Measure Asymptotic Separation Index and Hyperfiniteness

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Abstract

In this note, we show that modulo a null set, hyperfiniteness, finite asymptotic separation index (asi), asi 1, and existence of Borel toast are all equivalent. This is of interest as several of the directions of this equivalence are open problems in the Borel context.

1 Introduction

Let X be a set. By an *extended metric* on X, we mean a metric which is allowed to take the value ∞. If ρ is such a metric, the pair (X, ρ) is called an *extended metric space*. If X is a standard Borel space, such a ρ is called *Borel* if it is Borel as a function $X^2 \to \mathbb{R} \cup \{\infty\}$, and in this case (X, ρ) is called a *Borel extended metric space*.

We call an extended metric *locally finite* if every finite radius ball with respect to it is finite.

For (X, ρ) an extended metric space, we write E_ρ to denote the equivalence relation $\{(x, y) \in X^2 \mid \rho(x, y) < \infty\}$. It is Borel if (X, ρ) is. To avoid trivialities in what follows, we will always assume that each E_ρ class is infinite.

This note concerns several notions of what might be called "finitization" in the Borel context. The first is the following well studied definition: A countable Borel equivalence relation E on a space X is called *hyperfinite* if it can be written as the increasing union of a countable sequence of finite Borel equivalence relations on X.

The second was defined recently in [1]. First, for (X, ρ) an extended metric space and $r > 0$, we define the *r-jump graph* G_ρ^r as the graph on X for which two distinct points $x, y \in X$ are adjacent iff $\rho(x, y) \leq r$. It is Borel if (X, ρ) is.

We then say $\mathrm{asi}(X, \rho) \leq s$, for $s \in \omega$, if for each $r > 0$, X can be partitioned into sets $U_0, \ldots, U_s$ such that for each i, the connected components of $G^r_\rho \restriction U_i$ all have finite ($\rho-$) diameter. Note that this is equivalent to those components being finite when ρ is locally finite. This defines $\mathrm{asi}(X, \rho)$, called the *asymptotic separation index* of (X, ρ), as an element in $\omega \cup \{\infty\}$.

Replacing "finite diameter" with "uniformly bounded diameter" in the above definition, one recovers a notion due to Gromov [6] called *asymptotic dimension*. Asymptotic separation index as we have defined it, though, is uninteresting: It is always 1, as can be witnessed by alternating annulli around a base point in each E_ρ-class.

Both asymptotic dimension and separation index, though, were studied in [1] in the Borel context (where they are both interesting). When (X, ρ) is Borel, we define the *Borel asymptotic dimension separation index*, denoted $\mathrm{asi}_B(X, \rho)$, as we did $\mathrm{asi}(X, \rho)$, but where the U_i's are required to be Borel.

The condition that asi_B be finite has proven to have much utility in Borel combinatorics. For example, in [1] it is shown that in this case, asymptotic dimension and Borel asymptotic dimension are equal, and an upper bound for Borel chromatic numbers of graphs with finite asi_B (for their path metrics) is obtained. This bound is better for lower asi_B. In [11], Certain Borel Schreier graphs are shown to admit Borel degree-plus-one edge colorings if $\mathrm{asi}_B = 1$. The last two sentences may help motivate the following question from [1]:

Problem 1. *Is there a locally finite Borel extended metric space* (X, ρ) *for which* $1 < \mathrm{asi}_B(X, \rho) < \infty$*?*

In [1], a negative answer is found under the additional assumption that (X, ρ) has finite asymptotic dimension

This next question, though not mentioned in [1], also seems to be open.

Problem 2. *Let* (X, ρ) *be a locally finite Borel extended metric space for which* $\mathrm{asi}_B(X, \rho) = 1$. *Is* E_ρ *hyperfinite? What if only* $\mathrm{asi}_B(X, \rho) < \infty$*?*

Again, in [1], a positive answer is found under the additional assumption that (X, ρ) has finite asymptotic dimension.

The third notion has its origins in [3] and [4], though it is typically phrased in the less general language of graphs. Our formalization is based on that from [5]. For (X, ρ) an extended metric space and $r > 0$, let us say an *r-toast* is a collection $\mathcal{C} \subset [E_\rho]^{<\omega}$ (this means a collection of finite subsets of X, each contained in a single E_ρ-class) satisfying:

- For all $(x, y) \in E_\rho$, there is some $C \in \mathcal{C}$ with $x, y \in C$.

- For distinct $C, D \in \mathcal{C}$, either $\rho(C, D) > r$, $B(C, r) \subset D$, or $B(D, r) \subset C$.

