}(Su)$. □

Remark 1.20. *A simple computation (exercise!) shows that*
$$\partial_t\left(\frac{\rho |u|^2}{2}\right) + \operatorname{div}\left(\frac{\rho |u|^2}{2} u\right) = f \cdot u - \operatorname{div}(Su) + S : Du,$$
and hence
$$\rho(\partial_t\varepsilon + u \cdot \nabla\varepsilon) = \partial_t(\rho\varepsilon) + \operatorname{div}(\rho\varepsilon u) \leq -S : Du$$
is equivalent to the energy dissipation inequality.

Proof. We first check the second equality. Indeed, write

$$\begin{aligned}\partial_t(\rho\varepsilon) + \operatorname{div}(u\rho\varepsilon) &= (\partial_t\rho)\varepsilon + \rho(\partial_t\varepsilon) + [\varepsilon \operatorname{div}(\rho u) + (\rho u)\cdot\nabla\varepsilon] \\ &= \rho[\partial_t\varepsilon + u\cdot\nabla\varepsilon] + \varepsilon[\partial_t\rho + \operatorname{div}(\rho u)] = \rho(\partial_t\varepsilon + u\cdot\nabla\varepsilon).\end{aligned}$$

It thus remains to prove the simple computation. To do this, we will show that both sides are equal to $(\rho u)\cdot\partial_t u + \rho|u|^2\nabla u$. We first demonstrate this with the left hand side. To begin, recall that

$$\partial_t|u|^2 = \partial_t(u\cdot u) = 2u\cdot\partial_t u.$$

This means that

$$\partial_t(\rho|u|^2) = |u|^2\partial_t\rho + \rho\cdot\partial_t|u|^2 = |u|^2\cdot\partial_t\rho + 2(\rho u)\cdot\partial_t u,$$

and so

$$\partial_t\left(\frac{\rho|u|^2}{2}\right) = \frac{\partial_t\rho\cdot|u|^2}{2} + (\rho u)\cdot\partial_t u.$$

For the second term, we do something similar: write

$$\operatorname{div}(\rho|u|^2u) = \operatorname{div}((\rho u)|u|^2) = |u|^2\operatorname{div}(\rho u) + \rho u\cdot\nabla(|u|^2) = |u|^2\operatorname{div}(\rho u) + 2\rho|u|^2\cdot\nabla u,$$

which means in turn that

$$\operatorname{div}\left(\frac{\rho|u|^2}{2}u\right) = \frac{|u|^2\operatorname{div}(\rho u)}{2} + \rho|u|^2\cdot\nabla u.$$

Thus adding yields

$$\begin{aligned}\partial_t\left(\frac{\rho|u|^2}{2}\right) + \operatorname{div}\left(\frac{\rho|u|^2}{2}u\right) &= \frac{|u|^2}{2}[\partial_t\rho + \operatorname{div}(\rho u)] + (\rho u)\cdot\partial_t u + \rho|u|^2\cdot\nabla u \\ &= (\rho u)\cdot\partial_t u + \rho|u|^2\cdot\nabla u,\end{aligned}$$

proving the left hand side of the equality.

We now turn to the right hand side. Our first goal is to remove the presence of S. To do this, recall (i.e. guess) that since S is a tensor field we have $\operatorname{div}(Su) = u\cdot\operatorname{div}S + S : Du$. Thus, with the aide of Balance of Momentum II, we have

$$f\cdot u - \operatorname{div}(Su) + S : Du = f\cdot u - u\cdot\operatorname{div}S = u\cdot\partial_t(\rho u) + u\cdot\operatorname{div}(\rho u\otimes u).$$

It remains to simplify this expression. To do this, first recall that

$$\operatorname{div}(\rho u\otimes u) = u\operatorname{div}(\rho u) + (\partial_{\rho u}u)|\rho u| = u\operatorname{div}(\rho u) + \nabla u\cdot\rho u = u\operatorname{div}(\rho u) + \rho(u\cdot\nabla u).$$

Further, note that $u\cdot\partial_t(\rho u) = |u|^2(\partial_t\rho) + (\rho u)\cdot\partial_t u$ by product rule. Thus, we have

$$\begin{aligned}u\cdot\partial_t(\rho u) + u\cdot\operatorname{div}(\rho u\otimes u) &= |u|^2[\partial_t\rho + \operatorname{div}(\rho u)] + (\rho u)\cdot\partial_t u + \rho|u|^2\cdot\nabla u \\ &= (\rho u)\cdot\partial_t u + \rho|u|^2\cdot\nabla u,\end{aligned}$$

and so the right hand side of the equality has been established as well. We are done. $\square$

1.5 Synthesis

We record the equations of motion:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \operatorname{div} S = f \\ \rho(\partial_t \varepsilon + u \cdot \nabla \varepsilon) \leq -S : Du \\ S = S^\top \end{cases}.$$

These are the equations of purely mechanical continuum mechanics. Note that this is far from a closed system. We need to specify the form of S and ε. This specification depends on the material the comprises the continuum. For example S is quite different in an elastic solid from S in a gas. We will focus exclusively on fluid flow. Before moving on, we turn to a discussion that will help us determine S and ε.

