Lecture 10

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1. Find a formula for the sequence $\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \dots\}$.

   Notice that $a_{n+1} = -\frac{2}{3}a_n$. Therefore, $a_n = (-\frac{2}{3})^{n-1}(-3) = (-\frac{2}{3})^n \frac{9}{2}$.

2. Determine whether the sequence converges or diverges

$$\left\{ \frac{(2n-1)!}{(2n+1)!} \right\}$$

   Note that
$$\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)} = \frac{1}{4n^2+2n}$$
   and clearly $\lim_{n\to\infty} 4n^2 + 2n = \infty$, so $\lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = 0$.

3.
$$a_n = \left(1 + \frac{2}{n}\right)^n$$

   Recall the limit definition of e (p.418)

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

   Observe that the sequence
$$\sqrt{a_n} = \left(1 + \frac{2}{n}\right)^{n/2}$$
   and the function $f(x) = (1+x)^{1/x}$ is continuous, therefore
$$\lim_{n\to\infty} \sqrt{a_n} = \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^{n/2} = \lim_{n\to\infty} f(\frac{2}{n})$$
   Since f is continuous, and $\lim_{n\to\infty} \frac{2}{n} = 0$, so $\lim_{n\to\infty} \sqrt{a_n} = \lim_{x\to 0} f(x) = e$ so $\lim_{n\to\infty} a^n = e^2$ (it is clear that it has to be the positive square root).

4.
$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}$$
   Note that
$$a_n = \frac{1}{2n}(\frac{3 \cdot 5 \cdots (2n-1)}{(2n)^{n-1}})$$
   and the fraction on the right is less than 1 since the denumerate is greater than or equal to the numerator. Therefore
$$0 \leq a_n \leq \frac{1}{2n}$$
   so by the Squeez Theorem $\lim_{n\to\infty} a_n = 0$.

5.
$$3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)^{n-1} 3$$
   so it is divergent, since the ratio $|r| \geq 1$.

6.

$$10 - 2 + 0.4 - 0.08 + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{5}\right) 10 = \frac{10}{1 - (-\frac{1}{5})} = \frac{50}{6} = \frac{25}{3}$$

using the formula $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ for $|r| < 1$.

7.

$$\sum_{n=1}^{\infty} (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = (\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}) + (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}) + \cdots = \frac{1}{\sqrt{1}} = 1$$