Lecture 12

Enoch Cheung

February 26, 2014

1. $$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n} \leq \sum_{n=1}^{\infty} \frac{9^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$$

   converges since $|\frac{9}{10}| < 1$.

2. $$\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2} \geq \sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^n$$

   diverges since $|\frac{4}{3}| \geq 1$.

3. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

   let $a_n = \frac{1}{\sqrt{n^2+1}}$ and compare it to $b_n = \frac{1}{n}$, then

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

   converges to a number $\neq 0$. Since $\sum_{n=1}^{\infty} b_n$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent.

4. $$\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$$

   let $a_n = \frac{5+2n}{(1+n^2)^2}$ and compare it to $b_n = \frac{n}{n^4} = \frac{1}{n^3}$.

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5+2n}{(1+n^2)^2} \frac{n^4}{n} = \lim_{n\to\infty} \frac{\frac{5}{n}+2}{(\frac{1}{n^2}+1)^2} = 2$$

   so since $\sum_{n=1}^{\infty} b_n$ converges, it converges.

5. Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$ then $\sum a_n$ is divergent.

   Let $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} na_n \neq 0$. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a number $\neq 0$, then by Limit Comparison Test, since $\sum b_n$ is divergent, $\sum a_n$ is divergent. On the other hand, if $\lim_{n\to\infty} na_n$ is divergent, then $\lim_{n\to\infty} a_n$ is divergent, so $\sum a_n$ is divergent.

6. $$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$$

   by Ratio test

   $$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k}\right| = \lim_{k\to\infty} \frac{2}{3}\frac{k+1}{k} = \frac{2}{3} < 1$$

   so it is absolutely convergent.

7.

$$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$$

by Ratio test

$$\lim_{n\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{n\to\infty} \left| \frac{10^{n+1}}{(n+2)4^{2n+3}} \frac{(n+1)4^{2n+1}}{10^n} \right| = \lim_{n\to\infty} \frac{n+1}{n+2} \frac{10}{4^2} = \frac{10}{16} < 1$$

so it is absolutely convergent.

8.

$$\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1} \right)^n$$

by Ratio test

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{n\to\infty} \left(\frac{(n+1)^2+1}{2(n+1)^2+1} \right)^{n+1} \left(\frac{2n^2+1}{n^2+1} \right)^n \\
&= \lim_{n\to\infty} \left(\frac{(n+1)^2+1}{2(n+1)^2+1} \right) \left(\frac{(n+1)^2+1}{n^2+1} \right)^n \left(\frac{2n^2+1}{2(n+1)^2+1} \right)^n \\
&= \lim_{n\to\infty} \left(\frac{(1+\frac{1}{n})^2+\frac{1}{n^2}}{2(1+\frac{1}{n})^2+\frac{1}{n^2}} \right) \left(\frac{(1+\frac{1}{n})^2+\frac{1}{n}}{1+\frac{1}{n^2}} \right)^n \left(\frac{2+\frac{1}{n^2}}{2(1+\frac{1}{n})^2+\frac{1}{n}} \right)^n \\
&= \frac{1}{2} < 1
\end{aligned}$$

so it is absolutely convergent.