1.6 Frame indifference

Let's say an observer (A) sees $\Omega(t)$ evolve in Eulerian coordinates (x, t). Suppose now that a second observer (B) sees the evolution of $\Omega(t)$ but utilizes a different spatial coordinate system. Further suppose A and B have the same measuring systems, which means for $t \in \mathbb{R}^+$ there exists an isometry $\Psi(\cdot, t) : \mathbb{R}^3_A \to \mathbb{R}^3_B$ between the coordinate systems. Then there exists $z : \mathbb{R}^+ \to \mathbb{R}^3, R : \mathbb{R}^+ \to O(3)$ such that $\Psi(x,t) = z(t) + R(t)x$. Let's write $\tilde{x} = \Psi(x,t)$. Then $(\tilde{x}, t)$ is B-coordinates and A-coordinates are (x,t). Extending this notation, we put a $\sim$ on quantities measured by B, e.g. $\tilde{u}, \tilde{\rho}, \tilde{\Omega}(t)$, etc.

Definition.

1. *We say a scalar $\varphi(\cdot,\cdot)$ if frame-indifferent if $\tilde{\varphi} = \varphi$, i.e. if $\tilde{\varphi}(\tilde{x}, t) = \varphi(x,t)$ for $t \in \mathbb{R}^+, x \in \Omega(t), \tilde{x} = \Psi(x,t) \subset \tilde{\Omega}(t)$.*

2. *Note that $\tilde{x} - \tilde{y} = R(t)(x, y)$, so displacement transforms by $R(t)$ multiplication. We say a vector $w \in \mathbb{R}^3$ is frame-indifferent if $\tilde{w} = R(t)w$, i.e. $\tilde{w}(\tilde{x}, t) = R(t)w(x,t)$.*

3. *Say that $w(x,t), v(x,t) \in \mathbb{R}^3$ are frame-indifferent and that $v = Mw$ for a tensor field $M(\cdot, t) : \Omega(t) \to \mathbb{R}^{3\times 3}$. Then*

$$\tilde{v} = (\tilde{Mw}) = RMw = RMR^\top Rw = RMR^\top \tilde{w} =: \tilde{M}\tilde{w}.$$

A 2-tensor $M \in \mathbb{R}^{3\times 3}$ is frame-indifferent if $\tilde{M}(\tilde{x}, t) = R(t)M(x,t)R^\top(t)$.

As an example, velocity is <u>not</u> frame-indifferent:

$$\tilde{u}(\tilde{x}, t) = R(t)u(x,t) + \dot{R}(t)R^{-1}(t)(\tilde{x} - z(t)) + \dot{z}(t)$$

nor its derivative:

$$D_{\tilde{x}}\tilde{u}(\tilde{x}, t) = R(t)D_x u(x,t)R^{-1}(t) + \dot{R}(t)R^{-1}(t).$$

Proof. Set $\tilde{\eta}(y,t) = z(t) + R(t)\eta(y,t)$. Then $\tilde{v}(y,t) = R(t)v(y,t) + \dot{R}(t)\eta(y,t) + \dot{z}(t)$. Note that

$$\begin{aligned} \tilde{x} = \tilde{\eta}(y,t) = R(t)\eta(y,t) + z(t) &\implies x = \eta(y,t) = R^{-1}(t)(\tilde{x} - z(t)) \\ &\implies \tilde{\eta}^{-1}(\tilde{x}, t) = y = \eta^{-1}(R^{-1}(t)\tilde{x} - R^{-1}(t)z(t), t). \end{aligned}$$

Then

$$\tilde{u}(\tilde{x},t) = \tilde{v}(\tilde{\eta}^{-1}(\tilde{x},t),t) = R(t)u\left(R^{-1}(t)(\tilde{x}-z(t)),t\right) + \dot{R}(t)R^{-1}(t)(\tilde{x}-z(t)) + \dot{z}(t).$$

□

We make the following frame-indifference assumptions:

1. The scalars ρ and ε are frame-indifferent.
2. The vector force f is frame-indifferent.
3. The 2-tensor S is frame-indifferent.

