Meaning in mathematics
–or– Belief as Irrefutability

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Goal Supply meaning for higher math
by defining heuristics to learn truth.

Start with how skeptical computer scientists
imagine knowledge accumulates.

Generalize to how physicists/scientists imagine
knowledge accumulates.

Seek heuristics for mathematical intuition.

Knowledge as sets of facts

Crow Arithmetic. (how many farmers are in the barn)

$0+1=1 \quad 1+1=2 \quad 2+1=3 \quad 3+1=4$

$4-1=3 \quad 3-1=2 \quad 2-1=1 \quad 1-1=0$

this is static hard-wired knowledge

Learning as deduction

Presburger Arithmetic.

$$\frac{}{0 \neq x + 1} \qquad \frac{x + 1 = y + 1}{x = y} \qquad \frac{}{x + 0 = x}$$

$$\frac{}{(x + y) + 1 = x + (y + 1)} \qquad \frac{P(0) \qquad P(x) \implies P(x + 1)}{P(y)}$$

$$\frac{}{(0 + 1) + (0 + 1) = ((0 + 1) + 1) + 0} \qquad \frac{}{\text{“}1 + 1 = 2 + 0\text{”}}$$

$$\frac{}{(0 + 1) + (0 + 1) = (0 + 1) + 1} \qquad \frac{}{\text{“}1 + 1 = 2\text{”}}$$

A maximal deductive theory

Peano Arithmetic. (now with quantifiers)

...first order equational logic...

$$\frac{}{0 \neq \mathsf{x} + 1}$$

$$\frac{\mathsf{x} + 1 = \mathsf{y} + 1}{\mathsf{x} = \mathsf{y}} \qquad \frac{\mathsf{P}(0) \qquad \forall \mathsf{x}.\ \mathsf{P}(\mathsf{x}) \implies \mathsf{P}(\mathsf{x} + 1)}{\forall \mathsf{y}.\ \mathsf{P}(\mathsf{y})}$$

PA is analogy-complete among deductive systems...

Knowledge relates via analogy

Interpretation of rationals $\langle \mathbb{Q}, \leq, +, \times \rangle$ in PA.
(define addition, multiplication, division, then pairing)

$$\langle \mathsf{x}, \mathsf{y} \rangle = \mathsf{y} + (\mathsf{x} + \mathsf{y})(\mathsf{x} + \mathsf{y} + 1)/2$$

$$\mathsf{rational}(\langle \mathsf{x}, \mathsf{y} \rangle) \iff \mathsf{y} \neq 0$$

$$\begin{aligned} \mathsf{less}(\langle \mathsf{w}, \mathsf{x} \rangle, \langle \mathsf{y}, \mathsf{z} \rangle) \iff & \ \mathsf{rational}(\langle \mathsf{w}, \mathsf{x} \rangle) \\ \text{and} & \ \mathsf{rational}(\langle \mathsf{y}, \mathsf{z} \rangle) \\ \text{and} & \ \mathsf{wz} \leq \mathsf{xy} \end{aligned}$$

$$\mathsf{add}(\langle \mathsf{w}, \mathsf{x} \rangle, \langle \mathsf{y}, \mathsf{z} \rangle) = \langle \mathsf{wz} + \mathsf{xy}, \mathsf{xz} \rangle$$

$$\mathsf{mult}(\langle \mathsf{w}, \mathsf{x} \rangle, \langle \mathsf{y}, \mathsf{z} \rangle) = \langle \mathsf{wy}, \mathsf{xz} \rangle$$

How far can deduction get us?

Far there are deduction systems into which all others can be interpreted.
(deduction = Σ^0_1,
and there are Σ^0_1-complete sets)

...but not so far...
(by Gödel's 1st incompleteness theorem)

- No decidable system can explain all other deductive systems.
  (Σ^0_1-complete is beyond Δ^0_1)
- Every analogy-complete system expresses unprovable statements.
  (Σ^0_1-complete is beyond Π^0_1)

When deduction fails, Induce

But as scientists, we can learn such facts using the scientific method.

(1) Make a guess / hypothesis
(here, a set of theories)
(2) Perform an experiment
(here, deduce consequences)
(3) Update belief in hypothesis

What is belief?

