Plan:

4.3. Options with random maturity in binomial model

4.4. American options in binomial model

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Options with random maturity in binomial model.

Example (Rebate option)

Pays \$1 as soon as the stock price exceeds barrier U

4.3

General definition:

$\tau = \tau(\omega)$: random maturity

$V_\tau = V_\tau(\omega_1 \dots \omega_\tau)$: payoff at τ

A random maturity is formally defined as a stopping time.

Definition A 4.4

random variable $\tau = \tau(\omega)$ is called a stopping time if

(i) $\tau = \tau(\omega)$ takes values in $\{0, 1, 2, \dots, N\}$

$$\left(\Longleftrightarrow \quad \sum_{k=0}^{N} I(\tau(\omega) = k) = 1 \right)$$

(ii) $\forall\ 0 \le n \le N$ the set $\{\omega : \tau(\omega) = n\}$ is determined by $\{\omega_1 \dots \omega_n\}$

$\left(\Longleftrightarrow \quad X_n = \sum_{k=0}^{n} I(\tau = k),\ 0 \le n \le N, \text{ is adapted} \right)$

Problem Which random variable defines a stopping time? 4.5

(i) $\tau = \min\{0 \le n \le N : S_n \le h\}$

(ii) $\sigma = \max\{0 \le n \le N : S_n \le h\}$

where h is a lower barrier

Arbitrage-free pricing. 4.6

Consider a European option with random maturity $\tau = \tau(\omega)$ and payoff Π_τ

AFP = Replication

Goal: construct a replication strategy:

$$X_0 \longrightarrow X_\tau = \Pi_\tau$$

(unknown) (known)

Key idea: think "conditionally"!

Denote

$X_n = X_n(\omega_1 \ldots \omega_n)$: capital of replication strategy at time n [illegible] under the condition that $\{\tau \geq n\}$

Backward induction:

time N : on $\{\tau = N\}$

$$X_N = \nabla_N$$

$$\vdots$$

Assume that X_{n+1} is computed

4.8

time n: on $\{\tau \geq n\}$

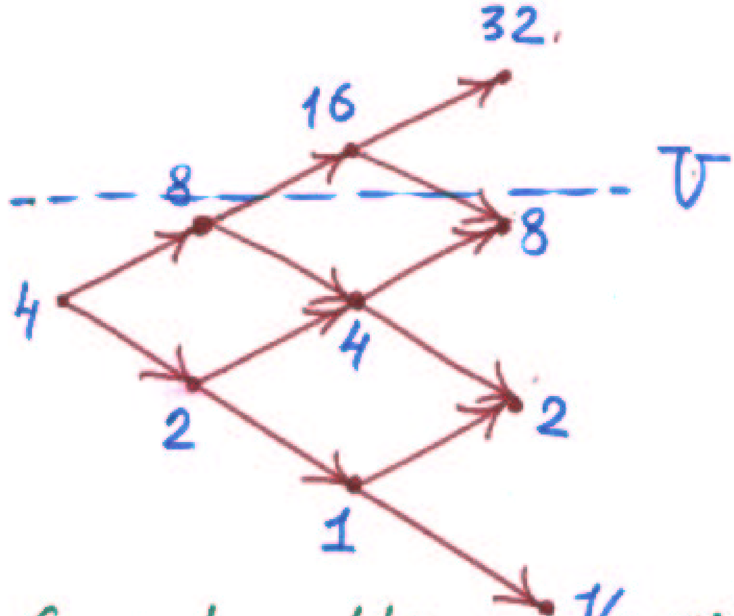
(a) if $\{\tau = n\}$, then

$$X_n = V_n$$

(b) if $\{\tau > n\}$, then

$$X_n = \frac{1}{1+r}\Big[\tilde{p}\, X_{n+1}(\omega_1 \ldots \omega_n H) + \tilde{q}\, X_{n+1}(\omega_{n+1} = T)\Big]$$

Problem Consider $N=3$-period model with $S_0=4$, $u=2$, $d=\frac{1}{2}$, $r=0.25$ [4.9]



Compute the arbitrage-free price for rebate option that pays \$1 at

4.10

$$\tau = \min\{n : S_n \geq U\}$$

where $U = 9$.

(If the barrer is not crossed, then payoff is zero).

Solution

4 11

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1}{2} \quad \tilde{q} = \frac{1}{2}$$

X_n : capital of replicating strategy if $\tau \geq n$

Look for $(X_n)_{0 \leq n \leq N}$ in the form

$$X_n = f_n(S_n), \quad 0 \leq n \leq N,$$

where $f_n = f_n(x)$ is a deterministic function

Backward induction :

Time N: on $\{\tau = N\}$ | 4 12

$$X_N = I(S_N \geq h)$$

⋮

time n: on $\{\tau \geq n\}$

(a) if $\{S_n \geq n\} = \{\tau = n\}$

then

$$X_n = 1$$

(b) else ($\tau > n$)

$$X_n = \frac{1}{1+r}\left[\tilde{p}\, X_{n+1}(\omega_{n+1} = H) + \tilde{q}\, X_{n+1}(\omega_{n+1} = T)\right]$$

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time N:

$$f_N(x) = I(x \geq h)$$

$$\vdots$$

time n:

$$f_n(x) = \frac{1}{1+r}\left[\tilde{p}\, f_{n+1}(ux) + \tilde{q}\, f_{n+1}(dx)\right] I(x < h) + I(x \geq h)$$

For our concrete example we have: 4.14

time 3:

$$f_3(32) = 1$$

$$f_3(8) = f_3(2) = f_3\left(\tfrac{1}{2}\right) = 0$$

time 2:

$$f_2(16) = 1$$

$$f_2(4) = \frac{2}{5}\left(f_3(8) + f_3(2)\right) = 0$$

$$f_2(1) = 0$$

time 1:

$$f_1(8) = \frac{2}{5} \qquad f_1(2) = 0$$

time 0: $f_0(4) = 0.16$

4.15

American options in binomial model

Example (American put)

An owner of the option can exercise it at any time before maturity. If he exercises at time n, then he receives

$$G_n = \max(K - S_n, 0)$$

strike

General description: 4 16

$G = (G_n)_{0 \le n \le N}$: payment process (adapted process)

$G_n = G_n(\omega_1 \dots \omega_n)$: "intrinsic" value at time n

Exercise policy: any stopping time $\tau = \tau(\omega)$

Questions:

(a) Optimal τ^* ?