Here and throughout, $B(A, r)$ denotes the set of points with distance at most r from A. If (X, ρ) is Borel, we call this toast *Borel* if its Borel as a subset of $[X]^{<\omega}$.

This note is concerned with these notions in the measurable setting. Let (X, μ) be a standard Borel probability space. We say a countable Borel equivalence relation E on X is *μ-hyperfinite* if $(X', E \upharpoonright X')$ is hyperfinite for some Borel E-invariant μ-conull set X'. Similarly, if ρ is a Borel extended metric on X, we say $\mathrm{asi}_\mu(X, \rho) \leq s$ if there is some X' as above (with $E = E_\rho$) for which $\mathrm{asi}_B(X', \rho \upharpoonright X') \leq s$, and we say (X, ρ) admits a μ-measurable r-toast if there is some such X' for which $(X', \rho \upharpoonright X')$ admits a Borel r-toast.

In Section 3 of this note, we will show the following:

Lemma 1. *Let (X, ρ) be a locally finite Borel extended metric space and μ a Borel probability measure on X. If $\mathrm{asi}_\mu(X, \rho) < \infty$, then E_ρ is μ-hyperfinite.*

This answers Problem 2 in the measurable setting. Perhaps more interestingly, we combine this lemma with the work from [3] (which we will rephrase in our setting in Section 2) to answer Problem 1 as well, plus a little more:

Theorem 1. *Let (X, ρ) be a locally finite Borel extended metric space and μ a Borel probability measure on X. The following are equivalent.*

1. *E_ρ is μ-hyperfinite*
2. *For all $r > 0$, (X, ρ) admits a μ-measurable r-toast.*
3. $\mathrm{asi}_\mu(X, \rho) = 1$
4. $\mathrm{asi}_\mu(X, \rho) < \infty$

It is interesting to ask how well this equivalence holds up in the Borel context. Our proof for $2 \implies 3$ will done in the Borel context, and of course $3 \implies 4$ holds. In [2], the authors construct hyperfinite acyclic bounded degree Borel graphs with aribtrarily large Borel chromatic numbers, which can be seen to rule out $1 \implies 4$ (and therefore $1 \implies 2, 3$ as well). $4 \implies 3$ and $3, 4 \implies 1$ were already mentioned as open problems. Toast is essentially defined as a witness to hyperfiniteness with nice metric properties, and so we have $2 \implies 1$ (in fact, the existence of r-toast for *any* r implies 1). Finally $3 \implies 2$ seems to be open, so we mention it here:

Problem 3. *Let (X, ρ) be a locally finite Borel extended metric space for which $\mathrm{asi}_B(X, \rho) = 1$. Does (X, ρ) admit a Borel r-toast for every $r > 0$?*

Again, it follows from results in [1] that this has a positive answer under the additional assumption that (X, ρ) has finite asymptotic dimension.

It is worth mentioning that all of these notions hold automatically modulo a E_ρ-invariant meager set, as shown in [3][7].

Finally, Theorem 1 provides additional motivation for studying Problem 2: A major open question in the field of countable Borel equivalence relations is whether such a relation which is μ-hyperfinite for each μ is hyperfinite. By our theorem, a negative answer to Problem 2 would imply a negative answer tot his question.

2 $1 \implies 2 \implies 3$

As was suggested in the introduction, the proof of these directions is essentially from [3]. Our reasons for reproducing it here are to phrase it in the language of locally finite metrics, and to isolate the construction of toast as a key intermediate step.

Throughout this section, (X, ρ) will be some locally finite Borel extended metric space, and μ will be a Borel probability measure on X. We'll start with $1 \implies 2$:

Proof. Fix r. By removing a E_ρ-invariant null set, we may assume E_ρ is hyperfinite, say as witnessed by the sequence E_n of finite Borel equivalence relations. Since ρ is locally finite, for all x and n, there is some $m > n$ such that $B([x]_{E_n}, r) \subset [x]_{E_m}$. Let A_n be the set of x for which this holds for $m = n+1$. By passing to a subsequence, we may then assume $\mu(A_n) > 1 - 2^{-n}$ for each n. Let X' be the set of $x \in X$ in all but finitely many A_n's. By the Borel-Cantelli lemma, $\mu(X') = 1$. If $x \in X'$ and $y \in [x]_{E_\rho}$, find n large enough so that $x \in A_m$ for all $m \geq n$ and $(x, y) \in E_n$. Then since A_m is E_m-invariant for each m, $y \in A_m$ for all $m \geq n$, so $y \in X'$. That is, X' is E_ρ-invariant.

