Automated Equational Reasoning in Nondeterministic λ-Calculi Modulo Theories $\mathcal{H}^*$

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Motivation

to semi-automatically build

a complete knowledge base

of mathematical facts

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What do we want of a foundation?

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Simple syntax

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Simple syntax *combinators*

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Full extensionality

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Add probing term $\top\ x = \top$.

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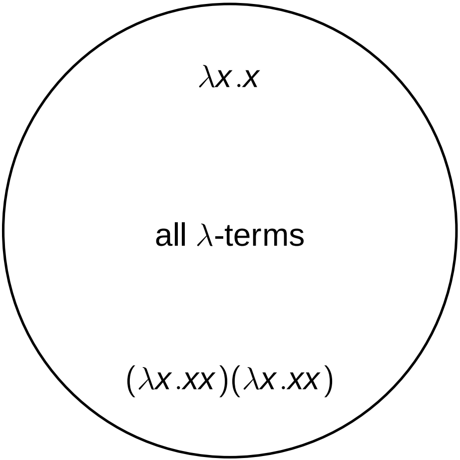

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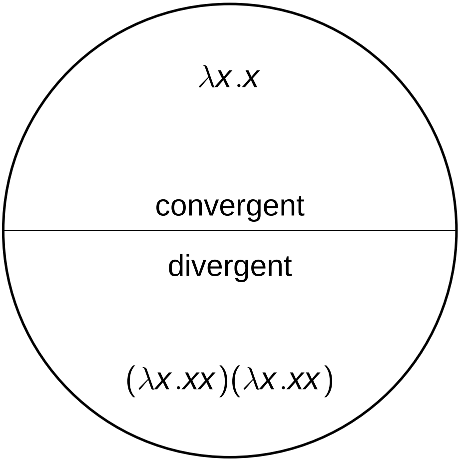

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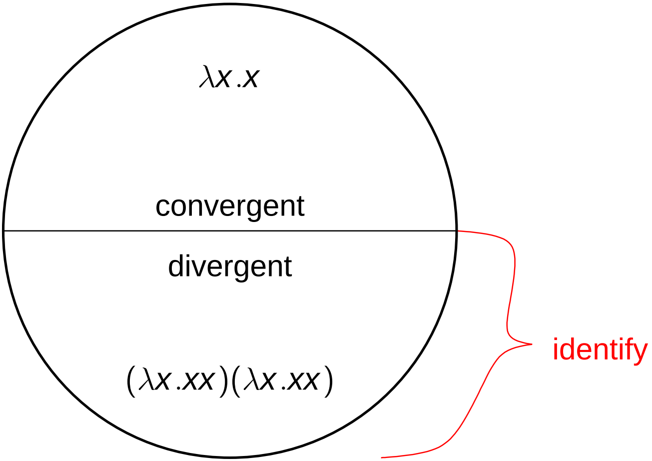

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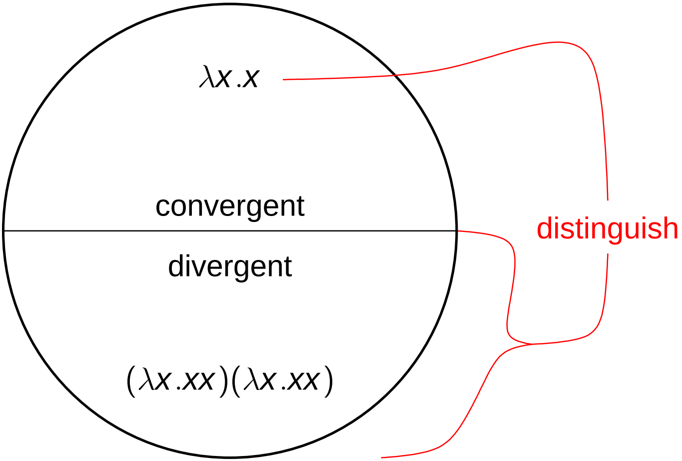

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A universe for all math

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Full extensionality	$\mathcal{H}^*$
A universe for all math	*varies...*