We now discuss the differential geometry interpretation of frame-indifference. Let T be a 2-tensor field on Ω_0. Since η is a diffeomorphism, we can push it forward:

$$\begin{aligned}\eta_* T(x)(v,w) &= T(\eta^{-1}(x))(D\eta^{-1}(x)v, D\eta^{-1}(x)w)\\ &= T_{ij}A_{ik}v_kA_{j\ell}w_\ell = \left(A^\top_{ki}T_{ij}A_{j\ell}\right)v_kw_\ell\\ &= \left(A^\top TA\right)_{k\ell}v_kw_\ell = \left(A^\top TA\right)(v,w),\end{aligned}$$

where we write A for $D\eta^{-1}$, and so $S(x) = \eta_*T := D\eta^{-\top}(x)T(\eta^{-1}(x))D\eta^{-1}(x)$. Now suppose we also have $\tilde{\eta}, \tilde{\Omega}(t)$, where $\eta, \tilde{\eta}$ are related by $\tilde{x} = z + Rx$. Then

$$\tilde{\eta}^{-1}(\tilde{x}) = y = \eta^{-1}(x) = \eta^{-1}(R^\top(\tilde{x}-z)) \implies D_{\tilde{x}}\tilde{\eta}^{-1}(\tilde{x}) = D_x\eta^{-1}(R^\top(\tilde{x}-z))R^\top = D_x\eta^{-1}(x)R^\top.$$

So,

$$\tilde{S}(\tilde{x}) = D_{\tilde{x}}\tilde{\eta}^{-\top}(\tilde{x})T(y)D_{\tilde{x}}\tilde{\eta}^{-1}(\tilde{x}) = RD_x\eta^{-\top}(x)T(y)D_x\eta^{-1}(x)R^\top = RS(x)R^\top$$

and

$$\tilde{u}(\tilde{x}) = \left(D_{\tilde{x}}\eta^{-1}(\tilde{x})\right)^{-1}v(y) = R\left(D_x\eta^{-1}(x)\right)^{-1}v(y) = Ru(x).$$

We can play a similar game with scalar fields, i.e. 0-tensors.

Note that the equations of motion are <u>not</u> frame-indifferent. They are only indifferent with respect to Galilean transformations: $\ddot{z} = 0, \dot{R} = 0$.

Remark 1.21. *The above shows that rules for frame-indifference follow naturally from notions in differential geometry, namely pushforward and pushbackward maps.*

2 Fundamentals of fluid mechanics

2.1 Constitutive relations in compressible fluids

In order to close the equations of motion we must assume constitutive laws: equations relating ε and S to ρ, η, and u.

Informally, a fluid is a continuum that experiences stress due to density and local spatial variations in the velocity. The latter we can imagine as a sort of friction: $u = \lambda\rho$.

Definition. *A compressible fluid is a continuum for which there exist functions* $\beta : (0,\infty)\times\mathbb{R}^{3\times3}\to\mathbb{R}, \Gamma : (0,\infty)\times\mathbb{R}^{3\times3}\to \mathrm{Sym}(3)$ *such that* $\varepsilon(x,t) = \beta\left(\rho(x,t), Du(x,t)\right)$ *and* $S(x,t) = \Gamma\left(\rho(x,t), Du(x,t)\right)$.

To determine β and Γ we will enforce the frame-indifference: $\beta(\tilde{\rho}, D\tilde{u}) = \tilde{\varepsilon} = \varepsilon = \beta(\rho, Du)$ for all frame changes. Similarly, $\Gamma(\tilde{\rho}, D\tilde{u}) = \tilde{S} = RSR^\top = R(t)\Gamma(\rho, Du)R^\top(t)$. We previously computed $D\tilde{u} = R(t)DuR^\top(t) + \dot{R}(t)R^\top(t)$, and by assumption $\tilde{\rho} = \rho$. Thus,

$$\begin{aligned}\beta(\rho, Du) &= \beta\left(\rho, R(t)DuR^\top(t) + \dot{R}(t)R^\top(t)\right)\\ R(t)\Gamma(\rho, Du)R^\top(t) &= \Gamma\left(\rho, R(t)DuR^\top(t) + \dot{R}(t)R^\top(t)\right).\end{aligned}$$

These must hold for all possible fluid motions, and so we require that

$$\begin{aligned}\beta(r, M) &= \beta\left(r, R(t)MR^\top(t) + \dot{R}(t)R^\top(t)\right)\\ R(t)\Gamma(r, M)R^\top(t) &= \Gamma\left(r, R(t)MR^\top(t) + \dot{R}(t)R^\top(t)\right)\end{aligned}$$

for all $r \in (0, \infty)$ and $M \in \mathbb{R}^{3\times 3}$.