Thus, we may assume $X = X'$. For each n, let $\mathcal{C}_n \subset [E_\rho]^{<\omega}$ be the set of nonempty sets of the form $A_n \cap [x]_{E_{n+1}}$ for $x \in X$. Let $\mathcal{C} = \bigcup_n \mathcal{C}_n$. This is clearly Borel, and we claim it is an r-toast. For the first condition, suppose $(x, y) \in E_\rho$. Find n large enough so that $(x, y) \in E_n$ and, using the hypothesis $X = X'$, $x, y \in A_n$. Then $x, y \in A_n \cap [x]_{E_{n+1}} \in \mathcal{C}_n$.

Now suppose $C \in \mathcal{C}_n$, $D \in \mathcal{C}_m$, and $C \neq D$. WLOG assume $n \leq m$. Observe that $B(C, r)$ is contained a single E_{n+1}-class while D is E_m-invariant. Thus, if $n < m$, $B(C, r)$ is either contained in D or disjoint from it, giving our second condition in the definition of toast. If $n = m$, C and D are contained in different E_{n+1}-classes, and so $B(C, r)$ and $B(D, r)$ are disjoint, and the second condition is again satisfied.

□

As promised in the introduction, the proof of $2 \implies 3$ does not require throwing away a null set:

Proposition 1. *Suppose (X, ρ) admits a Borel r-toast for every $r > 0$. Then* $\mathrm{asi}_B(X, \rho) = 1$.

Proof. Fix r. Let $\mathcal{C}$ be a Borel $2r$-toast. Let $U \subset X$ be the set of x such that for some $C \in \mathcal{C}$, $x \in C$, but $B(x, r) \not\subset C$. This is Borel. We claim U and $X \setminus U$ witness $\mathrm{asi}_B = 1$ for the distance r.

By König's lemma, it suffices to show that G^r_ρ does not admit any infinite injective paths contained in either U or $X \setminus U$. Suppose first that $(x_n)_{n \in \omega}$ is an infinite injective path in $G^r_\rho \upharpoonright (X \setminus U)$. Pick $C \in \mathcal{C}$ with $x_0 \in C$. For each n, if $x_n \in C$, then since $B(x_n, r) \subset C$, we conclude $x_{n+1} \in C$. Thus each x_n is in C by induction, contradicting the finiteness of C.

Now suppose $(x_n)_{n \in \omega}$ is an infinite injective path in $G^r_\rho \upharpoonright U$. Let $C \in \mathcal{C}$ such that $x_0 \in C$. As before, it suffices to show that if $x_n \in C$, so is x_{n+1}. If not, there must be some $D \neq C$ with $x_{n+1} \in D$ but $B(x_{n+1}, r) \not\subset D$. But since $B(D, r)$ contains $x_n \in C$, and x_{n+1} witnesses $D \not\subset C$, it must be the case that $B(C, 2r) \subset D$, but $x_{n+1} \in B(C, r)$, so $B(x_{n+1}, r) \subset B(C, 2r) \subset D$, a contradiction. □

3 $4 \implies 1$

This section contains our result that $4 \implies 1$. Again, throughout, (X, ρ) is a locally finite Borel extended metric space, and μ is a Borel probability measure on X.

We start with the following: Call a set $A \subset X$ an *r-barrier*, for $r > 0$, if $G^r_\rho \upharpoonright (X \setminus A)$ has only finite connected components.

Lemma 2. *E_ρ is μ-hyperfinite if and only if for all $r > 0$ and $\epsilon > 0$, There is a Borel r-barrier A with $\mu(A) < \epsilon$.*

This equivalence is basically well known. See for example Lemma 3.1 in [9] for a very similar statement in the context of a countable group action (whose proof our proof below is mostly a rephrasing of), or Proposition 9.12 in [10] and [8] for similar statements in the more familiar context of a Borel graph, but where the universal quantifier on r is traded for an assumption of invariance or quasi-invariance for μ. Only the reverse direction will be used.

Proof. Suppose E_ρ is μ-hyperfinite, as witnessed by a conull E_ρ-invariant X' and a sequence of finite Borel equivalence relations $(E_n)_{n\in\omega}$ on X'. Fix r. For each n, let A_n be the set of $x \in X$ for which $B(x, r) \not\subset [x]_{E_n}$. Each A_n is an r-barrier for $(X', \rho' \restriction X')$ since if $(x_n)_{n\in\omega}$ is an infinite injective path in $G^r_\rho \restriction (X' \setminus A_n)$, it can clearly never leave $[x_0]_{E_n}$, a contradiction. Thus $A_n \cup (X \setminus X')$ is a Borel r-barrier (of the same measure as A_n).