Outline

$\mathcal{H}^*$ for λ-join-calculus

Generalize to order

$$\frac{\forall \mathbf{C}[\ ].\ \mathbf{C}[x]\ \text{conv} \iff \mathbf{C}[y]\ \text{conv}}{\mathcal{H}^* \ \vdash\ x = y}$$

$\mathcal{H}^*$ for λ-join-calculus

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Generalize to order, traces $\mathbf{C}[x] = x\ M_1\ \ldots\ M_n$

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$x \mid y$ conv whenever x conv or y conv

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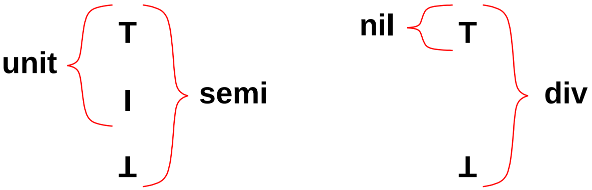
lower powerdomains: Sset

Dropped and Lifted types



nullary products

nullary coproducts

Simple types are definable

Theorem

For each simple type τ*,*
there is an SKJ-definable closure $[\tau]$ *s.t.*
for any SK-term x
x : τ *syntactically iff* $[\tau]\mathrm{x} = \mathrm{x}$ *mod* $\mathcal{H}^*$*.*

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Theorem

For each simple type τ,
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x $: \tau$ *syntactically iff* $[\tau]$x $=$ x *mod* $\mathcal{H}^*$.

Conjecture

...for any SKJ-term x*...*

Trick: Close over section-retract pairs

$$\begin{aligned}
&\text{Simple } f = \mathbf{V}\ (\\
&\quad f\ \mathbf{I}\ \mathbf{I}\\
&\quad |\ f\ (\lambda x.\ x\ \top)\ (\lambda x, y.\ x)\\
&\quad |\ f\ (\lambda x.\ x\ \bot)\ (\lambda x, y.\ x \mid \text{div}\ y)\\
&\quad |\ \text{Simple}\lambda a, a'.\ \text{Simple}\lambda b, b'.\ f\ (a' \to b)\ (a \to b')\\
&)
\end{aligned}$$

e.g. booleans $\tau = a \to a \to a...$

$$\bot,\ \mathbf{K},\ \mathbf{F},\ \top\ :\ \text{Simple}\lambda a, a'.\ a \to a \to a'$$

Trick: Disambiguation

$$\mathbf{K} \mid \mathbf{F} \;:\; \mathsf{Simple}\,\lambda \mathsf{a}, \mathsf{a}'.\; \mathsf{a} \to \mathsf{a} \to \mathsf{a}'$$

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$$\mathbf{K} \mid \mathbf{F} \ : \ \text{Simple}\lambda a, a'.\ a \to a \to a'$$

$$\text{disambiguate} \ := \ \lambda q.\ q(q\ \bot\ \top)(q\ \top\ \bot)$$

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Feature: Problems of specific complexity

$x = \mathbf{I}, x = \mathbf{K}$ are Π^0_2-complete

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Feature: Problems of specific complexity

$x = \mathbf{I}$, $x = \mathbf{K}$ are Π^0_2-complete

unit $x = \mathbf{I}$ is only Π^0_1

semi $x = \mathbf{I}$ is only Δ^0_2

bool $x = \mathbf{K}$ is only Δ^0_1 in test_bool $x = \mathbf{I}$

$\mathcal{H}^*$ for randomness

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Generalize to random convergence

$$\frac{\forall \underline{M}.\ \ x\ \ \underline{M}\ \ \text{conv} \quad \Longrightarrow \quad y\ \ \underline{M}\ \ \text{conv}}{\mathcal{H}^* \ \vdash \ x \sqsubseteq y}$$

