Variational Aspects of Steady Irrotational Water Wave Theory

John Toland

ONEPAS Online Seminar 22nd September 2020

Abstract

Among the many modern approaches to abstract nonlinear problems, those based on the implicit function theorem, real-analytic function theory, Nash-Moser theory and topological degree theory have made significant contributions to water-wave theory in recent years. However, the same cannot be said of variational methods (min/max, mountain-pass, Morse index, Lyusternik-Schnirelman genus etc) even though, when the viscosity of water is ignored and the flow is assumed to be irrotational, there are several attractive ways to formulate the equations of wave motion variationally.

*On the 100th anniversary of the first proof that the equations of motion have non-zero, small-amplitude solutions, this talk will briefly survey these issues and advocate variational methods for analyzing water waves that are 2π-periodic in space. Below are some publications which, with the references therein, are related to these topics; those marked * are useful places to start.*

Local and Global Bifurcation Theory

Real-Analytic Bifurcation Theory - local and global

1. **Analytic Theory of Global Bifurcation - An Introduction.* (With B. Buffoni.) Princeton University Press (ISBN 0 691 11298 3), Princeton, 2003.
2. Lecture Notes in French *Introduction á la Théorie Globale des Bifurcations* (With B. Buffoni.) Presses Polytechniques et Universitaires Romandes, 2002. (http://ppur.epfl.ch/livres/2-88074-494-6.html) ISBN 2 88074 4946

Global Topological Theory of Stokes Waves

3. * Stokes waves. *Topological Methods in Nonlinear Analysis*, **7**(1) and **8**(2), (1996), 1-48 and 413-414.
4. On the existence of a wave of greatest height and Stokes's conjecture. *Proc. R. Soc. Lond.* A **363** (1978), 469-485.
5. On the Stokes conjecture for the wave of extreme form. (With C.J. Amick and L.E. Fraenkel.) *Acta mathematica,* **148** (1982), 193-214.
6. The Fourier coefficients of Stokes waves. (With P. I. Plotnikov.) *Nonlinear Problems in Mathematical Physics and Related Topics I*, A Volume dedicated to Olga A. Ladyzhensk'aya on her 80th birthday. Eds. M. Sh. Birman, S Hildebrandt, V. A. Solonnikov, N. N. Uraltseva. Pages 303-316. Kluwer, New York, 2002.
7. Convexity of Stokes waves of extreme form. (With P. I. Plotnikov.) *Arch. Rational Mech. Anal.* **171** (2004), 349-416.

Global Topological Theory of Solitary Waves

8. On solitary water-waves of finite amplitude. (With C.J. Amick.) *Arch. Rational Mech. Anal.,* **76** (1) (1981), 9-95.
9. On periodic water-waves and their convergence to solitary waves in the long-wave limit. (With C.J. Amick.) *Phil. Trans. R. Soc. Lond.* A **303** (1981), 633-669.
10. Solitary waves with surface tension I: trajectories homoclinic to periodic orbits in four dimensions. (With C.J. Amick.) *Arch. Rational Mech. Anal.*, **118** (1992), 37-69.

Local Bifurcation of Waves with Surface Tension

11. The bifurcation and secondary bifurcation of capillary-gravity waves. (With M.C.W. Jones.) *Proc. R. Soc. Lond.* A **399** (1985), 391-417.
12. Symmetry and the bifurcation of capillary-gravity waves. (With M.C.W. Jones.) *Arch. Rational Mech. Anal.,* **96** (1) (1986), 29-53.

