Fluctuations of the corner growth model with geometric weights

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Abstract

This is an expository note on the classical corner growth model with i.i.d. geometrically distributed weights. A well-known theorem of K. Johansson states that suitably rescaled last-passage times converge in distribution to the Tracy-Widom GUE distribution. We reprove this result by adapting a steepest-descent analysis of J.Gravner, C.Tracy and H.Widom.

1 Introduction

The corner growth model describes a random region growing over time in the first quadrant of the plane, and is closely related to totally asymmetric simple exclusion process (TASEP), queues in series and last-passage percolation; we refer the reader to [26] for a detailed introduction. The discrete-time version of the model made its early appearances in [5], [19] and [25], and can be formulated as follows. Represent the first quadrant with $\mathbb{N}^2$. Each site $(i,j) \in \mathbb{N}^2$ receives a separate coin with tails probability $q \in (0,1)$. The region is initially empty and evolves according to the following rule. Each (i,j) waits for $(i-1,j)$ (if $i > 1$) and $(i,j-1)$ (if $j > 1$) to be in the region, then flips its coin at each time step onwards until heads comes up and immediately joins the region. Thus, if $W(i,j)$ is the amount of time (i,j) spends for the coin flips, then the random variables $\{W(i,j) : i,j \in \mathbb{N}\}$ are independent and their joint distribution $\mathbf{P}$ satisfies

$$\mathbf{P}(W(i,j) = k) = (1-q)q^k \quad \text{for } i,j \in \mathbb{N} \text{ and } k \in \mathbb{Z}_+ . \tag{1.1}$$

Furthermore, writing $G(i,j)$ for the time when (i,j) joins the region, we have the recursion

$$G(i,j) = G(i-1,j) \vee G(i,j-1) + W(i,j) \quad \text{for } i,j \in \mathbb{N}, \tag{1.2}$$

with boundary values defined as $G(i,0) = G(0,j) = 0$ for $i,j \in \mathbb{N}$. This leads to the last-passage formula

$$G(m,n) = \max_{\pi \in \Pi_{m,n}} \sum_{(i,j)\in\pi} W(i,j) \quad \text{for } m,n \in \mathbb{N} \tag{1.3}$$

where $\Pi_{m,n}$ is the set of all directed paths from $(1,1)$ to (m,n) (all sequences $((i_k,j_k))_{k\in[l]}$ in $\mathbb{N}^2$ such that $(i_1,j_1) = (1,1)$, $(i_l,j_l) = (m,n)$, $(i_k)_{k\in[l]}$ and $(j_k)_{k\in[l]}$ are nondecreasing and $i_{k+1} - i_k + j_{k+1} - j_k = 1$ for $1 \leqslant k < l$). We will refer to $\{W(i,j) : i,j \in \mathbb{N}\}$ as weights and $\{G(i,j) : i,j \in \mathbb{N}\}$ as the last-passage times. Typically, the statistical properties of the last-passage times are the issue of interest.

While we will not consider it here, there is also a continuous-time version of the corner growth model in which each (i,j) receives a separate Poisson clock with rate $\lambda > 0$ instead of a coin. When $(i-1,j)$ and $(i,j-1)$ (if they exist) are both in the region, (i,j) starts its clock and joins the region when the clock rings. In other words, we replace (1.1) with

$$\mathbf{P}(W(i,j) \geqslant x) = e^{-\lambda x} \text{ for } i,j \in \mathbb{N} \text{ and } x \geqslant 0. \tag{1.4}$$

This version appeared in the influential work of H. Rost [22].

The KPZ (Kardar-Parisi-Zhang) universality class is a conjectural collection of statistical models including various growth processes, interacting particle systems and directed polymers in random media, see [6] and the references therein. It is expected that the fluctuations of the interesting observables in the KPZ-models scale with exponent 1/3 and, after rescaling, converge to a Tracy-Widom distribution in the limit. In a breakthrough work of K. Johansson these supposed universal features were confirmed for the corner growth model; the relevant theorem is [16, Theorem 1.2] stated below.

Theorem 1.1. *Let $r > 0$. Then*

$$\lim_{n\to\infty} \mathbf{P}(G(\lfloor nr \rfloor, n) \leqslant n\gamma_r + n^{1/3}\sigma_r s) = F_{\mathrm{GUE}}(s) \tag{1.5}$$

for any $s \in \mathbb{R}$, where

$$\gamma_r = \frac{q(1+r) + 2\sqrt{qr}}{1-q}, \tag{1.6}$$

$$\sigma_r = \frac{1}{1-q}\left(\frac{q}{r}\right)^{1/6}(\sqrt{q}+\sqrt{r})^{2/3}(1+\sqrt{qr})^{2/3} \tag{1.7}$$

and F_{GUE} denotes the c.d.f. of the Tracy-Widom GUE distribution defined in Section 6.

If we assume (1.4) instead of (1.1) then (1.5) still holds with different explicit constants γ_r and σ_r, [16, Theorem 1.6].

In [8], we considered a particular generalization of the corner growth model in which parameter q is replaced with $a_i b_j$ for each $(i,j) \in \mathbb{N}^2$ for some random sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ in $(0,1)$ whose joint distribution satisfies certain ergodicity assumptions. (For fixed sequences, this model appeared in [17] and [18], see Section 6.) We obtained a variational formula for the law of large numbers limit $\lim_{n\to\infty} n^{-1}G(\lfloor nr \rfloor, n)$, namely, the constant analogous to (1.6). Corresponding large deviation properties were subsequently studied in [9], which suggested that the limit fluctuations of $G(\lfloor nr \rfloor, n)$ obey the Tracy-Widom GUE distribution only if $r_1 < r < r_2$ for some critical values $0 \leqslant r_1 < r_2 \leqslant \infty$.[1] We have developed some arguments in the course of an ongoing project to verify this prediction. To be used for future reference, the present note records these arguments in the simpler setting of classical corner growth model.

In a series of papers J.Gravner, C.Tracy and H.Widom carried out a similar program for a variant of the corner growth model known as oriented digital boiling [12], [13],[14], which is equivalent to a first-passage percolation model introduced in [24]. The recursion for $\{G(i,j) : i,j \in \mathbb{N}\}$ is now

$$G(i,j) = G(i-1,j) \vee (G(i,j-1) + W(i,j)) \quad \text{for } i,j \in \mathbb{N}, \tag{1.8}$$

instead of (1.2). The weights $\{W(i,j) : i,j \in \mathbb{N}\}$ are independent and each $W(i,j)$ is Bernoulli-distributed with parameter p_j for some i.i.d. sequences $(p_j)_{j\in\mathbb{N}}$. This note mainly builds on the work in [12], which deals with the basic case when $(p_j)_{j\in\mathbb{N}}$ is a constant sequence.

The original proof of Theorem 1.1 expresses the point distributions of last-passage times as a Fredholm determinant of the Meixner kernel and uses properties of the Meixner polynomials for asymptotic analysis [16], see also the exposition in [26, Chapter 5]. In this note, we present another proof by adapting an approach from [12], which involves steepest-descent analysis of an alternative Fredholm determinant represented in terms of contour integrals. As usual, one needs to find suitable deformations of the contours to make analysis possible. The appealing side of the approach taken is that one does not need explicit parametrization of the contours; rather, some useful properties of the contours are observed from general considerations. This enables us to use the same approach to study the inhomogeneous corner growth model described above.

[1] In fact, [9] only considered a generalization of the continous-time model but we expect similar large deviation results in the discrete-time.

In a recent work of I.Corwin, Z.Liu and D.Wang [7] that came to our attention while this note was under preparation, the limit fluctuations of the last-passage times were identified for some generalized corner growth models in which parameter q is perturbed for finitely many rows and columns. Lemma 2.2 there establishes a concentration inequality for the last-passage times and also derives elegantly in its proof the main asymptotic result (Theorem 4.1 below) needed to obtain Theorem 1.1. The proof is based on analysis of the same contour integrals as in here but the contours are deformed differently. However, we were able to utilize an idea from the proof that led to significant departure from [12] in the derivation of some bounds such as Theorem 4.1b. While an additional steepest-descent analysis is performed for the analogous bounds in [12, pp 20-23], we were able to get around this step by showing that the chosen contour lies inside a certain circle, see Lemma 4.2 below.

Outline. In Section 2, we recall the definitions of the Tracy-Widom GUE distribution as well as the Airy function and the Airy kernel. Section 3 gives a standard derivation of a Fredholm determinant for the distribution of the last-passage times in the more general case of site-dependent parameters. Some preliminaries for the steepest-descent analysis including the choice of suitable contours are carried out in Section 4. Detailed justification of the properties of these contours based on standard considerations from ODE theory is in the appendix. The proofs of the main asymptotic lemmas are given in Section 5. Finally, Theorem 1.1 is proved in Section 6.

Notation and conventions. Some standard notation that appears in this note are listed below.