(b) Arbitrage-free price V_0 ?

4.17

Key idea: think "conditionally"!

Denote

$V_n = V_n(\omega_1 \dots \omega_n)$: price of the option under the condition $\{\tau \geq n\}$ (option has not been exercised before)

Backward induction:

time N:

$$V_N = G_N$$

$\vdots$

time n:

$$V_n^{stop} = G_n \quad \text{(value to stop)}$$

$$V_n^{cont} = \frac{1}{1+r}\left[\tilde{p}\, V_{n+1}(\omega_{n+1} = H) + \tilde{q}\, V_{n+1}(\omega_{n+1} = T)\right]$$

(value to continue)

$$V_n = \max\left(V_n^{stop}, V_n^{cont}\right)$$

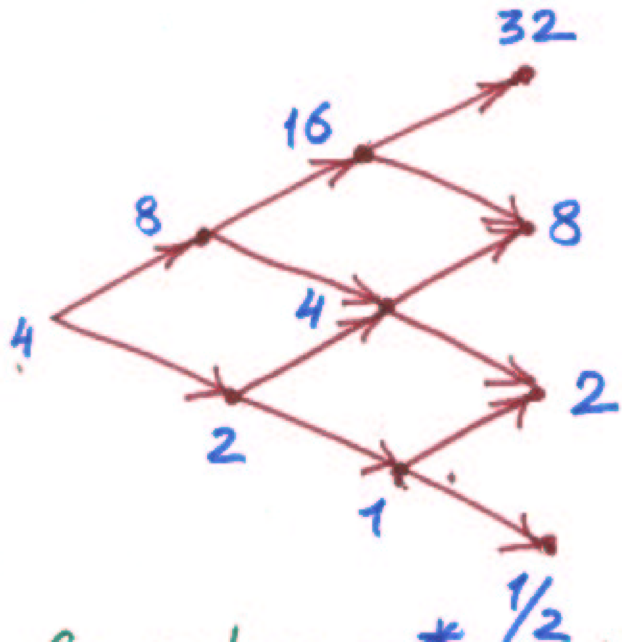
$\vdots$

First optimal exercise time : 4.19

$$\tau^* = \min\{0 \le n \le N : V_n = G_n\}$$

Example $N=3$, $S_0=4$, 4 20

$u=2$, $d=\frac{1}{2}$, $r=\frac{1}{4}$



Compute τ^* and V_0 for American put with strike $K=5$.

Solution Since 4.21

$$G_n = g(S_n), \text{ where}$$

$$g(x) = \max(K - x, 0)$$

we look for

$V_n = V_n(\omega_1 \dots \omega_n)$: the value of the option at n if $\{\tau \geq n\}$

in the form:

$$V_n = v_n(S_n), \quad 0 \leq n \leq N,$$

where $v_n = v_n(x)$ are deterministic functions.

Backward induction: 4.22

time N:

$$v_N(x) = g(x) = \max(x-K, 0)$$

$$\vdots$$

time n:

$$v_n(x) = \max\left(g(x), \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(ux) + \tilde{q}\, v_{n+1}(dx)\right]\right)$$

4.23

For our concrete example:

time 3:

$$v_3(32) = v_3(8) = 0$$

$$v_3(2) = 3 \qquad v_3\left(\tfrac{1}{2}\right) = 4\tfrac{1}{2}$$

time 2: $\left(\tilde{p} = \tilde{q} = \frac{1}{2} \quad \frac{\tilde{p}}{1+r} = \frac{2}{5}\right)$

$$v_2(16) = \max\left(0, \tfrac{2}{5}(0+0)\right) = 0$$

$$v_2(4) = \max\left(1, \tfrac{2}{5}(0+3)\right) = 1.2$$

$$v_2(1) = \max\left(4, \tfrac{2}{5}(3+4.5)\right) = 4$$

time 1:

$$v_1(8) = \max\left(0, \tfrac{2}{5}(0+1.2)\right) = \frac{12}{25}$$

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$$v_1(2) = \max\left(3, \tfrac{2}{5}(4 + 1.2)\right) = 3$$

time 0 :

$$v_0(4) = \max\left(1, \tfrac{2}{5}\left(\tfrac{12}{25} + 3\right)\right)$$
$$= 1.392$$

Optimal stopping:

time 0 : $g(4) = 1 < v_0(4)$

continue $\qquad\qquad v_0(4) = 1.392$

time 1 :

$\omega_1 = H \qquad g(8) = 0 < v_1(8) = \frac{12}{25}$

continue

$\omega_1 = T \qquad g(1) = \cancel{3}\, 4 = v_1(1)$

stop

time 2:

4.25

$\omega_1 \omega_2 = HT$

$g(4) = 1 < v(4) = 1.2$

continue

$\omega_1 \omega_2 = HH$

$g(8) = 0 = v(8)$

stop or continue