Lecture 18

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1.

$$\begin{aligned} f(x) &= \frac{x}{9+x^2} \\ &= \frac{x}{9}\frac{1}{1+\frac{x^2}{9}} \\ &= \frac{x}{9}\frac{1}{1-(-\frac{x^2}{9})} \\ &= \frac{x}{9}\sum_{n=0}^{\infty}(-\frac{x^2}{9})^n \\ &= \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{9^n} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}} \end{aligned}$$

note the substitution is for $|-\frac{x^2}{9}| < 1$, equivalently $|x| < 3$. Thus $R = 3$.

2.

$$\begin{aligned} f(x) &= \frac{x}{x^2+16} \\ &= \frac{x}{16}\frac{1}{\frac{x^2}{16}+1} \\ &= \frac{x}{16}\frac{1}{1-(-\frac{x^2}{16})} \\ &= \frac{x}{16}\sum_{n=0}^{\infty}(-\frac{x^2}{16})^n \\ &= \frac{x}{16}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{16^n} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{16^{n+1}} \end{aligned}$$

note the substitution is for $|-\frac{x^2}{16}| < 1$, equivalently $|x| < 4$. Thus $R = 4$.

3.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}(\frac{1}{1-x}) = \frac{d}{dx}(\sum_{n=0}^{\infty} x^n) = \sum_{n=0}^{\infty} nx^{n-1}$$

$$
\begin{aligned}
f(x) &= \frac{1+x}{(1-x)^2} \\
&= \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} \\
&= \sum_{n=0}^{\infty} nx^{n-1} + x\sum_{n=0}^{\infty} nx^{n-1} \\
&= \sum_{n=1}^{\infty} nx^{n-1} + \sum_{n=0}^{\infty} nx^{n} \\
&= \sum_{n=0}^{\infty} (n+1)x^{n} + \sum_{n=0}^{\infty} nx^{n} \\
&= \sum_{n=0}^{\infty} (2n+1)x^{n}
\end{aligned}
$$

4.

$$
\ln(1+x) = \int \frac{1}{1+x}dx = \int \sum_{n=0}^{\infty}(-1)^n x^n dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}
$$

The substitution was with $|x| < 1$. Now we need to determine what C is. For $x = 0$, $\ln(1+0) = C + 0$, thus $C = \ln(1) = 0$.

Therefore,

$$
\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}
$$

with $R = 1$.

Now

$$
\begin{aligned}
\int x^2 \ln(1+x)dx &= \int x^2 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}dx \\
&= \int \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+2}}{n}dx \\
&= C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+3}}{n(n+3)}
\end{aligned}
$$

5. Using $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ with $R = \infty$.

$$
\begin{aligned}
f(x) &= x\cos(\frac{1}{2}x^2) \\
&= x\sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{2}x^2)^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n)!2^{2n}}
\end{aligned}
$$

with $R = \infty$