$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{Z}_+$	the set of nonnegative integers $\{0, 1, 2, \dots\}$
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{H}$	the set of $z \in \mathbb{C}$ with $\Im z > 0$.
$\mathbf{i}$	the imaginary unit
$[n]$	the set $\{1, \dots, n\}$ for $n \in \mathbb{N}$
$a \vee b$	the maximum of $a, b \in \mathbb{R}$
$a \wedge b$	the minimum of $a, b \in \mathbb{R}$
$\#S$	the number of elements in the set S
$\overline{z}$	the complex conjugate of $z \in \mathbb{C}$
$\lfloor x \rfloor$	the largest integer not exceeding $x \in \mathbb{R}$
$D(z, r)$	the (open) disk of radius r centered at $z \in \mathbb{C}$
$\delta_{i,j}$	the Kronecker delta function

Let f and g be complex-valued functions defined on a set X. We write $f = O(g)$ to assert existence of a constant $C > 0$ such that

$$|f(x)| \leqslant C|g(x)| \quad \text{for all } x \in X.$$

We refer to a particular choice of C as the implicit constant. We also define 0^0 as 1.

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2 Tracy-Widom GUE distribution

The Airy function can be defined as the contour integral

$$\mathrm{Ai}(s) = \frac{1}{2\pi\mathbf{i}} \int\limits_{\mathcal{C}} e^{z^3/3 - sz} dz \quad \text{for } s \in \mathbb{R}, \tag{2.1}$$

where contour $\mathcal{C}$ consists of the rays from $\infty e^{-\mathbf{i}\theta}$ to 0 and from 0 to $\infty e^{\mathbf{i}\theta}$ for some $\theta \in (\pi/6, \pi/2)$. This integral is absolutely and uniformly convergent on compact subsets of $\mathbb{R}$. Up to a constant factor, the Airy function is the unique solution of the ODE

$$\frac{d^2u}{ds^2} = su \quad s \in \mathbb{R},$$

known as the Airy equation, subject to the condition $u(s) \to 0$ as $s \to +\infty$, [21, Chapter 9]. In the sequel, we will use continuity of the Airy function and the following bound. Given $T > 0$, there exists a constant $C > 0$ such that

$$|\operatorname{Ai}(s)| \leqslant Ce^{-s} \qquad \text{for } s \geqslant -T. \tag{2.2}$$

These properties can be derived from (2.1); we suggest [26, Chapter 4] for details.

One way to define the Airy kernel is by

$$\mathrm{A}(s,t) = \int_0^\infty \operatorname{Ai}(s+x)\operatorname{Ai}(t+x)dx \qquad \text{for } s,t \in \mathbb{R}, \tag{2.3}$$

where the absolute and the uniform convergence of the integral over compact subsets of $\mathbb{R}^2$ are ensured by (2.2). Moreover, the Airy kernel is continuous and for each $T > 0$ there is a constant $C > 0$ such that

$$|\mathrm{A}(s,t)| \leqslant Ce^{-s-t} \qquad \text{for } s,t \geqslant -T. \tag{2.4}$$

For any $a \in \mathbb{R}$, we can view the Airy kernel as the kernel of the integral operator on $L^2((a,\infty))$ that maps f to the function

$$s \mapsto \int_a^\infty \mathrm{A}(s,t)f(t)dt.$$

That this image is in $L^2((a,\infty))$ comes from (2.4) and an application of the Cauchy-Schwarz inequality.

The $n \times n$ Gaussian Unitary Ensemble (GUE) is the distribution of the random matrix $X = [X(i,j)]_{i,j\in[n]}$ with the following properties.

(i) $X(i,i)$ is distributed as the real normal distribution with mean 0 and variance 1 for $i \in [n]$.

(ii) $X(i,j)$ is distributed as the complex normal distribution with mean 0 and variance 1 for distinct $i,j \in [n]$

(iii) The entries $\{X(i,j) : 1 \leqslant i \leqslant j \leqslant n\}$ are independent.

(iv) X is Hermitian, that is, $X(i,j) = \overline{X(j,i)}$ for $i,j \in [n]$.

Being a Hermitian matrix, X has n real eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$. The Tracy-Widom GUE distribution arises as the distributional limit of the rescaled largest eigenvalue

$$n^{1/6}(\lambda_1 - 2n^{1/2})$$

as $n \to +\infty$. Its cumulative distribution function is given by the following Fredholm determinant of the integral operator whose kernel is the Airy kernel

$$F_{\mathrm{GUE}}(s) = 1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \int_{[s,\infty)^l} \det[\mathrm{A}(x_i,x_j)]_{i,j\in[l]}dx_1 \ldots dx_l \qquad \text{for } s \in \mathbb{R}, \tag{2.5}$$

[10]. The absolute convergence of the series above follows from (2.4) and Hadamard's inequality, see Lemma 6.2 below. Another characterization of the Tracy-Widom distribution is

$$F_{\mathrm{GUE}}(s) = \exp\left(-\int_s^\infty (t-s)q(t)^2dt\right) \qquad \text{for } s \in \mathbb{R},$$

where q is the unique solution of the Painlevé II equation

$$\frac{d^2u}{ds^2} = 2u^3 + su \qquad \text{for } s \in \mathbb{R}$$

subject to the condition $u(s)/\operatorname{Ai}(s) \to 1$ as $s \to \infty$, see [28].

3 Distribution of last-passage times

In this section, we consider a generalization of the corner growth model introduced in [17] and [18]. Let $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ and $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$ be sequences in the interval $[0, 1)$. Suppose that $\{W(i,j) : i, j \in \mathbb{N}\}$ are independent and

$$\mathbf{P}(W(i,j) = k) = (1 - a_i b_j) a_i^k b_j^k \quad \text{for } i, j \in \mathbb{N} \text{ and } k \in \mathbb{Z}_+ . \tag{3.1}$$

Recall our convention $0^0 = 1$; thus, if $a_i = 0$ or $b_j = 0$, the right-hand equals 1 when $k = 0$. For this model, there are exact formulas for the distribution of $G(m, n)$ (still defined by (1.2)) for each $m, n \in \mathbb{N}$. For completeness, we include a derivation of these formulas based on the discussion in [17]; we refer the reader to [2], [4], [18] for more detailed accounts.

One of the main tools utilized in the argument is the Robinson-Schensted-Knuth (RSK) correspondence. To state it, some definitions are in order. A *weak composition* $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ is a sequence in $\mathbb{Z}_+$ with finitely many nonzero terms. Define the *length* and the *size* of α as

$$l(\alpha) = \max\{i \in \mathbb{N} : \alpha_i > 0\} \qquad \text{and} \qquad |\alpha| = \sum_i \alpha_i,$$

respectively. A *partition* $\lambda = (\lambda_i)_{i \in \mathbb{N}}$ is a nonincreasing weak composition. Each λ_i is called a *part* of λ. To each partition λ, we associate a *Young diagram*

$$Y(\lambda) = \{(i,j) \in \mathbb{N}^2 : i \leqslant \lambda_j \text{ and } j \leqslant l(\lambda)\}.$$

A *semi-standard Young tableau (SSYT)* of *shape* λ is a map $P : Y(\lambda) \to \mathbb{N}$ such that $P = P(i,j)$ is nondecreasing in i and (strictly) increasing in j. We write $\lambda = \text{shape}(P)$. Also, define the *type* of P as the weak composition

$$\text{type}(P) = (\#\{(i,j) : P(i,j) = k\})_{k \in \mathbb{N}}.$$

See Figure 3.1 for a visualization of a Young diagram and an SSYT. A *generalized permutation* ς (of *length* $l \in \mathbb{N}$) is a finite sequence $\varsigma = ((i_k, j_k))_{k \in [l]}$ in $\mathbb{N}^2$ that is nondecreasing with respect to the lexicographic order ($i_k \leqslant i_{k+1}$ and if $i_k = i_{k+1}$ then $j_k \leqslant j_{k+1}$ for all $1 \leqslant k < l$). We write $L(\varsigma)$ for the maximal length of a nondecreasing subsequence of $(j_k)_{k \in [l]}$. Let $\mathcal{P}$ denote the set of all generalized permutations and $\mathcal{T}$ denote the set of all pairs of SSYTs (P, Q) such that $\text{shape}(P) = \text{shape}(Q)$.

(a) (b)

Figure 3.1: (a) The Young diagram $Y(\lambda)$ for $\lambda = (4, 2, 2, 1, 0, \dots)$ viewed as the set of unit squares with upper-right corners at $(i,j) \in Y(\lambda)$. (b) An SSYT P of shape $\lambda = (4, 2, 2, 1, 0, \dots)$. The value $P(i,j)$ is written inside the corresponding unit square. For example, $P(1, 2) = 4$ and $P(2, 3) = 6$. Note that the numbers are nondecreasing along rows and increasing along columns. Also, $\text{type}(P) = (1, 2, 1, 1, 1, 2, 1, 0, \dots)$.

Theorem 3.1 (RSK correspondence)**.** *There exists a bijection* $\text{RSK} : \mathcal{P} \to \mathcal{T}$ *with the following property: If* $\varsigma = ((i_k, j_k))_{k \in [l]} \in \mathcal{P}$ *and* $(P, Q) = \text{RSK}(\varsigma)$ *then*

$$\text{type}(P) = (\#\{k : j_k = n\})_{n \in \mathbb{N}}$$

$$\text{type}(Q) = (\#\{k : i_k = n\})_{n\in\mathbb{N}}$$

Furthermore, if $\lambda = (\lambda_i)_{i\in\mathbb{N}} = \text{shape}(P) = \text{shape}(Q)$ *then* $\lambda_1 = L(\varsigma)$.