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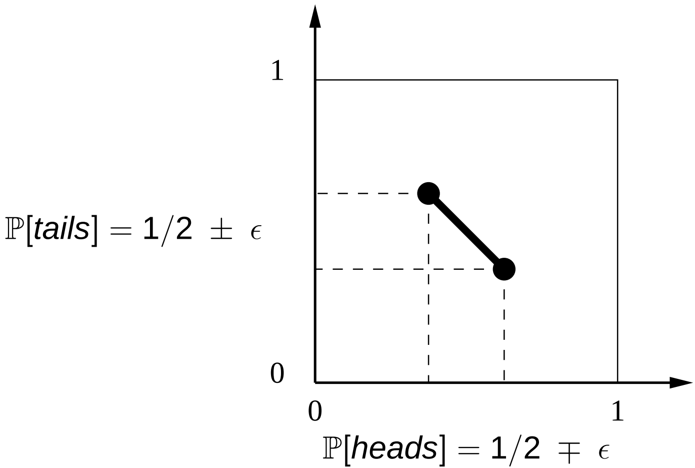
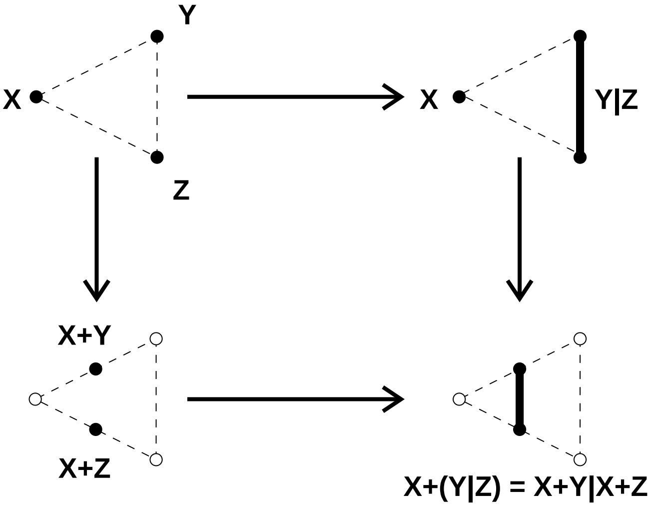
$$\mathbb{P}[x + y\ \text{conv}] = \frac{\mathbb{P}[x\ \text{conv}] + \mathbb{P}[y\ \text{conv}]}{2}$$

Convex Sets of Probability Distributions



Mixture Distributes over Join



A Monad for CSPDs

lower powerdomain	probability
$\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$	$\bigcirc(\phi \to \psi) \to \bigcirc\phi \to \bigcirc\psi$
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$\Box\bigcirc$ = Fuzzy, for CSPDs

Lifting Church-style terms

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compatible with simple types

Loose end: SKRJ-definable simple types?

Disambiguation fails:

$$\begin{aligned}\text{disambiguate}(\mathbf{K}+\mathbf{F}) &= (\top+\bot) \;+\; (\bot+\top)\\ &= \bot+\top\\ &\neq \top\end{aligned}$$

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Section-retract pairs must be head-affine:

$$\begin{array}{rcll}(\lambda x.\ x\ \top) & , & (\lambda x,y.\ x) & \textit{ok}\\ (\lambda x,y.\ x\ y\ y) & , & (\lambda x,y,z.\ x(y \mid z)) & \textit{ok}\\ (\lambda x.\ x\ \bot) & , & (\lambda x,y.\ x \mid \text{div}\ y) & \textit{bad}\end{array}$$

e.g., $\mathbf{K} \mid \bot+\top = \mathbf{K}+\top$

Adding logical strength (SKJO)

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ϕ : nat → bool is total

$$\forall x :: \text{test\_nat}.\ \phi\ x = \mathbf{K} \iff \bigsqcup_n\ \phi\ n\ \bot\ \mathbf{I} = \bot$$

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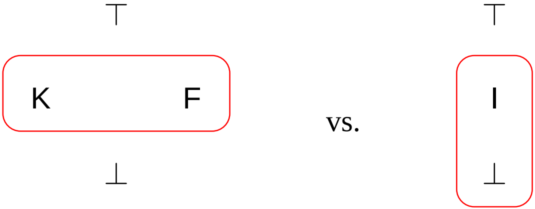

A Π^1_1-complete semi-oracle

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Comonadic codes: problem

Gödel codes take up space

$\mathbf{S}\ \mathbf{K} = \mathbf{K}\ \mathbf{I}$ but $\{\mathbf{S}\ \mathbf{K}\} \neq \{\mathbf{K}\ \mathbf{I}\}$

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...but that is inconsistent...