Let $A, R \in \mathbb{R}^{3\times 3}$ be such that $R_0 \in O(3), A^\top = -A$ and set $R(t) = e^{tA}R_0 \in O(3)$. Note that $\dot{R}(t) = Ae^{tA}R_0$, and so $R(0) = R_0, \dot{R}(0)R^\top(0) = AR_0R_0^\top = A$. Hence

$$\begin{aligned}\beta(r, M) &= \beta(r, RMR^\top + A)\\ R\Gamma(r, M)R^\top &= \Gamma(r, RMR^\top + A)\end{aligned}$$

for all $r \in (0, \infty)$, $M \in \mathbb{R}^{3\times 3}$, and all $R \in O(3)$ such that $A = -A^\top$.

Choose $R = I, A = (M^\top - M)/2$. Then

$$\begin{aligned}\beta(r, M) &= \beta(r, M + A) = \beta\left(r, \frac{M + M^\top}{2}\right)\\ \Gamma(r, M) &= \Gamma(r, M + A) = \Gamma\left(r, \frac{M + M^\top}{2}\right).\end{aligned}$$

Thus, β and Γ only depend on the symmetric part of M, so by abuse of notation and rearranging things, we then have that

$$\varepsilon(x, t) = \beta(\rho(x, t), \mathbb{D}u(x, t)), \qquad S(x, t) = \Gamma(\rho(x, t), \mathbb{D}(x, t))$$

where $\mathbb{D}u = Du + Du^\top$ and $\beta : (0, \infty) \times \mathrm{Sym}(3) \to \mathbb{R}, \Gamma(0, \infty) \times \mathrm{Sym}(3) \to \mathbb{R}^{3\times 3}$ are such that $\beta(r, RMR^\top) = \beta(r, M), R\Gamma(r, M)R^\top = \Gamma(r, RMR^\top)$ for all $r \in (0, \infty), M \in \mathrm{Sym}(3), R \in O(3)$, and $A = -A^\top$.

Note that if we set $M = 0$, then

$$\Gamma(r, 0) = R\Gamma(r, 0)R^\top$$

for all $R \in O(3)$, which implies (exercise) that $\Gamma(r, 0) = p(r)I$ for $p : (0, \infty) \to \mathbb{R}$, called the pressure. We can then decompose Γ as

$$\Gamma(r, M) = \Gamma(r, 0) + \Gamma(r, M) - \Gamma(r, 0) := p(r)I + \Gamma_0(r, M).$$

Frame-indifference then requires that

$$R\Gamma_0(r, M)R^\top = \Gamma_0(r, RMR^\top)$$

for all r, M, R.

Next we examine the energy-dissipation inequality:

$$\rho(\partial_t \varepsilon + u \cdot \nabla \varepsilon) \leq -S : Du.$$

We compute

$$\partial_t \varepsilon = \partial_t \beta(\rho(x,t), \mathbb{D}u(x,t)) = \partial_r \beta(\rho, \mathbb{D}u) \partial_t \rho + \frac{\partial \beta}{\partial M_{ij}}(\rho, \mathbb{D}u) \partial_t \mathbb{D}u_{ij} + u \cdot \nabla \mathbb{D}u_{ij}$$

and similarly

$$-S : Du = -(pI + \Gamma_0) : Du = -p(\rho) \operatorname{div} u - \Gamma_0(\rho, \mathbb{D}u) : Du,$$

so

$$\rho \frac{\partial \beta}{\partial M_{ij}}(\rho, \mathbb{D}u) \left[\partial_t \mathbb{D}u_{ij} + u \cdot \nabla \mathbb{D}u_{ij}\right] + \left[-\partial_r \beta(\rho, \mathbb{D}u)\rho^2 + p(\rho)\right] \operatorname{div} u + \frac{\Gamma_0(\rho, \mathbb{D}u) : \mathbb{D}u}{2} \leq 0.$$

This holds for all possible flows, and we then can specify $\partial_t \mathbb{D}u + u \cdot \nabla \mathbb{D}u = N$ arbitrarily. Then,

$$r \frac{\partial \beta}{\partial M_{ij}}(r, M) N_{ij} + \left[p(r) - r^2 \partial_r \beta(r, M)\right] \operatorname{tr} L + \Gamma_0(\rho, M) : L \leq 0$$

for all r, N, M, where $L = M + M^\top$. Since N appears linearly, we can contradict the inequality by scaling N appropriately, unless $\partial \beta / \partial M_{ij}(r, M) = 0$ for all i, j. Then $\partial \beta / \partial M_{ij}(r, M) = 0$ for all i, j and hence $\beta = \beta(r)$.