Standing Waves

Nash-Moser Theory

13. * The semi-analytic theory of standing waves. (With C.J. Amick.) *Proc. R. Soc. Lond.* A **411** (1987), 123-137.
14. A Nash-Moser theorem for standing water waves. (With P. I. Plotnikov.) *Proc. Equadiff'99*, Vol. II (Eds. B. Fiedler, K. Gröger & J. Sprekels, ISBN 981-02-4989-6), World Scientific, Singapore, 2000, 1379-1384.
15. Nash-Moser theory for standing water waves. (With P. I. Plotnikov.) *Arch. Rational Mech. Anal.* 159 (2001), 1-83
16. Standing waves problem on infinite depth. (With G. Iooss and P. I. Plotnikov.) C. R. .Acad. Sci. Paris, 338 (5), (2004), 425-431.
17. Ondes de gravité stationnaire en profondeur infinie. (With G. Iooss and P. I. Plotnikov.) *Comptes Rendus Acad. Sci. Paris*, Vol 338/5 (2004), 425-431.
18. Standing waves on an infinitely deep perfect fluid under gravity. (With G. Iooss and P. I. Plotnikov.) *Arch. Rational Mech. Anal.* **177** (3), (2005), 367-478.
19. * Riemann-Hilbert and variational structure for standing waves. (With G. Iooss.) *Far East J. Appl. Math.* **15** (3), (2004), 459-488.

Variational Theories

Travelling Waves - Babenko's Equation

20. A variational theory of Stokes waves and their sub-harmonic bifurcations. (With B. Buffoni and E. N. Dancer.) *University of Bath Preprint January, 1998.* (Accessible anonymously via ftp.maths.bath.ac.uk:pub/preprints.
21. *The regularity and local bifurcation of steady periodic water waves. (With B. Buffoni & E. N. Dancer.) *Arch. Rational Mech. Anal.* **152** (3), (2000), 207-240.
22. *The sub-harmonic bifurcation of Stokes waves. (With B. Buffoni & E. N. Dancer.) *Arch. Rational Mech. Anal.* **152** (3), (2000), 241-270.
23. *Stokes waves in Hardy spaces and as distributions. *J. Math. Pure et Appl.* **79** (9), (2000), 901-917.

24. *Dual free boundaries for Stokes waves. (With B. Buffoni.) *Comptes Rendus Acad. Sci. Paris*, Ser. I **332**, (2001), 73-78.

25. * On a pseudo-differential equation for Stokes waves. *Arch. Rational. Mech. Anal.*, 162 (2002) 179-189.

26. A Riemann-Hilbert problem and the Bernoulli boundary condition in the variational theory of Stokes waves. (With E Shargorodsky.) *Annales Institut H. Poincaré, Analyse Nonlineare,* 20 (1) (2003), 37-52.

27. Riemann-Hilbert theory for problems with vanishing coefficients that arise in nonlinear hydrodynamics. (With E. Shargorodsky.) *J. Funct. Anal.* 197 (1) (2003), 283-300

28. **Bernoulli-Free Boundary Problems.* (With E. Shargorodsky.) Memoir Amer. Math. Soc. MS, **196** (914) (2008), viii+70 pp. ISBN: 978-0-8218-4189-1

Miscellaneous Variational Approaches

29. On variational formulations of steady water waves. (With M.D. Groves.) *Arch. Rational Mech. Anal.* **137** (1997), 203-226.

30. Surface waves as saddle points of the energy. (With B. Buffoni & E. Séré.) *Calculus of Variations*, **17** (2003), 199-220.

31. Minimization methods for quasi-linear problems. with an application to periodic water waves. (With B. Buffoni and E. Séré.) *SIAM J. Math. Anal.* **36** (4), (2005), 1080-1094.

32. Surface waves on steady perfect-fluid flows with vorticity. (With G. R. Burton.) *Comm. Pure Appl. Math.* **64** (2011), 975-1007.

33. Modelling hydroelastic waves. (With P. I. Plotnikov.) *Phil. Trans. Royal Soc.* A **369** (2011), 2942-2956.

34. Non-existence of global minimisers of energy in Stokes-wave problems. *Discrete Continuous Dynam. Systems - A* **34** (2014), 3211-3217.

35. Energy-minimising parallel flows with prescribed vorticity distribution. *Discrete Continuous Dynam. Systems - A* **34** (2014), 3193-3210.

36. Variational problems in the theory of hydroelastic waves. (With P. I. Plotnikov.) *Phil. Trans. Royal Soc.* A **376**:20170343 (2018).