Comonadic codes: solution

Quotient WRT provable equality

$$\Pr(x = y) \iff \{x\} = \{y\}$$

Comonadic codes: solution

Quotient WRT provable equality

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Let $a, b : \mathbf{V}$, and $\Box a = \mathrm{Code}\{a\}$.

$$\begin{aligned} \mathrm{Apply}\{a\}\{b\} &: \Box(a \to b) \to \Box a \to \Box b \\ \mathrm{Eval}\{a\} &: \Box a \to a \\ \mathrm{Quote}\{a\} &: \Box a \to \Box\Box a \\ \mathrm{Code}\{a\}\{x\} &: \Box a \end{aligned}$$

Comonadic codes: solution

Quotient WRT provable equality

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Implementation

Adapts Todd-Coxeter to SKJ, $\sqsubseteq$

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Saturation complexity: $(|A|^2 \text{ space}) \times (|A|^3 \text{ time})$

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$$\sqsubseteq \subseteq A \times A$$
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1 day, $|A|$=10k, 5% application + 95% order known

Translating SK$\leftrightarrow\lambda$-**let**-calculus: problem

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Easy to reason about

$$\frac{}{\mathbf{S}\ x\ y\ z = x\ z(y\ z)} \qquad \frac{}{\mathbf{K}\ x\ y = x} \qquad \frac{f = g \qquad x = y}{f\ x = g\ y}$$

Translating SK↔λ-**let**-calculus: problem

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Easy to write

$\mathbf{let}\ s := (\lambda n, f, x.\ f(n\ f\ x)).\quad \mathbf{let}\ z := (\lambda -, x.\ x).\quad s(s(s(z)))$

$\mapsto$

$\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}\ \mathbf{K}\ \mathbf{K}))(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K}))(\mathbf{S}\ \mathbf{K})$

Translating SK↔λ-**let**-calculus: problem

Easy to reason about

$$\frac{}{\mathbf{S}\ x\ y\ z = x\ z(y\ z)} \qquad \frac{}{\mathbf{K}\ x\ y = x} \qquad \frac{f = g \qquad x = y}{f\ x = g\ y}$$

Easy to write

$\textbf{let}\ s := (\lambda n, f, x.\ f(n\ f\ x)).\quad \textbf{let}\ z := (\lambda -, x.\ x).\quad s(s(s(z)))$

$\mapsto$

$\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}\ \mathbf{K}\ \mathbf{K}))(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K}))(\mathbf{S}\ \mathbf{K})$

Impossible to read

Translating SK↔λ-**let**-calculus: solution

(1) find simple/simplest I,K,F,B,C,W,S-term

$$
\begin{aligned}
&\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}\ \mathbf{K}\ \mathbf{K}))(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K}))(\mathbf{S}\ \mathbf{K})\\
&\quad = \ \mathbf{S}\ \mathbf{B}(\mathbf{W}\ \mathbf{B})(\mathbf{S}\ \mathbf{B})\mathbf{F}.
\end{aligned}
$$

Translating SK$\leftrightarrow\lambda$-**let**-calculus: solution

(1) find simple/simplest I,K,F,B,C,W,S-term

$$\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K})(\mathbf{S}\ \mathbf{K}\ \mathbf{K}))(\mathbf{S}(\mathbf{S}(\mathbf{K}\ \mathbf{S})\mathbf{K}))(\mathbf{S}\ \mathbf{K}) = \mathbf{S}\ \mathbf{B}(\mathbf{W}\ \mathbf{B})(\mathbf{S}\ \mathbf{B})\mathbf{F}.$$

