Lecture 21

Enoch Cheung

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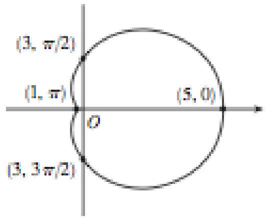
1. Sketch and find the area of $r = 3 + 2\cos\theta$.

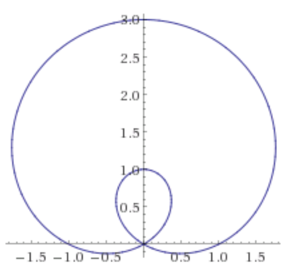


Notice that $r \geq 0$ for all θ. Thus we can get the area by consider $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2}(3 + 2\cos\theta)^2 d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(9 + 12\cos\theta + 4\cos^2\theta) d\theta \\
&= \int_0^{2\pi} \frac{9}{2} + 6\cos\theta + \cos(2\theta) + 1 d\theta \\
&= \left[\frac{11}{2}\theta + 6\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \\
&= 11\pi + 0 + 0 = 11\pi
\end{aligned}$$

2. Find the area of $r = 1 + 2\sin\theta$ (inner loop).

Think about what the graph looks like:



and the inner loop happens when $r < 0$. Consider $0 = r = 1 + 2\sin\theta$, then $-\frac{1}{2} = \sin\theta$, then $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$. Notice that the region we are considering is $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$.

Therefore, the area of the inner loop is given by

$$\begin{aligned}
A &= \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(1+2\sin\theta)^2 d\theta \\
&= \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(1+4\sin\theta+4\sin^2\theta) d\theta \\
&= \int_{7\pi/6}^{11\pi/6} \frac{1}{2}+2\sin\theta+(1-\cos 2\theta) d\theta \\
&= \left[\frac{3}{2}\pi - 2\cos\pi - \frac{1}{2}\sin 2\theta\right]_{7\pi/6}^{11\pi/6} \\
&= \frac{3}{2}(\frac{11\pi}{6}-\frac{7\pi}{6}) - 2(\frac{\sqrt{3}}{2}-(-\frac{\sqrt{3}}{2}) - \frac{1}{2}(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}) \\
&= \pi - \frac{3}{2}\sqrt{3}
\end{aligned}$$

3. Find all common points of $r = 2\sin 2\theta$ and $r = 1$.

   First, consider the points where $2\sin 2\theta = 1$, so $\sin 2\theta = \frac{1}{2}$, so $2\theta = \frac{\pi}{6} + k2\pi, \frac{5\pi}{6} + k2\pi$ for $k \in \mathbb{Z}$, so $\theta = \frac{\pi}{12} + k\pi, \frac{5\pi}{12} + k\theta$. This gives $(1, \theta)$ for $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

   We also need to consider the case where $2\sin 2\theta = -1$, because those points will are intersections of the lines too (despite having different θ at the points). $\sin 2\theta = \frac{1}{2}$, so $2\theta = \frac{7\pi}{6} + k2\pi, \frac{11\pi}{6} + k2\pi$ for $k \in \mathbb{Z}$, so $\theta = \frac{7\pi}{12} + k\pi, \frac{11\pi}{12} + k\theta$. This gives $(-1, \theta)$ for $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

4. Find all common points of $r = \sin\theta$ and $r = \sin 2\theta$.

   First, observe consider situation where $\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta$, so $\frac{1}{2} = \cos\theta$, so we are considering when $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\sin\theta = 0$. Plugging in gives us $r = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$ respectively. This gives us the points $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$ and $(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}) = (\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$ and the pole. We can also observe that since both r are 0 at some θ, the pole is an intersection (they intersect at the pole as long as they are 0 at any θ).

   To see that there are no other points, observe that the other way they can intersect is when $-\sin(\theta+\pi) = \sin 2\theta$. However, since $\sin\theta = -\sin(\theta+\pi)$, we have actually already considered these cases.

   Therefore, the points are $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$ and $(\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$ and the pole.

5. Find the length of $r = 2\cos\theta$ for $0 \le \theta \le \pi$.

$$L = \int_0^\pi \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta = \int_0^\pi \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2} d\theta = \int_0^\pi \sqrt{4(\cos^2\theta + \sin^2\theta)} d\theta = \int_0^\pi 2 d\theta = 2\pi$$