For a proof, see [11], [27]. We will use the corollary below noted in [16]. For $m, n \in \mathbb{N}$, define

$$\mathcal{P}_{m,n} = \{((i_k, j_k))_{k\in[l]} \in \mathcal{P} : i_k \in [m] \text{ and } j_k \in [n] \text{ for } k \in [l]\}$$
$$\mathcal{T}_{m,n} = \{(P, Q) \in \mathcal{T} : l(\text{type}(P)) \leqslant n \text{ and } l(\text{type}(Q)) \leqslant m\}.$$

Corollary 3.2. *Let* $m, n \in \mathbb{N}$. *There exists a bijection* $f_{m,n} : \mathbb{Z}_+^{[m]\times[n]} \to \mathcal{T}_{m,n}$ *such that if* $A \in \mathbb{Z}_+^{[m]\times[n]}$ *and* $(P, Q) = f_{m,n}(A)$ *then*

(a) $|\lambda| = \sum_{i=1}^{m}\sum_{j=1}^{n} A(i, j)$, *where* $\lambda = \text{shape}(P) = \text{shape}(Q)$.

(b) $\text{type}(P)_j = \sum_{i=1}^{m} A(i, j)$ *for* $j \in [n]$.

(c) $\text{type}(Q)_i = \sum_{j=1}^{n} A(i, j)$ *for* $i \in [m]$.

(d) $\lambda_1 = \max_{\pi\in\Pi(m,n)} \sum_{(i,j)\in\pi} A(i, j)$, *where* λ_1 *is the largest part of* λ *in (a).*

Proof. For each $A \in \mathbb{Z}_+^{[m]\times[n]}$, define $g_{m,n}(A)$ as the unique $\varsigma = ((i_k, j_k))_{k\in[l]} \in \mathcal{P}_{m,n}$ such that each $(i, j) \in [m] \times [n]$ is repeated exactly $A(i, j)$ times in ς. Note that $g_{m,n}$ is a bijection and $l = \sum_{i=1}^{m}\sum_{j=1}^{n} A(i, j)$. Moreover, the lengths of the maximal nondecreasing subsequences of $(j_k)_{k\in[l]}$ are given by $\sum_{(i,j)\in\pi} A(i, j)$ for various $\pi \in \Pi_{m,n}$. Hence, $L(\varsigma)$ equals the last-passage time $\max_{\pi\in\Pi(m,n)} \sum_{(i,j)\in\pi} A(i, j)$. It follows from Theorem 3.1 that the map RSK restricts to a bijection $\text{RSK}_{m,n}$ between $\mathcal{P}_{m,n}$ and $\mathcal{T}_{m,n}$. Now, the composition $f_{m,n} = \text{RSK}_{m,n} \circ g_{m,n}$ is a bijection between $\mathbb{Z}_+^{[m]\times[n]}$ and $\mathcal{T}_{m,n}$ with properties (a)-(d). □

We will also rely on the following generalization of the Cauchy-Binet identity [18, Proposition 2.10].

Proposition 3.3. *Let* (X, μ) *be a measure space,* $n \in \mathbb{N}$ *and* $f_i, g_i : X \to \mathbb{C}$ *be measurable functions for* $i \in [n]$ *such that* $f_i g_j$ *is integrable for any* $i, j \in [n]$. *Then*

$$\det\left[\int_X f_i(x)g_j(x)\mu(dx)\right]_{i,j\in[n]} = \frac{1}{n!}\int_{X^n} \det[f_i(x_j)]_{i,j\in[n]} \det[g_i(x_j)]_{i,j\in[n]}\mu(dx_1)\dots\mu(dx_n).$$

We next obtain a Fredholm determinant representation for the distribution of $G(n, n)$ in the case of injective $(a_i)_{i\in[n]}$ and $(b_j)_{j\in[n]}$ (terms do not repeat). A more general version of the following proof can also be found in [4] and [18].

Theorem 3.4. *Let* $n \in \mathbb{N}$. *Suppose that* $(a_i)_{i\in[n]}$ *and* $(b_j)_{j\in[n]}$ *are injective sequences. Define*

$$K_n(x, y) = \sum_{i,j\in[n]} \frac{a_i^x b_j^y}{1 - a_i b_j} \frac{\prod\limits_{k\in[n]} (1 - a_i b_k)(1 - a_k b_j)}{\prod\limits_{\substack{k\in[n]\\k\neq i}} (a_k - a_i) \prod\limits_{\substack{k\in[n]\\k\neq j}} (b_k - b_j)} \quad \textit{for } x, y \in \mathbb{Z}_+ . \tag{3.2}$$

Then

$$\mathbf{P}(G(n, n) \leqslant k) = 1 + \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{x_1,\dots,x_l\geqslant k+n} \det\left[K_n(x_i, x_j)\right]_{i,j\in[l]} \quad \textit{for } k \in \mathbb{Z}_+ . \tag{3.3}$$

Proof. Let Φ denote the map that sends $A \in \mathbb{Z}_+^{[n]\times[n]}$ to the common shape of the corresponding SSYT pair under the bijection in Corollary 3.2, and define $\Lambda = \Phi([W(i, j)]_{i,j\in[n]})$. Then $\mathbf{P}(G(n, n) \leqslant k) = \mathbf{P}(\Lambda_1 \leqslant k)$. Moreover, for any partition λ, we have

$$\mathbf{P}(\Lambda = \lambda) = \sum_{A:\Phi(A)=\lambda} \mathbf{P}(W(i, j) = A(i, j) \text{ for } i, j \in [n])$$

$$
\begin{aligned}
&= \sum_{A:\Phi(A)=\lambda} \prod_{i,j\in[n]} (1-a_i b_j) a_i^{A(i,j)} b_j^{A(i,j)} \\
&= \prod_{i,j\in[n]} (1-a_i b_j) \sum_{A:\Phi(A)=\lambda} \prod_{i\in[n]} a_i^{\sum_{j\in[n]} A(i,j)} \prod_{j\in[n]} b_j^{\sum_{i\in[n]} A(i,j)} \\
&= \prod_{i,j\in[n]} (1-a_i b_j) \sum_{\substack{P:\text{shape}(P)=\lambda \\ l(\text{type}(P))\leqslant n}} \prod_{j\in[n]} b_j^{\text{type}(P)_j} \sum_{\substack{Q:\text{shape}(Q)=\lambda \\ l(\text{type}(Q))\leqslant n}} \prod_{i\in[n]} a_i^{\text{type}(Q)_i} \qquad (3.4)
\end{aligned}
$$

Note the inequality $l(\text{type}(P)) \geqslant l(\text{shape}(P))$ for any SSYT P; hence, (3.4) is zero unless $l(\lambda) \leqslant n$.

We now use the polynomial identity

$$
\sum_{\substack{P:\text{shape}(P)=\lambda \\ l(\text{type}(P))\leqslant n}} \prod_{j\in[n]} X_j^{\text{type}(P)_j} = \frac{\det[X_i^{\lambda_j - j+n}]_{i,j\in[n]}}{\det[X_i^{-j+n}]_{i,j\in[n]}}, \qquad (3.5)
$$

either side of which is the Schur polynomial indexed by λ in n variables $X_1, \ldots, X_n$. For a proof of (3.5), see [27, Chapter 7]. Since $(a_i)_{i\in[n]}$ and $(b_j)_{j\in[n]}$ are injective, the Vandermonde determinant

$$
\det[X_i^{-j+n}]_{i,j\in[n]} = \prod_{1\leqslant i<j\leqslant n} (X_i - X_j)
$$

is nonzero when evaluated by setting $X_i = a_i$ for $i \in [n]$ or $X_i = b_i$ for $i \in [n]$. Hence, by (3.4) and (3.5), we obtain

$$
\mathbf{P}(\Lambda = \lambda) = Z_n^{-1} \det[a_i^{\lambda_j - j+n}]_{i,j\in[n]} \det[b_i^{\lambda_j - j+n}]_{i,j\in[n]}, \qquad (3.6)
$$

where the normalization constant is given by

$$
Z_n = \frac{\prod_{1\leqslant i<j\leqslant n} (a_i - a_j)(b_i - b_j)}{\prod_{i,j\in[n]} (1-a_i b_j)}. \qquad (3.7)
$$

The probability $\mathbf{P}(\Lambda_1 \leqslant k)$ can then be written as

$$
\begin{aligned}
\mathbf{P}(\Lambda_1 \leqslant k) &= \frac{1}{Z_n} \sum_{\substack{\lambda:\lambda_1\leqslant k \\ l(\lambda)\leqslant n}} \det[a_i^{\lambda_j - j+n}]_{i,j\in[n]} \det[b_i^{\lambda_j - j+n}]_{i,j\in[n]} \\
&= \frac{1}{n! Z_n} \sum_{\substack{m_1,\ldots,m_n\in\mathbb{Z}_+ \\ m_j \leqslant k+n-1}} \det[a_i^{m_j}]_{i,j\in[n]} \det[b_i^{m_j}]_{i,j\in[n]}. \qquad (3.8)
\end{aligned}
$$

For the last equality, first change the summation index from λ to the decreasing sequence $(m_j)_{j\in[n]} = (\lambda_j - j + n)_{j\in[n]}$ in $\mathbb{Z}_+$ with $m_1 \leqslant k + n - 1$. (Note that λ is uniquely determined from $(m_j)_{j\in[n]}$ because $l(\lambda) \leqslant n$). Then remove the ordering of $(m_j)_{j\in[n]}$, which introduces the factor $1/n!$ Finally, allow repeats in $(m_j)_{j\in[n]}$, which does not alter the sum because the added terms are all zero.

