Lecture 15

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1.
$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2x^n}{2^n}$$

Let $a_n = (-1)^n\frac{n^2x^n}{2^n}$. By Ratio Test, consider

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}}{2^{n+1}}\frac{2^n}{n^2x^n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\frac{|x|}{2} = \frac{|x|}{2}$$

so the limit is < 1 when $|x| < 2$ and > 1 when $|x| > 2$. Thus by the Ratio Test the series converges when $|x| < 2$ and diverges when $|x| > 2$.

Now to consider the end points of $x = 2$ and $x = -2$. When $x = 2$, we are considering

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2 2^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n n^2$$

which diverges since $|a_{n+1}| > |a_n|$. Similarly, when $x = -2$,

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2(-2)^n}{2^n} = \sum_{n=1}^{\infty}n^2$$

which also diverges.

Therefore, the interval of convergence is $(-2, 2)$ and the radius of convergence is 2.

2.
$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}x^n$$

By Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{1.5}}\frac{n^{1.5}}{(-3)^nx^n}\right| = \lim_{n\to\infty}\left|(-3)x\left(\frac{n}{n+1}\right)^{1.5}\right| = \lim_{n\to\infty}|3x|$$

thus by the Ratio test, this converges when $|x| < \frac{1}{3}$ and diverges when $|x| > \frac{1}{3}$.

Now to look at the endpoints $x = \frac{1}{3}$ and $x = -\frac{1}{3}$. When $x = \frac{1}{3}$,

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}(\frac{1}{3})^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1.5}}$$

converges absolutely by p-test, so converges.

Similarly, when $x = -\frac{1}{3}$,

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}(-\frac{1}{3})^n = \sum_{n=1}^{\infty}\frac{1}{n^{1.5}}$$

converges by p-test.

Thus the interval is $[-\frac{1}{3}, \frac{1}{3}]$ and the radius is $\frac{1}{3}$.

3.

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$$

By Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\frac{n^2+1}{(x-2)^n}\right| = \lim_{n\to\infty} \left|\left(\frac{n^2+1}{(n+1)^2+1}\right)(x-2)\right| = \lim_{n\to\infty} |x-2|$$

so by the Ratio test, converges when $|x-2|<1$ and diverges when $|x-2|>1$.

Now consider the end points. When $x=1$,

$$\sum_{n=0}^{\infty} \frac{(1-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

When $x=3$,

$$\sum_{n=0}^{\infty} \frac{(3-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Thus both of them converges absolutely, by limit comparison test with $\sum \frac{1}{n^2}$.

Therefore the interval is when $|x-2|\leq 1$ so the interval is $[1,3]$ and the radius is 1.

4.

$$\sum_{n=1}^{\infty} \frac{x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$$

By Ratio Test

$$\lim_{n\to\infty} \left|\frac{x^{n+1}}{1\cdot 3\cdot 5\cdots(2n+1)}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{2n+1}\right| = 0$$

thus the series converges for every x, so the interval is $(-\infty,\infty)$ and the radius is ∞.

5. If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, does it follow that the following series are convergent?

(a)

$$\sum_{n=0}^{\infty} c_n(-2)^n$$

Consider the power series

$$\sum_{n=0}^{\infty} c_n x^n$$

which is centered around $a=0$. Since this is a power series, and $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, the radius of convergence is at least 4. So we know that since -2 is in the interval $(-4,4)$ of radius 4, $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b)

$$\sum_{n=0}^{\infty} c_n(-4)^n$$

There is not enough information to conclude that the series is convergent. Consider

$$c_n = (-1)^n\frac{1}{n}(\frac{1}{4})^n$$

so

$$\sum_{n=0}^{\infty} c_n 4^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$$

which converges because it is an alternating series.

However,

$$\sum_{n=0}^{\infty} c_n(-4)^n = \sum_{n=0}^{\infty} \frac{1}{n}$$

which diverges.