Now

$$p(r) - r^2 \beta'(r)) \operatorname{tr} M + \Gamma_0(r, M + M^\top) : M \leq 0$$

for all r, M. Let $M = \alpha M_0$ for some $M_0 \in \operatorname{Sym}(3)$. Then $(p(r) - r^2 \beta'(r)) \operatorname{tr} M_0 + \Gamma_0(r, \alpha(M_0 + M_0^\top)) : M_0 \leq 0$ for all α. Sending $\alpha \to 0$ and using $\Gamma_0(r, 0) = 0$ shows that

$$(p(r) - r^2 \beta'(r)) \operatorname{tr} M_0 \leq 0 \qquad \text{for all } M_0 \qquad \Longrightarrow p(r) - r^2 \beta'(r) = 0.$$

Thus

$$\beta(r) = \beta_0 + \int_1^r \frac{p(s)}{s^2} \, ds$$

for some $\beta_0 \in \mathbb{R}$. In turn we find that $\Gamma_0(r, M) : M \leq 0$ for all $M \in \operatorname{Sym}(3)$. We've proved the following theorem.

Theorem 2.1. *For compressible fluids, we must have*

$$S(x,t) = p(\rho(x,t))I + \Gamma(\rho(x,t)\mathbb{D}u(x,t)), \qquad \varepsilon(x,t) = \beta_0 + \int_1^{\rho(x,t)} \frac{\rho(s)}{s^2} \, ds$$

for $p : (0, \infty) \to \mathbb{R}$, *and* $\Gamma_0 : (0, \infty) \times \operatorname{Sym}(3) \to \operatorname{Sym}(3)$ *such that* $\Gamma_0(r, M) : M \leq 0$ *for all* $r, M \in \operatorname{Sym}(3)$ *and* $R\Gamma_0(r, M)R^\top = \Gamma_0(r, M)$ *for all* r, M, R.

We now quote two important linear algebra results without proof. For proofs, see, e.g., Gurtin's Introduction to Continuum Mechanics [Gur82], section 37.

Theorem 2.2. *Define the "invariant map" ($J(RMR^\top) = J(M)$ for all $M \in \mathbb{R}^{3\times 3}, R \in O(3)$) $J : \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ via*

$$J_1(M) = \operatorname{tr} M, \quad J_2(M) = \frac{(\operatorname{tr} M)^2 - \operatorname{tr}(M^2)}{2}, \quad J_3(M) = \det M.$$

Then the following are equivalent for $F : \mathrm{Sym}(3) \to \mathrm{Sym}(3)$:

1. *$RF(M)R^\top = F(RMR^\top)$ for all $M \in \mathrm{Sym}(3), R \in O(3)$.*
2. *There exist $\varphi_0, \varphi_1, \varphi_2 : \mathbb{R}^3 \to \mathbb{R}$ such that*

$$F(M) = \varphi_0(J(M))I + \varphi_1(J(M))M + \varphi_2(J(M))M^2.$$

Theorem 2.3. *Suppose $F : \mathrm{Sym}(3) \to \mathrm{Sym}(3)$ is linear. Then the following are equivalent:*

1. *$RF(M)R^\top = F(RMR^\top)$ for all $M \in \mathrm{Sym}(3), R \in O(3)$.*
2. *There exist $\lambda, \mu \in \mathbb{R}$ such that $F(M) = \mu M + \lambda(\operatorname{tr} M)I$.*

These two theorems completely characterize the possible forms of Γ_0. We will focus on a special type.

Definition. *A compressible fluid is Newtonian if for each $r \in (0, \infty)$ the map $\Gamma_0(r, \cdot) : \mathrm{Sym}(3) \to \mathrm{Sym}(3)$ is linear.*

Theorem 2.4. *For a Newtonian fluid we have that there exist $\mu, \lambda : (0, \infty) \to \mathbb{R}$ such that*

$$\Gamma_0(r, M) = -\mu(r)M - \frac{\lambda(r)}{2}(\operatorname{tr} M)I$$

and $\mu(r) \geq 0, \lambda(r) \geq -\frac{2}{3}M(R)$ for all $r \in (0, \infty)$.