It remains to turn (3.8) into the desired form. For any $l \in \mathbb{Z}_+$, by Proposition 3.3,

$$
\sum_{\substack{m_1,\ldots,m_n\in\mathbb{Z}_+ \\ m_i \leqslant l}} \det[a_i^{m_j}]_{i,j\in[n]} \det[b_i^{m_j}]_{i,j\in[n]} = n! \det\left[\sum_{m=0}^{l} a_i^m b_j^m\right]_{i,j\in[n]} = n! \det\left[\frac{1-a_i^{l+1} b_j^{l+1}}{1-a_i b_j}\right]_{i,j\in[n]}. \qquad (3.9)
$$

Letting $k \to +\infty$ in (3.8) and $l \to +\infty$ in (3.9) yield

$$
Z_n = \det\left[\sum_{m=0}^{\infty} a_i^m b_j^m\right]_{i,j\in[n]} = \det\left[\frac{1}{1-a_i b_j}\right]_{i,j\in[n]}. \qquad (3.10)
$$

Since $Z_n \neq 0$ by (3.7), the matrix $C = \big[(1-a_ib_j)^{-1}\big]_{i,j\in[n]}$ is invertible, and

$$\begin{aligned}
\mathbf{P}(\Lambda_1 \leqslant k) &= \frac{\det\big[C(i,j) - \sum_{m=k+n}^{\infty} a_i^m b_j^m\big]_{i,j\in[n]}}{\det[C(i,j)]_{i,j\in[n]}} \\
&= \det\left[\delta_{i,j} - \sum_{p=1}^{n} C^{-1}(i,p) \sum_{m=k+n}^{\infty} a_p^m b_j^m\right]_{i,j\in[n]} \\
&= \det\left[\delta_{i,j} - \sum_{m=k+n}^{\infty} b_j^m h_i(m)\right]_{i,j\in[n]},
\end{aligned}$$

where $h_i(m) = \sum_{p=1}^{n} C^{-1}(i,p)a_p^m$. Then, applying Proposition 3.3 twice, we obtain

$$\begin{aligned}
\mathbf{P}(\Lambda_1 \leqslant k) &= 1 + \sum_{q=1}^{n} \frac{(-1)^q}{q!} \sum_{i_1,\dots,i_q\in[n]} \det\left[\sum_{m=k+n}^{\infty} b_{i_s}^m h_{i_r}(m)\right]_{r,s\in[q]} \\
&= 1 + \sum_{q=1}^{n} \frac{(-1)^q}{q!} \sum_{i_1,\dots,i_q=1}^{n} \frac{1}{q!} \sum_{m_1,\dots,m_q\geqslant k+n} \det[b_{i_s}^{m_r}]_{r,s\in[q]} \det[h_{i_s}(m_r)]_{r,s\in[q]} \\
&= 1 + \sum_{q=1}^{n} \frac{(-1)^q}{q!} \sum_{m_1,\dots,m_q\geqslant k+n} \frac{1}{q!} \sum_{i_1,\dots,i_q=1}^{n} \det[b_{i_s}^{m_r}]_{s,r\in[q]} \det[h_{i_s}(m_r)]_{r,s\in[q]} \\
&= 1 + \sum_{q=1}^{n} \frac{(-1)^q}{q!} \sum_{m_1,\dots,m_q\geqslant k+n} \det\left[\sum_{i=1}^{n} h_i(m_r) b_i^{m_s}\right]_{r,s\in[q]} \\
&= 1 + \sum_{q=1}^{n} \frac{(-1)^q}{q!} \sum_{m_1,\dots,m_q\geqslant k+n} \det\left[\sum_{i=1}^{n}\sum_{p=1}^{n} a_p^{m_r} C^{-1}(i,p) b_i^{m_s}\right]_{s,r\in[q]}. \qquad (3.11)
\end{aligned}$$

Finally, we compute the inverse of C. Note that $C^{i,j}$, the i,j-minor of C, has the same structure (in terms of entries) as C; therefore, by (3.7) and (3.10),

$$\begin{aligned}
\det C^{i,j} &= \prod_{\substack{k,l\in[n]\\ k\neq i\\ l\neq j}} \frac{1}{1-a_kb_l} \prod_{\substack{k,l\in[n]\\ k<l\\ k,l\neq i}} (a_k - a_l) \prod_{\substack{k,l\in[n]\\ k<l\\ k,l\neq j}} (b_k - b_l) \\
&= \frac{\det C}{1-a_ib_j} \prod_{k\in[n]} (1-a_ib_k)(1-a_kb_j) \\
&\quad \cdot (-1)^{i+j} \prod_{\substack{k\in[n]\\ k\neq i}} (a_i-a_k)^{-1} \prod_{\substack{k\in[n]\\ k\neq j}} (b_j-b_k)^{-1}
\end{aligned}$$

Then, by Cramer's rule,

$$C^{-1}(i,j) = (-1)^{i+j}\frac{\det C^{j,i}}{\det C} = \frac{1}{1-a_jb_i} \frac{\prod_{k\in[n]} (1-a_jb_k)(1-a_kb_i)}{\prod_{\substack{k\in[n]\\ k\neq i}} (b_k-b_i) \prod_{\substack{k\in[n]\\ k\neq j}} (a_k-a_j)}. \qquad (3.12)$$

Inserting (3.12) into (3.11) completes the proof. □

Note that the assumption of independent weights distributed as (3.1) is crucial in the preceding proof to obtain (3.4), the representation of the distribution of the last-passage times in terms of the Schur polynomials. The probability measure (3.6) on the space of partitions is an example of a Schur measure introduced in [20].

Another probability measure of interest derived from (3.6) is the distribution of the random set $\mathcal{S} = \{\Lambda_j - j + n : j \in [n]\}$, which is given by

$$\mathbf{P}(\mathcal{S} = S) = \frac{1}{n! Z_n} \det[a_i^{m_j}]_{i,j\in[n]} \det[b_i^{m_j}]_{i,j\in[n]} \tag{3.13}$$

for any $S = \{m_1, \ldots, m_n\} \subset \mathbb{Z}_+$. By a general fact from the theory of point processes, for any distinct $x_1, \ldots, x_q \in \mathbb{Z}_+$,

$$\mathbf{P}(\{x_1, \ldots, x_q\} \subset \mathcal{S}\} = \det[K_n(x_r, x_s)]_{r,s\in[q]}. \tag{3.14}$$

In the language of the theory, $\mathcal{S}$ can be viewed as a determinantal point process on $\mathbb{Z}_+$ with correlation kernel K_n. Since $G(n,n) = \Lambda_1 = \max \mathcal{S} - n + 1$, a restatement of (3.3) is that

$$\mathbf{P}(\max \mathcal{S} \leqslant k + n - 1) = 1 + \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{x_1,\ldots,x_l \geqslant k+n} \mathbf{P}(\{x_1, \ldots, x_l\} \subset \mathcal{S}) \quad \text{for } k \in \mathbb{Z}_+,$$

which is an application of the inclusion/exclusion principle. This furnishes a probabilistic interpretation of (3.3). For a proof of (3.14) and a detailed discussion of the notions in this paragraph, we refer the reader to [3], [4], and [18].

A useful conclusion from (3.14) is that

$$\det[K_n(x_r, x_s)]_{r,s\in[q]} \geqslant 0 \quad \text{for any } x_1, \ldots, x_q \in \mathbb{Z}_+\,. \tag{3.15}$$

Moreover, this determinant equals 0 if $q > n$. One can also make these observations more directly using Proposition 3.3; we have

$$\begin{aligned} \det[K_n(x_r, x_s)]_{r,s\in[q]} &= \det\left[\sum_{i,j\in[n]} a_i^{x_r} C^{-1}(j,i) b_j^{x_s}\right]_{r,s\in[q]} \\ &= \frac{1}{(q!)^2} \sum_{i_1,\ldots,i_q\in[n]} \sum_{j_1,\ldots,j_q\in[n]} \det[a_{i_s}^{x_r}]_{s,r\in[q]} \det[C^{-1}(j_s, i_r)]_{r,s\in[q]} \det[b_{j_s}^{x_r}]_{r,s\in[q]} \qquad (3.16) \\ &= \frac{1}{q!} \sum_{i_1,\ldots,i_q\in[n]} \sum_{j_1,\ldots,j_q\in[n]} \frac{\det[a_{i_s}^{x_r}]_{s,r\in[q]} \det[b_{j_s}^{x_r}]_{r,s\in[q]}}{q! \det[C(j_s, i_r)]_{r,s\in[q]}}, \qquad (3.17) \end{aligned}$$

where the last equality requires $q \leqslant n$; otherwise the determinants in (3.16) are zero. For each choice of $i_1, \ldots, i_q$ and $j_1, \ldots, j_q$, the summand is a probability by (3.13) and, hence, the sum is nonnegative.