(2) decompile into λ-let-calculus

$$\mathbf{S}\ \mathbf{B}(\mathbf{W}\ \mathbf{B})(\mathbf{S}\ \mathbf{B})\mathbf{F} \mapsto \mathbf{let}\ \ a := (\lambda b, c, d.\ \ c(b\ \ c\ \ d)).\quad a(a(a\lambda e, f.\ \ f))$$

Translating SK$\leftrightarrow\lambda$-**let**-calculus: solution

(1) find simple/simplest I,K,F,B,C,W,S-term

$$\mathbf{S(S(K\ S)K)(S(S(K\ S)K)(S\ K\ K))(S(S(K\ S)K))(S\ K)} \\ = \mathbf{S\ B(W\ B)(S\ B)F}.$$

(2) decompile into λ-let-calculus

$$\mathbf{S\ B(W\ B)(S\ B)F} \\ \mapsto \mathbf{let}\ a := (\lambda b, c, d.\ c(b\ c\ d)).\quad a(a(a\lambda e, f.\ f))$$

- affine-β-η-reduce whenever possible
- combine copied subexpressions via **let**-in
- be generous with variable names;
  e.g. favor $\lambda a.\lambda b.b$ over $\lambda a.\lambda a.a$

Automated conjecturing

Problem: $\mathcal{H}^*$ is Π^0_2-complete.

Automated conjecturing

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Solution: Mine for missing equations.

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(1) Define probability distribution over inequality proofs.

Automated conjecturing

Problem: $\mathcal{H}^*$ is Π^0_2-complete.

Solution: Mine for missing equations.

(1) Define probability distribution over inequality proofs.

(2) Propagate evidence of unprovability among $\sim 10^8$ pairs.

Automated conjecturing

Problem: $\mathcal{H}^*$ is Π_2^0-complete.

Solution: Mine for missing equations.

(1) Define probability distribution over inequality proofs.

(2) Propagate evidence of unprovability among $\sim 10^8$ pairs.

(3) Find $\sim$ 100 pairs with most evidence.

Learning an optimal basis

Problem: sensitivity to basis weights

Learning an optimal basis

Problem: sensitivity to basis weights

$$\{\text{app}@0.5,\ \ \mathbf{S}@0.3,\ \ \mathbf{K}@0.2\}$$

–versus–

$$\{\text{app}@0.6,\ \ \mathbf{S}@0.2, \mathbf{K}@0.2\}$$

Learning an optimal basis

Problem: sensitivity to basis weights

$$\{\text{app@0.5}, \ \ \mathbf{S}\text{@0.3}, \ \ \mathbf{K}\text{@0.2}\}$$

–versus–

$$\{\text{app@0.6}, \ \ \mathbf{S}\text{@0.2}, \mathbf{K}\text{@0.2}\}$$

Solution: Locally minimize corpus complexity

Learning an optimal basis

Problem: sensitivity to basis weights

$$\{\text{app}@0.5,\ \ \mathbf{S}@0.3,\ \ \mathbf{K}@0.2\}$$

–versus–

$$\{\text{app}@0.6,\ \ \mathbf{S}@0.2, \mathbf{K}@0.2\}$$

Solution: Locally minimize corpus complexity

Empirically most useful terms:

B, **C**, **K**, (**C I**), **I**, **S**, **J**, **V**, **Y**, (**C B**), **F**, **P**, **W**

Loose End: Better Verification

Loose End: Better Verification

integrate with Knuth-Bendix

Loose End: Better Verification

integrate with Knuth-Bendix

additional knowledge representations

$$\circ : A \times A \longrightarrow A$$

$$| : A \times A \longrightarrow A$$

Loose End: Better Verification

integrate with Knuth-Bendix

additional knowledge representations

$$\circ : A \times A \to A$$

$$| : A \times A \to A$$

embarrassingly parallel rule search

Thanks to:

Rick Statman, Dana Scott,

Kevin Kelly, and James Cummings

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Questions?