Proof. Since the fluid is Newtonian we may write $\Gamma_0(r, M) = -\mu(r)M - \frac{1}{2}\lambda(r)(\operatorname{tr} M)I$. However, we must have that $\Gamma_0(r, M + M^\top) : M \leq 0$ for all m, r if and only if

$$-\mu(r)(M + M^\top) : M - \frac{1}{2}\lambda(r) \operatorname{tr}(M + M^\top) \operatorname{tr} M \leq 0$$

for all M, r. Choose $M = I$:

$$-\mu(r)2I : I - \lambda(r)\frac{1}{2} \operatorname{tr}(2I) \operatorname{tr} I \leq 0 \implies -6\mu(r) - 9\lambda(r) \leq 0 \implies \lambda(r) \geq -\frac{2}{3}\mu(r).$$

Also, if $\operatorname{tr} M = 0$, then

$$-\mu(r)\left|M + M^\top\right|^2 \leq 0$$

and hence $\mu(r) \geq 0$. $\square$

Corollary 2.5. *In a compressible Newtonian fluid we have*

$$S(x, t) = p(\rho(x, t))I - \mu(\rho(x, t))\mathbb{D}u(x, t) - \lambda(\rho(x, t))(\operatorname{div} u)I.$$

We now have the full equations of motion:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p(\rho) - \operatorname{div}\left(\mu(\rho)\mathbb{D}u + \lambda(p)(\operatorname{div} u)I\right) = f \end{cases}$$

where $p, \mu, \lambda : (0, \infty) \to \mathbb{R}$ with $\mu \geq 0, \lambda \geq -\frac{2}{3}\mu$. We typically assume $p \geq 0$ and increasing. We call $\mu \geq 0$ the shear viscosity, and $\lambda + \frac{2}{3}\mu \geq 0$ the bulk viscosity. We assume

$$\begin{cases} \mu = \lambda = 0 \iff \text{inviscid flow} \iff \text{compressible Euler} \\ \mu > 0, \lambda + \frac{2}{3}\mu \geq 0 \iff \text{viscous flow} \iff \text{compressible Navier-Stokes} \end{cases}.$$

Note: μ, λ constant $\implies \operatorname{div}(\mu\mathbb{D}u + \lambda \operatorname{div} uI) = \mu\Delta u + (\lambda + \mu)\nabla \operatorname{div} u$.

2.2 Incompressible fluids

Now we assume that the flow is incompressible, i.e. locally volume preserving. Then $\operatorname{div} u = 0$ and so $\partial_t \rho + u \cdot \nabla \rho = 0$. This means that ρ a constant solves this equation trivially. We will assume this, which means we should resally say incompressible, homogeneous fluids, but this is a standard abuse of notation.

The procedure we used for compressible fluids was

1. Assume constitutive equations.

2. Plug into energy-dissipation inequality to deduce structure.

Let's look at this under the incompressibility assumption. Write $S = (\operatorname{tr} S/3)I + S_0$. Then

$$\rho(\partial_t \varepsilon + u \cdot \nabla \varepsilon) \leq -S : Du = -S_0 : Du - \frac{\operatorname{tr} S}{3} \operatorname{tr} Du = -S_0 : Du.$$

This means that we have no hope of computing $\operatorname{tr} S$; we can only hope to determine S_0.

Definition. *An incompressible fluid is a continuum in which we assume*

1. $\operatorname{div} u = 0$*,* ρ *is constant.*

2. There exists a "pressure function" $p(\cdot, t) : \Omega(t) \to \mathbb{R}$ *and functions* $\beta : \mathbb{R}^{3\times 3} \to \mathbb{R}, \Gamma : \mathbb{R}^{3\times 3} \to \operatorname{Sym}_0(3) = \{M \in \operatorname{Sym}(3) : \operatorname{tr} M = 0\}, S(x, t) = p(x, t)I + \Gamma(Du(x, t))$ *(i.e.* $3p = \operatorname{tr} S, S_0 = \Gamma$*),* $\varepsilon(x, t) = \beta(Du(x, t))$*.*

Frame-indifference again says that $\beta(Du) = \beta(\mathbb{D}u), \Gamma(Du) = \Gamma(\mathbb{D}u)$. Then energy-dissipation says

$$\rho \frac{\partial \beta}{\partial M_{ij}}(\mathbb{D}u)\left(\partial_t \mathbb{D}u_{ij} + u \cdot \nabla \mathbb{D}u_{ij}\right) + \Gamma(\mathbb{D}u) : Du \leq 0,$$

from which we again deduce that

$$\begin{aligned} &\rho \frac{\partial \beta}{\partial M_{ij}}(M + M^\top)N_{ij} + \Gamma(M + M^\top) : M \leq 0 \quad \text{for all } N, M \in \mathbb{R}^{3\times 3} \\ \implies &\frac{\partial \beta}{\partial M_{ij}}(M + M^\top) = 0 \qquad\qquad\qquad\qquad \text{for all } i, j,\ \Gamma(M + M^\top) : M \leq 0 \text{ for all } M \\ \implies &\beta \text{ constant}, \Gamma(M + M^\top) : M \leq 0 \end{aligned}$$

Frame-indifference also says that $R\Gamma(M)R^\top = \Gamma(RMR^\top)$, so we know all possible Γ.

