Lecture 16

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1.
$$\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n \qquad b>0$$

By Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-a)^{n+1}}{b^{n+1}}\frac{b^n}{n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\frac{x-a}{b}\right| = \left|\frac{x-a}{b}\right|$$

thus when $|x-a|<b$ converges and when $|x-a|>b$ diverges.

Now consider the endpoints $x=a-b, a+b$. For $x=a-b$,

$$\sum_{n=1}^{\infty} \frac{n}{b^n}(a-b-a)^n = \sum_{n=1}^{\infty}(-1)^n n$$

diverges, and

$$\sum_{n=1}^{\infty} \frac{n}{b^n}(a+b-a)^n = \sum_{n=1}^{\infty} n$$

also diverges.

Therefore, the interval is $(a-b, a+b)$ and the radius is b.

2.
$$\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$$

By Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(5x-4)^{n+1}}{(n+1)^3}\frac{n^3}{(5x-4)^n}\right| = \lim_{n\to\infty}\left|(5x-4)\left(\frac{n}{n+1}\right)^3\right| = |5x-4| = \left|5(x-\frac{4}{5})\right|$$

Now consider the endpoints $x=\frac{4}{5}-\frac{1}{5}, \frac{4}{5}+\frac{1}{5}$ so $x=\frac{3}{5}, 1$. For $x=\frac{3}{5}$,

$$\sum_{n=1}^{\infty} \frac{(5\frac{3}{5}-4)^n}{n^3} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

converges by p-test. Similarly, for $x=1$,

$$\sum_{n=1}^{\infty} \frac{(5\cdot 1-4)^n}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

converges by p-test.

Thus the interval is $(\frac{3}{5}, 1)$ and the radius is $\frac{1}{5}$.

3. Consider
$$f(x) = 1+2x+x^2+2x^3+x^4+\cdots = \sum_{n=0}^{\infty} c_n x^n$$

where $c_{2n}=1, c_{2n+1}=2$. Find the interval of convergence and an explicit formula for $f(x)$.

$$f(x) = 1+2x+x^2+2x^3+x^4+\cdots = (1+x+x^2+\dots)+x(1+x^2+x^4+\dots) = \sum_{n=0}^{\infty} x^n + x\sum_{n=0}^{\infty} x^{2n}$$

and we know that the sums both diverge to ∞ when $x \geq 1$ and both converge when $x < 1$. When $x < 1$,

$$f(x) = \frac{1}{1-x} + \frac{x}{1-x^2}$$

4.

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$$

for x in $(-1, 1)$.

5.

$$f(x) = \frac{2}{3-x} = \frac{2/3}{1-\frac{x}{3}} = \frac{2}{3}\sum_{n=0}^{\infty}(\frac{x}{3})^n$$

for x in $(-3, 3)$.