For the purposes of asymptotics as well as to extend Theorem 3.4 to the case of noninjective parameters, it is useful to express (3.2) as a contour integral. Let us write $\mathbf{a}$ and $\mathbf{b}$ for the parameter sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$, respectively. For $m, n \in \mathbb{N}$, $x, y \in \mathbb{Z}_+$ and $z \in \mathbb{C} \smallsetminus \{a_1, \ldots, a_m\}$, define

$$F^{\mathbf{a},\mathbf{b}}_{m,n,x}(z) = \frac{\prod_{j=1}^{n}(1 - zb_j)}{\prod_{i=1}^{m}(z - a_i)} \cdot z^{m+x}. \tag{3.18}$$

and the contour integral

$$K^{\mathbf{a},\mathbf{b}}_{m,n}(x,y) = \frac{1}{(2\pi\mathbf{i})^2} \oint\limits_{|w|=\rho} \oint\limits_{|z|=\rho} \frac{F^{\mathbf{a},\mathbf{b}}_{m,n,x}(z) F^{\mathbf{b},\mathbf{a}}_{n,m,y}(w)}{1 - zw}\, dz\, dw, \tag{3.19}$$

where $\max_{1\leqslant i\leqslant m} a_i \vee \max_{1\leqslant j\leqslant n} b_j < \rho < 1$ and the circles of integration are oriented counterclockwise.

Theorem 3.5. *Let $m, n \in \mathbb{N}$. Then*

$$\mathbf{P}(G(m,n) \leqslant k) = 1 + \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{\substack{x_1,\ldots,x_l \in \mathbb{Z}_+ \\ x_i \geqslant k}} \det[K^{\mathbf{a},\mathbf{b}}_{m,n}(x_i,x_j)]_{i,j\in[l]} \quad \textit{for any } k \in \mathbb{Z}_+ . \tag{3.20}$$

Proof. If $(a_i)_{i\in[n]}$ and $(b_i)_{i\in[n]}$ are injective, for each $u \in D(0,1)$, the only singularities of the functions $z \mapsto \dfrac{F^{\mathbf{a},\mathbf{b}}_{n,n,x}(z)}{1-zu}$ and $w \mapsto \dfrac{F^{\mathbf{b},\mathbf{a}}_{n,n,y}(w)}{1-wu}$ inside $D(0,1)$ are simples poles at $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$, respectively. Therefore, by Cauchy's residue formula and (3.2),

$$\begin{aligned} K^{\mathbf{a},\mathbf{b}}_{n,n}(x,y) &= \sum_{i,j\in[n]} \frac{\operatorname{Res}_{a_i} F^{\mathbf{a},\mathbf{b}}_{n,n,x} \operatorname{Res}_{b_j} F^{\mathbf{b},\mathbf{a}}_{n,n,y}}{1-a_ib_j} \\ &= \sum_{i,j\in[n]} \frac{a_i^{x+n} b_j^{y+n}}{1-a_ib_j} \frac{\prod\limits_{k\in[n]}(1-a_ib_k)(1-a_kb_j)}{\prod\limits_{\substack{k\in[n]\\k\neq i}}(a_k-a_i)\prod\limits_{\substack{k\in[n]\\k\neq j}}(b_k-b_j)} \\ &= K_n(x+n, y+n) \end{aligned}$$

for $x, y \in \mathbb{Z}_+$ provided that $(a_i)_{i\in[n]}$ and $(b_i)_{i\in[n]}$ are injective. Then, by Theorem 3.4,

$$\mathbf{P}(G(n,n) \leqslant k) = 1 + \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{\substack{x_1,\ldots,x_l \in \mathbb{Z}_+ \\ x_i \geqslant k}} \det[K^{\mathbf{a},\mathbf{b}}_{n,n}(x_i,x_j)]_{i,j\in[l]} \quad \text{for } k \in \mathbb{Z}_+ . \tag{3.21}$$

We claim that both sides of (3.21) are continuous in parameters $(a_i)_{i\in[n]}$ and $(b_i)_{i\in[n]}$. Then (3.21) holds even if $(a_i)_{i\in[n]}$ or $(b_i)_{i\in[n]}$ has repeats. In particular, setting $a_i = 0$ for $m < i \leqslant n$ and $b_j = 0$ for $n < j \leqslant m$, we obtain the result. To prove the claim, note that $\mathbf{P}(G(n,n) \leqslant k)$ is continuous because it can be written as the finite sum of probabilities

$$\mathbf{P}(W(i,j) = A(i,j) \text{ for } i,j \in [n]) = \prod_{i,j\in[n]} (1-a_ib_j) a_i^{A(i,j)} b_j^{A(i,j)} \tag{3.22}$$

over matrices $A \in \mathbb{Z}_+^{[n]\times[n]}$ for which $\max_{\pi\in\Pi(m,n)} \sum_{(i,j)\in\pi} A(i,j) \leqslant k$, and (3.22) is continuous. Pick $\delta > 0$ small so that

$$\max_{1\leqslant i\leqslant n} a_i \vee \max_{1\leqslant j\leqslant n} b_j < \rho - \delta, \tag{3.23}$$

where ρ is as in (3.19). Then there exists $C > 0$ (which depends on n and δ) such that $K^{\mathbf{a},\mathbf{b}}_{n,n}(x,y) \leqslant C\rho^{x+y}$ for any $x, y \in \mathbb{Z}_+$, which leads to the bound

$$\det[K^{\mathbf{a},\mathbf{b}}_{n,n}(x_i,x_j)]_{i,j\in[l]} = \sum_{\sigma\in\mathcal{S}_l} \operatorname{sgn}(\sigma) \prod_{i=1}^{l} K^{\mathbf{a},\mathbf{b}}_{n,n}(x_i, x_{\sigma(i)}) \leqslant l! C^l \rho^{2\sum_{i=1}^{l} x_i},$$

which is summable over $x_1,\ldots,x_l \geqslant k$. Hence, the inner sum in (3.21) converges uniformly on the set $[0,\rho-\delta)^{2n}$. This and continuity of $K^{\mathbf{a},\mathbf{b}}_{n,n}$ imply that the right-hand side of (3.21) is also continuous on $[0,\rho-\delta)^{2n}$. Since $\rho - \delta$ can be chosen arbitrarily close to 1, the claim is proved. ☐

Since $\rho < 1$, using the identity $(1-zw)^{-1} = \sum_{l=0}^{\infty} z^l w^l$ and Fubini-Tonelli theorem, we can rearrange (3.19) as

$$K^{\mathbf{a},\mathbf{b}}_{m,n}(x,y) = \frac{1}{(2\pi\mathbf{i})^2} \oint\limits_{|w|=\rho} \oint\limits_{|z|=\rho} \sum_{l=0}^{\infty} F^{\mathbf{a},\mathbf{b}}_{m,n,x+l}(z) F^{\mathbf{b},\mathbf{a}}_{n,m,y+l}(w)\, dz\, dw$$

$$= \sum_{l=0}^{\infty} I^{\mathbf{a},\mathbf{b}}_{m,n,x+l} I^{\mathbf{b},\mathbf{a}}_{n,m,y+l}, \tag{3.24}$$

where

$$I^{\mathbf{a},\mathbf{b}}_{m,n,x} = \frac{1}{2\pi\mathbf{i}} \oint_{|z|=\rho} F^{\mathbf{a},\mathbf{b}}_{m,n,x}(z)dz = \frac{1}{2\pi\mathbf{i}} \oint_{|z|=1} F^{\mathbf{a},\mathbf{b}}_{m,n,x}(z)dz. \tag{3.25}$$

4 Steepest-descent analysis

Fix $q \in (0,1)$ and set $a_i = \sqrt{q}$ for $i \in [m]$ and $b_j = \sqrt{q}$ for $j \in [n]$. Then (3.18) becomes

$$F_{m,n,x}(z) = \frac{(1 - z\sqrt{q})^n}{(z - \sqrt{q})^m} \cdot z^{m+x} \tag{4.1}$$

for $m, n \in \mathbb{N}$, $x, y \in \mathbb{Z}_+$ and $z \in \mathbb{C} \smallsetminus \{\sqrt{q}\}$. In this special case, we will drop superscripts $\mathbf{a}$ and $\mathbf{b}$ in (3.19) and (3.25) as well. Theorem 3.5 asserts that

$$\mathbf{P}(G(m,n) \leqslant t) = 1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \sum_{\substack{x_1,\ldots,x_l \in \mathbb{Z}_+ \\ x_i \geqslant t}} \det[K_{m,n}(x_i, x_j)]_{i,j \in [l]}. \tag{4.2}$$

for any $m, n \in \mathbb{N}$ and $t \geqslant 0$. To prove Theorem 1.1, we will show that the right-hand side of (4.2) with $m = \lfloor nr \rfloor$ and $t = n\gamma + n^{1/3}\sigma s$ converges to that of (2.5). The main step here is to establish that appropriately rescaled $K_{m,n}(x,y)$ converges to the Airy kernel and obeys a uniform upper bound.

Let $(\eta_k)_{k \in \mathbb{N}}$ be a positive real sequence such that $\eta_k \to +\infty$. For $k \in \mathbb{N}$, set

$$n_k = \lfloor \eta_k \rfloor \quad \text{and} \quad m_k = \lfloor r\eta_k \rfloor. \tag{4.3}$$

Also, for $k \in \mathbb{N}$ and $s \in \mathbb{R}$, define

$$p_k(s) = \lfloor \eta_k \gamma + \eta_k^{1/3} \sigma s \rfloor. \tag{4.4}$$

Theorem 4.1. *Let $T > 0$.*

(a) For each $l \in \mathbb{N}$,

$$\lim_{k \to \infty} \det[K_{m_k,n_k}(p_k(s_i), p_k(s_j))]_{i,j \in [l]} \sigma^l \eta_k^{l/3} = \det[\mathrm{A}(s_i, s_j)]_{i,j \in [l]}$$

uniformly in $s_1, \ldots, s_l \in [-T, T]$.