Definition. *An incompressible Newtonian fluid is an incompressible fluid for which Γ is linear.*

Theorem 2.6. *In a Newtonian incompressible fluid we have that $S(x,t) = p(x,t)I - \mu\mathbb{D}u(x,t)$ for some $\mu \geq 0$.*

Proof. We can use the frame-indifference to find that $\Gamma(M) = -\mu M$ for all $M \in \mathrm{Sym}_0(3)$, and then energy-dissipation requires $-\mu(M + M^\top) : M \leq 0$ for all $M \in \mathrm{Sym}_0(3)$ and thus $\mu \geq 0$. $\square$

The full equations of motion are:

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) + \nabla p - \operatorname{div}(\mu\mathbb{D}u) = f \\ \operatorname{div} u = 0 \end{cases}.$$

Here,

$$\begin{cases} \mu = 0 \iff \text{inviscid} \iff \text{incompressible Euler} \\ \mu > 0 \iff \text{viscous} \iff \text{incompressible Navier-Stokes} \end{cases}.$$

Remark 2.7.

1. $\operatorname{div}(\mu\mathbb{D}u) = \mu\Delta u$

2. *We have 4 scalar unknowns and 4 equations, so we can hope to "solve for p":*

$$\begin{aligned} & \rho(\partial_t u + u \cdot \nabla u) + \nabla p = f + \mu\Delta u, \operatorname{div} u = 0 \\ \implies & \rho\partial_t \operatorname{div} u + \operatorname{div}(u \cdot \nabla u) + \Delta p = \operatorname{div} f + \mu\Delta \operatorname{div} u = 0 \\ \implies & -\Delta p = \operatorname{div}(u \cdot \nabla u) - \operatorname{div} f = \partial_i(u_j\partial_j u_i) - \operatorname{div} f = \operatorname{tr}(Du^2) - \operatorname{div} f \end{aligned}$$

so in some sense we should be able to compute p from u, f.

2.3 Boundary conditions

We will focus on two types of boundaries: rigid and moving/free.

2.3.1 Rigid

In this case we assume that $\partial\Omega(t)$ meets a rigid solid for all $t \in \mathbb{R}^+$. In this case the fluid cannot penetrate the solid, so we must have that

$$u \cdot \nu = v_{\mathrm{sol}} \cdot \nu$$

on $\partial\Omega(t)$, where v_{sol} is the velocity of the rigid solid. In particular, if the solid is stationary, then $v_{\mathrm{sol}} = 0$ and $u \cdot \nu = 0$.

If the fluid is viscous then we typically assume that actually

$$u = v_{\mathrm{sol}}$$

on $\partial\Omega(t)$ since otherwise slipping should generate "too much friction". In the stationary case $u = 0$.

2.3.2 Moving

For simplicity, let's only consider the case of two distinct fluids evolving together:

$$\mathbb{R}^3 = \Omega_1(t) \sqcup \Omega_2(t), \qquad \partial\Omega_1(t) \cap \partial\Omega_2(t) = \Sigma(t).$$

1. We assume the fluids remain in contact for all times.
2. The boundary force is $\psi(\cdot, t) : \Sigma(t) \to \mathbb{R}^3$.
3. Write $[\![g]\!] = g_2 - g_1$.

The first item is enforced via

$$\begin{cases} [\![u \cdot \nu]\!] = 0 & \text{inviscid} \\ [\![u]\!] = 0 & \text{viscous} \end{cases}.$$

Now consider $U_0 \subseteq \Omega_1(0)$ such that $\partial U_0 \cap \Sigma(0) \neq \varnothing$. By Cauchy-Euler 1, we have

$$\begin{aligned} \frac{d}{dt}\int_{U(t)} \rho_1 u_1 &= \int_{U(t)} f_1 + \int_{\partial U(t)\cap\Sigma(t)} -S_2\nu + \psi + \int_{\partial U(t)\setminus\Sigma(t)} -S_1\nu \\ &= \int_{U(t)} f_1 + \int_{\partial U(t)} -S_1\nu + \int_{\partial U(t)\cap\Sigma(t)} -[\![S\nu]\!] + \psi. \end{aligned}$$

and

$$0 = \int_{U(t)} \partial_t(\rho_1 u_1) + \operatorname{div}(\rho_1 u_1 \otimes u_1) + \operatorname{div} S_1 - f_1 = \int -\partial U(t) \cap \Sigma(t) - [\![S\nu]\!] + \psi.$$