Meaning as falsifiability (a la Popper)

We believe what has not been falsified;
formally consider refutation in the limit:

Belief Change mind arbitrarily many times,
but eventually settle on disbelief when false,
and maybe vacillate indefinitely when true.

(refutable-in-the-limit = Π^0_2)

How far can science get us?

Very Far ...but first some theory...

Aside: descriptive complexity

(hierarchy picture)

Δ^0_1: decidable
Σ^0_1: $\mathsf{t}_1(\mathsf{x}) = $ "does program x halt"
Π^0_1: $1 - \mathsf{t}_1(\mathsf{x}) = $ "does program x not halt"
Σ^0_2: $\mathsf{t}_2(\mathsf{x}) = $ "does program $\mathsf{x}(\mathsf{h}_1)$ halt"
(x can make calls to h_1)
Π^0_2: $1 - \mathsf{t}_2(\mathsf{x}) = $ "does program $\mathsf{x}(\mathsf{h}_1)$ not halt"
⋮
Δ^0_ω: $\mathsf{d}_\omega(\mathsf{x}, \mathsf{n}) = \mathsf{t}_\mathsf{n}(\mathsf{x})$
Σ^0_ω: $\mathsf{t}_\omega(\mathsf{x}) = $ "does program $\mathsf{x}(\mathsf{d}_\omega)$ halt"
⋮
Δ^1_1: "infinity"
Π^1_1: $\mathsf{T}_1(\mathsf{x}) = $ "does $\mathsf{x}(\mathsf{s})$ halt on every stream s"
⋮

How far can deduction get us?

Far there are deduction systems
into which all others can be interpreted.
(deductive = Σ_1^0,
and there are Σ_1^0-complete sets)

...but not so far...

- No decidable system can explain all other deduction systems. (Σ_1^0-complete is beyond Δ_1^0)
- Every analogy-complete deductive system expresses unprovable statements. (Σ_1^0 is not closed under complement)

How far can science get us?

Far there are refutation systems
into which all others can be interpreted.
(refutable = Π^1_1,
and there are Π^1_1-complete hypotheses)

...but not so far...

- No refutable theory can explain all other refutation systems. (Π^1_1-complete is beyond Δ^1_1)
- Every analogy-complete refutable theory expresses unrefutable statements. (Π^1_1 is not closed under complement)

Aside: implications for physics

physically meaningful = falsifiable
= refutable in the limit = Π^0_2-testable
$\implies \Delta^1_1$ predictions

But there is no Δ^1_1-complete theory.

hence, No GUTs:
every theory is either incomplete or non-physical
(expresses physically meaningless statements)

or maybe: there is no coordinate-free GUT

What I am doing...

Asking ...

...So Δ^1_1 sets are meaningful, right?

Can we learn them? (in any sense)

How does step (1) work, in the Scientific Method? (making a guess)

from Proof systems to Belief systems

formalizing...

Proof Systems $\langle \mathbb{T}, \ \mathsf{T}_0, \ +, \ \mathsf{con} : \Pi^0_1, \ \vdash \ : \Sigma^0_1 \rangle$

(lattice picture)

Belief Systems $\langle \mathbb{T}, \ \mathsf{T}_0, \ +, \ \mathsf{sensible} : \Pi^0_2, \ \models \ : \Pi^1_1 \rangle$

...completion, limits, forcing...

Science is possible

Theorem

For any Δ^1_1 *set (of statements)* X,
there is an unambiguous belief system whose limit is X.

Theorem

There is an ambiguous belief system
whose limits are uniformly Π^1_1*-complete.*

Science is tough

Theorem

Step (1) of the scientific method
is as hard as it gets (Δ^1_1-hard).

Proof.

If we had a method of guessing, we could construct a limit with only Π^0_2-much more effort.

$\square$

Heuristics to learn truth

Hope, à la Occam and Popper:
assume simple statements
that have not yet been decided;
(because they are easier to test)
scrap if ever to find an inconsistency; and
stick with the most plausible theory.

Problem how to balance simplicity and plausibility?
(complicated vs plausible picture)

Problem some assumptions only fail in their
lack of sensible complete extension