(b) There exist $N \in \mathbb{N}$ and constants $C, c > 0$ such that

$$\det[K_{m_k,n_k}(p_k(s_i), p_k(s_j))]_{i,j \in [l]} \leqslant C^l l^{l/2} \eta_k^{-l/3} e^{-c\sum_{i=1}^l s_i}$$

for all $l \in \mathbb{N}$, $k \geqslant N$ and $s_1, \ldots, s_l \geqslant -T$.

We give a proof of Theorem 4.1 in Section 5 via steepest descent analysis of the integral in (3.25). In order to find a suitable deformation of the contour, we now examine $F_{m_k,n_k,p_k(s)}$ as $k \to +\infty$.

Let D denote the disk having the interval $(\sqrt{q}, 1/\sqrt{q})$ as its diameter. We define the holomorphic function f on D by

$$f(z) = -r\log(z - \sqrt{q}) + \log(1 - \sqrt{q}z) + (r + \gamma)\log z, \tag{4.5}$$

where the logarithms are the principal branch. By (4.1), (4.3) and (4.4), we have

$$F_{m_k,n_k,p_k(s)}(z) = \exp\left(-m_k \log(z - \sqrt{q}) + n_k \log(1 - z\sqrt{q}) + (m_k + p_k(s))\log z\right) \tag{4.6}$$

$$= \exp\Big(\eta_k f(z) + \eta_k^{1/3}\sigma s \log z + O(1)(\log(1 - z\sqrt{q}) + \log(z - \sqrt{q}) + \log z)\Big)$$

for $z \in D$. This leads us to choose the contour in (3.25) based on the behavior of f.

Let u and v denote the real and the imaginary parts of f; hence,

$$u(z) = -r \log|z - \sqrt{q}| + \log|1 - \sqrt{q}z| + (r + \gamma)\log|z| \tag{4.7}$$

$$v(z) = -r \arg(z - \sqrt{q}) + \arg(1 - \sqrt{q}z) + (r + \gamma)\arg z. \tag{4.8}$$

Note that u and the derivative

$$f'(z) = \frac{\sqrt{q}}{\sqrt{q}z - 1} + \frac{r + \gamma}{z} - \frac{r}{z - \sqrt{q}} \tag{4.9}$$

both extend continuously to $\mathbb{C} \smallsetminus \{0, \sqrt{q}, 1/\sqrt{q}\}$ via the formulas in (4.7) and (4.9).

We can rewrite (4.9) as

$$f'(z) = \sqrt{q}(1 + \gamma)\frac{(z - \zeta)^2}{(z\sqrt{q} - 1)(z - \sqrt{q})z}, \tag{4.10}$$

where

$$\zeta = \frac{\sqrt{q} + \sqrt{r}}{1 + \sqrt{qr}}. \tag{4.11}$$

Hence, $f'(\zeta) = f''(\zeta) = 0$ and, in view of (1.7),

$$f'''(\zeta) = \frac{2\sqrt{q}(1 + \gamma)}{(\zeta\sqrt{q} - 1)(\zeta - \sqrt{q})\zeta} = -\frac{2\sigma^3}{\zeta^3}.$$

Then we have the expansion

$$f(z) = f(\zeta) + \frac{\sigma^3}{3}(1 - z/\zeta)^3 + O(|z - \zeta|^4) \tag{4.12}$$

around ζ.

We now discuss, somewhat informally, the nature of the steepest-descent and -ascent curves of u (i.e. curves that are tangent to the vector fields $-\nabla u$ and ∇u, respectively) emanating from ζ. A more rigorous version of the argument here is provided in the appendix.

It follows from (4.12) that there are three steepest-descent curves D_1, D_2, D_3 and three steepest-ascent curves A_1, A_2, A_3 of u which have ζ as one of their endpoints; D_i and A_i come in ζ at angles $i2\pi/3$ and $\pi - i2\pi/3$ with positive real axis for $i = 1, 2, 3$. We next deduce some global features of these curves depicted in Figure 4.1. By the Cauchy-Riemann equations, v equals the constant $v(\zeta) = 0$ along these curves. Since $\nabla u = \overline{f'}$ is real-valued on the real line and, by (4.10), has a zero only at ζ, the unique steepest-descent and -ascent curves passing through x are along the real line for any $x \in \mathbb{R} \smallsetminus \{0, \sqrt{q}, 1/\sqrt{q}, \zeta\}$. We conclude that D_1 and A_1, which contain points of the upper half-plane, cannot intersect the real line. D_1 and A_1 cannot intersect each other and themselves either due to strict monotonicity of u along these curves. In particular, D_1 and A_1 do not revisit a small neighborhood of ζ as the level set $v = 0$ in the upper half-plane nearby ζ consists of segments of D_1 and A_1.

If D_1 and A_1 are bounded then, along D_1 and A_1, ∇u is bounded away from 0 outside a small neighborhood of ζ and u approaches $-\infty$ and $+\infty$, respectively. We note from (4.7) that $u(z) \to +\infty$ as $z \to \sqrt{q}$ and $u(z) \to -\infty$ as $z \to 0$ or $1/\sqrt{q}$. Also, $u(z) \to +\infty$ as $|z| \to +\infty$; therefore, D_1 is bounded, which implies that D_1 approaches either 0 or $1/\sqrt{q}$. The latter is not possible because, otherwise, A_1 is trapped in the interior of the closed curve made up of D_1 and the line segment $[\zeta, 1/\sqrt{q}]$, and must approach the exterior point $\sqrt{q}$. This contradiction shows that D_1 ends at 0. Then A_1, being still cut off from $\sqrt{q}$, goes off to infinity. Due to the symmetry of u, D_2 and A_2 are mirror images with respect to the real axis of D_1 and A_1, respectively.

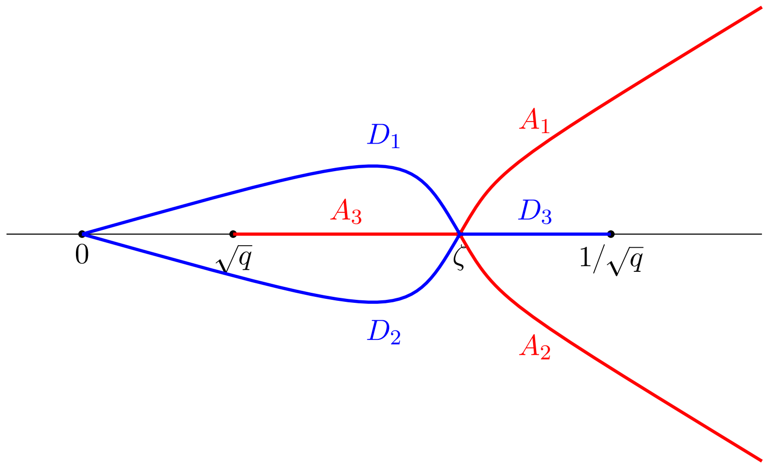


Figure 4.1: Steepest descent curves D_1, D_2, D_3 and steepest ascent curves A_1, A_2, A_3 for u at ζ.

Let Γ denote the contour consisting of D_1 and D_2 oriented counterclockwise. Since Γ encloses the singularity $\sqrt{q}$ of $F_{m,n,x}$, we can deform the contour in (3.25) to Γ.

In the proof of Theorem 4.1, we will need upper bounds on $I_{m_k,n_k,p_k(s)}$ that improve exponentially as $s \to +\infty$. The following lemma will be useful to establish these bounds. See Figure 4.2 below for an illustration of the lemma.

Lemma 4.2. *The circle $|z| = \zeta$ encloses $\Gamma \smallsetminus \{\zeta\}$.*

Proof. For $z \in D_1$, we have $v(z) = v(\zeta) = 0$. To prove the result, it suffices to show that the function $t \mapsto v(\zeta e^{\mathbf{i}t})$ is increasing on $[0, \pi)$. We have

$$\begin{aligned} \frac{d(v(\zeta e^{\mathbf{i}t}))}{dt} &= \nabla v(\zeta e^{\mathbf{i}t}) \cdot (\mathbf{i}\zeta e^{\mathbf{i}t}) = \Re(f'(\zeta e^{\mathbf{i}t})\zeta e^{\mathbf{i}t}) \\ &= \frac{(1+\gamma)\zeta^2 \Re(x(t)y(t))}{|\zeta e^{\mathbf{i}t} - 1/\sqrt{q}|^2 |\zeta e^{\mathbf{i}t} - \sqrt{q}|^2}, \end{aligned} \tag{4.13}$$

where $x(t) = (e^{\mathbf{i}t} - 1)(\zeta e^{-\mathbf{i}t} - 1/\sqrt{q})$ and $y(t) = (e^{\mathbf{i}t} - 1)(\zeta e^{-\mathbf{i}t} - \sqrt{q})$. Note that $\Re x(t) > 0, \Im x(t) < 0$, and $\Re y(t) > 0, \Im y(t) > 0$, which implies (4.13) is positive. $\square$

5 Asymptotics of the correlation kernel

In this section, we give a proof of Theorem 4.1. The argument is based on representation (3.19) for the correlation kernel and requires precise information about the behavior of the integrals $I_{m_k,n_k,p_k(s)}$ for large $k \in \mathbb{N}$. The following lemma shows that the Airy function arises as a uniform limit in the asymptotics as $k \to \infty$ and establishes a uniform bound, which will be sufficient for our purposes. The proof of the lemma comes from analysis of the integrands $F_{m_k,n_k,p_k(s)}$ along the steepest-descent contour Γ defined in Section 4.