Because U_0 was arbitrary, we find that $[\![S\nu]\!] = \psi$ on $\Sigma(t)$. Therefore, the inviscid free boundary conditions are

$$\begin{cases} [\![u \cdot \nu]\!] = 0 \\ [\![p]\!]\nu = \psi \end{cases}$$

the viscous free boundary conditions are

$$\begin{cases} [\![u]\!] = 0 \\ [\![S\nu]\!] = \psi \end{cases}$$

on $\Sigma(t)$, where ν points from domain 1 to domain 2. We can apply similar arguments to find that

$$\begin{aligned} &\int_{U(t)} x \times (\partial_t(\rho_1 u_1) + \operatorname{div}(\rho_1 u_1 \otimes u_1) + \operatorname{div} S_1 - f_1) = \int_{\partial U(t)\cap\Sigma(t)} x \times (-[\![S\nu]\!] + \psi) \\ &\int_{U(t)} \partial_t E + \operatorname{div}(Eu) - f \cdot u + \operatorname{div}(Su) \leq \int_{\partial U(t)\cap\Sigma(t)} -[\![Su \cdot \nu]\!] + \psi \cdot u. \end{aligned}$$

However,

$$\text{viscous} : [\![S\nu]\!] - \psi, [\![u]\!] = 0 \implies \begin{cases} x \times ([\![-S\nu]\!] + \psi) = 0 \\ [\![Su \cdot \nu]\!] = [\![\nu \cdot u]\!] = [\![S\nu]\!] \cdot u = \psi \cdot u \end{cases}$$

$$\text{inviscid} : [\![p]\!]\nu = \psi, [\![u \cdot \nu]\!] = 0 \implies \begin{cases} x \times ([\![-p]\!]\nu + \psi) = 0 \\ [\![Su \cdot \nu]\!] = [\![p]\!]u \cdot \nu = \psi \cdot u \end{cases}$$

and so in either case we gain no new info.

2.4 Scaling and the Reynold's number

Frame-indifference is about rigid changes of measuring frame, but we could also consider changes given by switching the units of measurement. This is related to scalings of the equations.

Consider a viscous incompressible fluid:

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) + \nabla p - \mu \Delta u = f \quad \text{in } \Omega(t) \\ \operatorname{div} u = 0 \end{cases}.$$

Fix $L, T > 0$ and consider

$$v(x,t) = \frac{T}{L} u(Lx, Tt), \quad q(x,t) = \frac{T^2}{L^2 \rho} p(Lx, Tt), \quad g(x,t) = \frac{T^2}{L\rho} f(Lx, Tt).$$

A simple computation shows that

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla q - \frac{1}{\mathcal{R}} \Delta v - g = \frac{T^2}{L\rho} \left(\rho \left(\partial_t u + u \cdot \nabla u \right) + \nabla p - \mu \Delta u - f \right) = 0 \\ \operatorname{div} v = 0 \end{cases}$$

in $\tilde{\Omega}(t) = L^{-1} \Omega(Tt)$, where

$$\mathcal{R} = \frac{L^2 \rho}{\mu T} > 0$$

is called Reynold's number.

The math upshots:

1. By rescaling we can reduce to a single parameter $\mathcal{R}$.
2. If we choose $L, T > 0$ we can force $\mathcal{R} = 1$, so there's no real loss of generality in studying

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p \cdot \Delta u = f \\ \operatorname{div} u = 0 \end{cases}.$$

Actually, the Reynolds number $\mathcal{R}$ has an important physical meaning. To see why, we have to recall the physical idea of units: mass (m), length (ℓ), time (τ). Write $[\cdot]$ for the units of something. Then

$$[u] = \frac{\ell}{\tau}, [\rho(\partial_t u + u \cdot \nabla u)] = [f] = \frac{1}{\mathcal{R}^3} \frac{m\ell}{\tau^2}, [p] = \frac{1}{\ell^2} \frac{m\ell}{\tau^2}, [\mu] = \frac{1}{\ell^3} \frac{m\ell^2}{\tau}.$$

Therefore, $\mathcal{R} = L^2 \rho / (\mu T)$ is <u>unitless</u> because $[L] = \ell, [\rho] = m/\ell^3, [T] = \tau$. The physical interpretation is that the rescaled problem

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla q - \frac{1}{\mathcal{R}} \Delta v = g \\ \operatorname{div} v = 0 \end{cases}$$

is unitless, and that all of the physics of scale is encoded in $\mathcal{R}$. In particular, any two flows with the same $\mathcal{R}$ are identical, up to rescaling.

References

[Gur82] Morton E Gurtin. *An introduction to continuum mechanics*, volume 158. Academic press, 1982.