The A_n's are decreasing since the E_n's are increasing, and they decrease to $\emptyset$ since ρ is locally finite, and so for all x there is some n for which $B(x, r) \subset [x]_{E_n}$. Thus the A_n's have measures approaching 0.

Conversely, suppose for each $n > 0$, we can find a Borel $r = 2n+1$-barrier A_n with $\mu(A_n) < 2^{-n}$. For each n, define a finite Borel equivalence relation F_n as follows: For each connected component C of $G^r_\rho \restriction (X \setminus A_n)$, $B(C, n)$ will be a class of F_n. Note that this is valid as ρ is locally finite, and distinct such classes have distance between them greater than $2n + 1$. Points not in $B(X \setminus A_n, n)$ will be F_n-related only to themselves. Let $E_m = \bigcap_{n\geq m} F_n$, so that the E_m's give an increasing sequence of finite Borel equivalence relations. Call their union E.

Obviously $E \subset E_\rho$. By the Borel-cantelli lemma, the set X' of x which belong to only finitely many A_n's is conull. Suppose $x \in X'$ and $\rho(x, y) < m \in \omega$. We may choose m large enough so that $x \notin A_n$ for all $n \geq m$. Then $(x, y) \in F_n$ for all such n, so $(x, y) \in E$. Thus $[X']_{E_\rho}$ is a Borel conull E_ρ-invariant set on which $E = E_\rho$, so since E is hyperfinite by construction we are done. □

Now we can show $4 \implies 1$ (Also referred to as Lemma 1 in the introduction):

Proof. We will show the equivalent condition from Lemma 2. As in the forward direction of the lemma, we can get a Borel barrier with small measure by finding a Borel barrier with small measure for the restriction of ρ to some conull invariant set, then adding in the complement of that invariant set. Thus we may assume that $\mathrm{asi}_B(X, \rho)$ is some $s \in \omega$.

Fix $r > 0$ and $N \in \omega$ large. For $j \in N$, let $I_j = [rj, r(j + 1))$, and let $I_N = [rN, \infty]$, so that the I_j's partition the extended non-negative reals. Let $U_0, \ldots, U_s$ be Borel sets covering X such that each $G^{(2N+1)r}_\rho \restriction U_i$ has only finite connected components. For each i, define a Borel $d_i : X \to N + 1$ by letting $d_i(x)$ be the unique j so that $\rho(x, U_i) \in I_j$. For each $t \in N^{s+1}$, let $A_t = \{x \in X \mid \exists i\ d_i(x) = t(i)\}$, and $B_t = \{x \in X \mid \forall i\ d_i(x) = t(i)\}$.

We first claim each A_t is an r-barrier. Fix t. Suppose $(x_n)_{n\in\omega}$ is an infinite injective path in G^r_ρ. The U_i's cover X, so fix an i with $x_0 \in U_i$, so $d_i(x_0) = 0$.

We claim that for some n, we must have $\rho(x_n, U_i) > rN$. Suppose not. Then for each n, find $y_n \in U_i$ with $\rho(x_n, y_n) \leq rN$. Then for each n, $\rho(y_n, y_{n+1}) \leq rN + r + rN = (2N+1)r$, since $\rho(x_n, x_{n+1}) \leq r$. Thus the y_n's are all in the same $G_\rho^{(2N+1)r} \restriction U_i$ component, so there are only finitely many, but this contradicts local finiteness since $\{x_n\} \subset B(\{y_n\}, rN)$.

Now, $d_i(x_n) = N$, but also since the distance between adjacent x_m's is at most r, $\rho(x_m, U_i)$ can change by at most r when m increments by 1, and so $d_i(x_m)$ can change by at most 1. Thus there must be some $0 \leq m < n$ for which $d_i(x_m) = t(i)$, and so $x_m \in A_t$ as desired.

Now it suffices to show some A_t has small measure. We will do this by showing their average measure is small. For $t, t' \in N^{s+1}$, write $t \sim t'$ if $t(i) = t'(i)$ for some i. Then $A_t = \bigsqcup_{t' \sim t} B_{t'}$ for each t. Let $c = N^{s+1} - (N-1)^{s+1} = O(N^s)$, so that c is the number of t''s related to each t. Now

$$\sum_t \mu(A_t) = \sum_t \sum_{t' \sim t} \mu(B_{t'}) = \sum_{t'} c\mu(B_{t'}) \leq c,$$

where the last inequality follows from the disjointness of the $B_{t'}$'s. Therefore the average of $\mu(A_t)$ over all t is at most $c/N^{s+1} = O(N^{-1})$. Since N was arbitrarily large, we are done. □

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