Lemma 5.1. *Let $T > 0$.*

(a)

$$\lim_{k\to\infty} \frac{\sigma \eta_k^{1/3} I_{m_k,n_k,p_k(s)}}{\zeta F_{m_k,n_k,p_k(s)}(\zeta)} = \mathrm{Ai}(s)$$

uniformly in $s \in [-T, T]$.

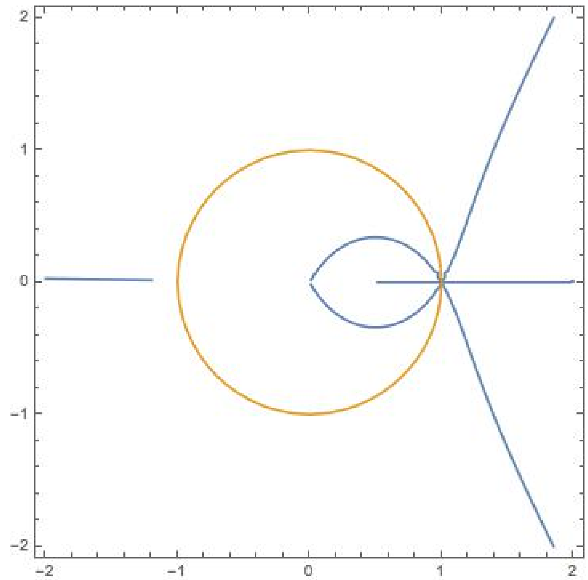


Figure 4.2: A plot of the contour $v(z) = 0$ (light blue) and the circle $|z| = \zeta$ (brown) for $q = 1/4$ and $r = 1$.

(b) There exist $N \in \mathbb{N}$ and constants $C, c > 0$ such that

$$|I_{m_k,n_k,p_k(s)}| \leqslant C\eta_k^{-1/3} e^{-cs} |F_{m_k,n_k,p_k(s)}(\zeta)|$$

for $k \geqslant N$ and $s \geqslant -T$.

Proof of (a). Because Γ is symmetric about the real axis and the upper half of Γ makes an angle of $2\pi/3$ with the positive real axis at ζ, if $\epsilon > 0$ is sufficiently small, the points $\zeta' = \zeta + \zeta\epsilon\sigma^{-1}e^{\mathbf{i}(2\pi/3+\phi)}$ and $\overline{\zeta'}$ are on Γ for some $\phi \in (-\pi/6, \pi/6)$. Split Γ into two parts Γ_1 and Γ_2 with endpoints at ζ' and $\overline{\zeta'}$, where Γ_1 is the part passing through ζ. Note that we can change the contour of integration Γ by deforming Γ_1 into Γ_1', the broken line segment with vertices at ζ, ζ' and $\overline{\zeta'}$ oriented from $\overline{\zeta'}$ to ζ', see Figure 5.1. The proof is complete once we show that

$$\lim_{k\to\infty} \frac{\sigma\eta_k^{1/3}}{\zeta F_{m_k,n_k,p_k(s)}(\zeta)} \cdot \frac{1}{2\pi\mathbf{i}} \int\limits_{\Gamma_1'} F_{m_k,n_k,p_k(s)}(z)dz = \mathrm{Ai}(s) \tag{5.1}$$

$$\lim_{k\to\infty} \eta_k^{1/3} \int\limits_{\Gamma_2} \frac{F_{m_k,n_k,p_k(s)}(z)}{F_{m_k,n_k,p_k(s)}(\zeta)} dz = 0, \tag{5.2}$$

both uniformly in $s \in [-T, T]$.

If ϵ is sufficiently small, we have for z on Γ_1'

$$\begin{aligned}
\frac{F_{m_k,n_k,p_k(s)}(z)}{F_{m_k,n_k,p_k(s)}(\zeta)} &= \exp\left(\eta_k(f(z) - f(\zeta)) + \eta_k^{1/3}\sigma s \log(z/\zeta)\right) \\
&\quad \cdot \exp\left(-(m_k - r\eta_k)\log\left(\frac{z-\sqrt{q}}{\zeta-\sqrt{q}}\right) + (n_k - \eta_k)\log\left(\frac{1-z\sqrt{q}}{1-\zeta\sqrt{q}}\right)\right) \\
&\quad \cdot \exp\left((p_k(s) - \eta_k\gamma - \eta_k^{1/3}\sigma s)\log(z/\zeta)\right) \\
&= \exp\left(\frac{\eta_k\sigma^3}{3}(1 - z/\zeta)^3 - \eta_k^{1/3}\sigma s(1 - z/\zeta)\right)
\end{aligned}$$

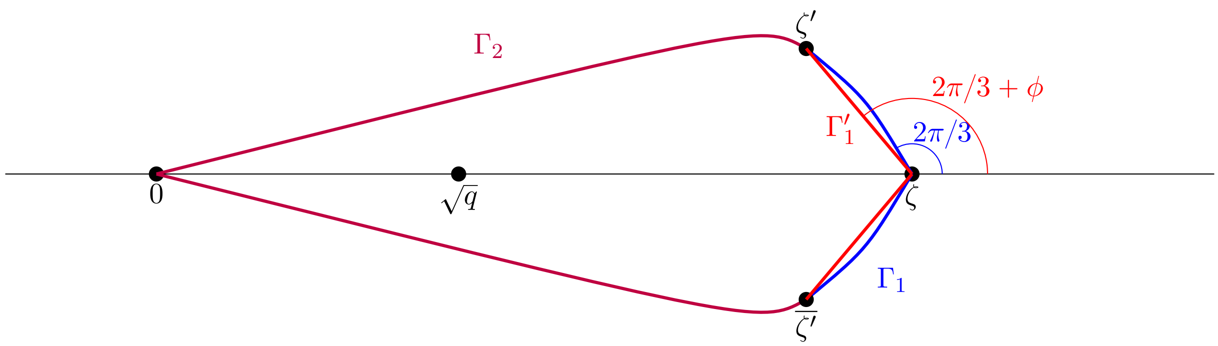


Figure 5.1: Contours Γ_1 (blue), Γ_2 (purple) and Γ_1'(red).

$$\cdot \exp\left(O(|z-\zeta|)+O(\eta_k^{1/3}|s||z-\zeta|^2)+O(\eta_k|z-\zeta|^4)\right).$$

This comes from (4.5), (4.6), (4.12) and the expansion

$$\log(z/\zeta)=-(1-z/\zeta)+O(|z-\zeta|^2).$$

Hence, changing the variables via $z=\zeta(1-\sigma^{-1}\eta_k^{-1/3}u)$ and rearranging terms, we arrive at

$$\frac{\sigma\eta_k^{1/3}}{\zeta F_{m_k,n_k,p_k(s)}(\zeta)}\cdot\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_1} F_{m_k,n_k,p_k(s)}(z)dz=\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_k} e^{u^3/3-su+O(\eta_k^{-1/3}(|u|+|s||u|^2+|u|^4))}du, \quad (5.3)$$

where contour $\mathcal{C}_k$ consists of the line segments from $\epsilon\eta_k^{1/3}e^{-\mathbf{i}(\pi/3-\phi)}$ to 0 and from 0 to $\epsilon\eta_k^{1/3}e^{\mathbf{i}(\pi/3-\phi)}$. See Figure 5.2.

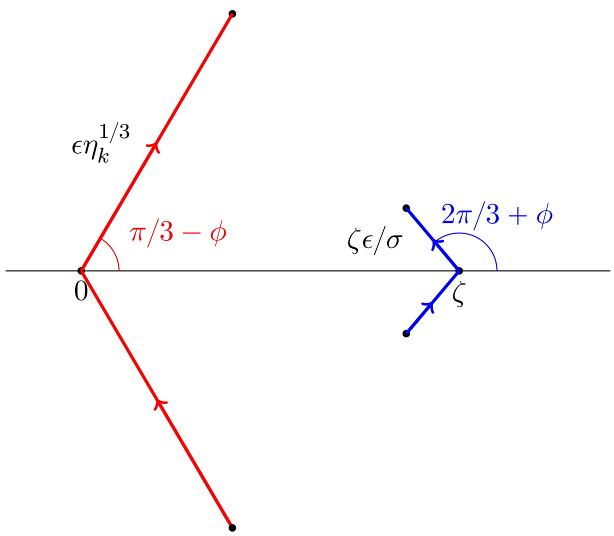


Figure 5.2: Contours Γ_1' (blue) and $\mathcal{C}_k$ (red).

We now consider the difference of the right-hand sides of (2.1) (with $\theta=\pi/3-\phi$) and (5.3). Let C denote a common implicit constant in the error terms in (5.3). Put $a=\cos(\pi-3\phi)/3$ and $b=\cos(\pi/3-\phi)$. Note that $\phi\to 0$ as $\epsilon\to 0$; therefore, a and b can be made arbitrarily close to $-1/3$ and $1/2$, respectively. For $u\in\mathcal{C}_k$,

$$|e^{u^3/3-su}(e^{O(\eta_k^{-1/3}(|u|+|s||u|^2+|u|^4))}-1)|=e^{a|u|^3-sb|u|}|e^{O(\eta_k^{-1/3}(|u|+|s||u|^2+|u|^4))}-1| \quad (5.4)$$

$$
\begin{aligned}
&\leqslant e^{a|u|^3 - sb|u| + C\eta_k^{-1/3}(|u|+|s||u|^2+|u|^4)} \\
&\cdot C\eta_k^{-1/3}(|u| + |s||u|^2 + |u|^4) && (5.5) \\
&\leqslant e^{a|u|^3 + Tb|u| + C\eta_k^{-1/3}(|u|+T|u|^2+|u|^4)} \\
&\cdot C\eta_k^{-1/3}(|u| + T|u|^2 + |u|^4) && (5.6) \\
&\leqslant C\epsilon e^{C\epsilon}(1 + T|u| + |u|^3) \\
&\cdot \exp((a + C\epsilon)|u|^3 + (Tb + C\epsilon T)|u|). && (5.7)
\end{aligned}
$$

Here, (5.4) follows because u has direction $e^{\pm\mathbf{i}(\pi/3-\phi)}$. For (5.5), we use the inequality $|e^z - 1| \leqslant |z|e^{|z|}$ for $z \in \mathbb{C}$. Finally, (5.6) and (5.7) are due to $|s| \leqslant T$ and $|u| \leqslant \epsilon\eta_k^{1/3}$. Note that the right-hand side of (5.6) does not depend on s and converges to 0 as $k \to \infty$ pointwise. Moreover, the right-hand side of (5.7) does not depend on k and is integrable provided that $a + C\epsilon < 0$, which is the case if $\epsilon > 0$ is small enough. Hence, (5.1) follows from dominated convergence.

We now turn to (5.2). Since $\log|1 - z\sqrt{q}|$ and $\log|z - \sqrt{q}|$ are bounded on Γ, we obtain

$$
\frac{|F_{m_k,n_k,p_k(s)}(z)|}{|F_{m_k,n_k,p_k(s)}(\zeta)|} = \exp\Big(\eta_k(u(z) - u(\zeta)) + (\eta_k^{1/3}\sigma s + O(1))\log|z/\zeta| + O(1)\Big) \tag{5.8}
$$

for $z \in \Gamma \setminus \{0\}$. Introduce a small parameter $\delta \in (0, 1)$. Since u is decreasing along Γ away from ζ, we have $\sup_{\Gamma_2} u(z) < u(\zeta)$. Also, $\log|z/\zeta|$ is bounded on Γ minus the disk $|z| \leqslant \delta$. These observations, assumption $|s| \leqslant T$ and (5.8) imply existence of constants $C, c > 0$ such that

$$
\frac{|F_{m_k,n_k,p_k(s)}(z)|}{|F_{m_k,n_k,p_k(s)}(\zeta)|} \leqslant Ce^{-c\eta_k}, \tag{5.9}
$$

for $z \in \Gamma_2$ with $|z| \geqslant \delta$. By boundedness of the first two terms in (4.7),

$$
u(z) - u(\zeta) = (\gamma + r)\log|z/\zeta| + O(1) \tag{5.10}
$$

for $0 < |z| \leqslant \delta$. Using this and $|s| \leqslant T$ leads to

$$
\eta_k(u(z) - u(\zeta)) + (\eta_k^{1/3}\sigma s + O(1))\log|z/\zeta| = \eta_k\Big((\gamma + r + O(\eta_k^{-2/3}))\log|z/\zeta| + O(1)\Big)
$$

The right-hand side is less than $-c'\eta_k$ for some constant $c' > 0$ for all large enough k provided that δ is small enough. Hence, an equality of the form (5.9) also holds for $|z| \leqslant \eta$ and k large enough. It follows that

$$
\eta_k^{1/3}\left|\int_{\Gamma_2} \frac{F_{m_k,n_k,p_k(s)}(z)}{F_{m_k,n_k,p_k(s)}(\zeta)}dz\right| \leqslant C\eta_k^{1/3}\,\mathrm{length}(\Gamma_2)e^{-c\eta_k}, \tag{5.11}
$$

which implies the uniform convergence in (5.2). □

Proof of (b). It suffices to show that there exist $C, c > 0$ and $N \in \mathbb{N}$ such that

$$
\left|\int_{\mathcal{C}_k} e^{u^3/3 - su + O(\eta_k^{-1/3}(|u|+|s||u|^2+|u|^4))}du\right| \leqslant Ce^{-cs} \tag{5.12}
$$

$$
\left|\int_{\Gamma_2} \frac{F_{m_k,n_k,p_k(s)}(z)}{F_{m_k,n_k,p_k(s)}(\zeta)}dz\right| \leqslant C\eta_k^{-1/3}e^{-cs} \tag{5.13}
$$

for all $s \geqslant -T$ and $k \geqslant N$.

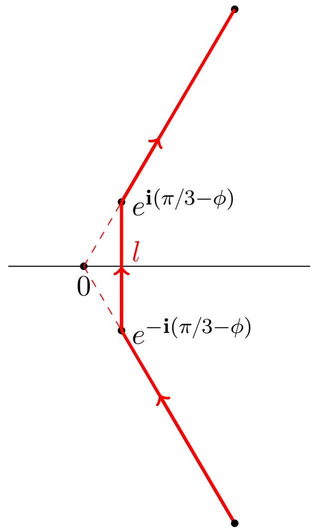


Figure 5.3: Contour obtained from deforming a part of $\mathcal{C}_k$ (dashed) into line segment l.

We first consider the case $s \geqslant 0$. Choose k large and deform the part of the contour $\mathcal{C}_k$ in (5.3) that contains 0 and has endpoints at $e^{\pm\mathbf{i}(\pi/3-\phi)}$ into the line segment l from $e^{-\mathbf{i}(\pi/3-\phi)}$ to $e^{\mathbf{i}(\pi/3-\phi)}$ as in Figure 5.3. Recall a and b from the proof of (a). For $w = b + \mathbf{i}y \in l$, we have

$$\begin{aligned}\Re\left(w^3/3 - sw + O(\eta_k^{-1/3}(|w| + s|w|^2 + |w|^4))\right) &= b^3/3 - by^2 - bs + O(\eta_k^{-1/3}(1+s)) \\ &\leqslant 1/3 - bs + C\eta_k^{-1/3}(1+s) \leqslant 1 - bs/2 \quad (5.14)\end{aligned}$$

for some constant $C > 0$ and sufficiently large k. Similarly, for $w \in \mathcal{C}_k$ with $|w| \geqslant 1$ and $\epsilon > 0$ sufficiently small, we get

$$\begin{aligned}\Re\left(w^3/3 - sw + O(\eta_k^{-1/3}(|w| + s|w|^2 + |w|^4))\right) &= a|w|^3 - sb|w| + O(\epsilon(1 + s|w| + |w|^3)) \\ &\leqslant (a + C\epsilon)\,|w|^3 + C\epsilon - (b - C\epsilon)s \qquad (5.15)\end{aligned}$$

for some constant $C > 0$. Bounds (5.14) and (5.15) imply (5.12) for $s \geqslant 0$.

Recall $\delta > 0$ introduced after (5.8) and that $\sup_{z\in\Gamma_2} u(z) < u(\zeta)$. Also, it follows from Lemma 4.2 that $\sup_{z\in\Gamma_2} \log|z/\zeta| < 0$. Then, by (5.8), there exist constants $C, c > 0$ such that

$$\frac{|F_{m_k,n_k,p_k(s)}(z)|}{|F_{m_k,n_k,p_k(s)}(\zeta)|} \leqslant Ce^{-c\eta_k - c\eta_k^{1/3}s} \qquad (5.16)$$

for $z \in \Gamma_2$ with $|z| \geqslant \delta$. Moreover, choosing k large, δ small and using (5.8) and (5.10), we obtain

$$\begin{aligned}\frac{|F_{m_k,n_k,p_k(s)}(z)|}{|F_{m_k,n_k,p_k(s)}(\zeta)|} &\leqslant C\exp\left(\eta_k((\gamma + r + O(\eta_k^{-1}))\log|z/\zeta| + O(1)) + \eta_k^{1/3}\sigma s\log|z/\zeta|\right) \\ &\leqslant Ce^{-c\eta_k - c\eta_k^{1/3}s} \qquad (5.17)\end{aligned}$$

for $z \in \Gamma_2$ with $0 < |z| \leqslant \delta$, for some constants $C, c > 0$. Then (5.13) for $s \geqslant 0$ follows from (5.16) and (5.17).

Let C and c refer to the constants in (5.12) and (5.13) for $s \geqslant 0$. Suppose now that $s \in [-T, 0)$. Then the integrand in (5.12) is bounded by

$$\exp(a|u|^3 + Tb|u| + C_1\epsilon(1 + T|u| + |u|^3)) \qquad (5.18)$$

for some constant $C_1 > 0$. If ϵ is small enough, (5.18) is integrable over $\mathcal{C}_k$. It follows from this and (5.11) that (5.12) and (5.13) still hold after replacing C by a larger constant if necessary. $\square$

Proof of Theorem 4.1. We will first obtain (b). By Lemma 5.1(b), there exist constants $C, c > 0$ such that

$$|I_{m_k,n_k,p_k(s)+l}| = \left|I_{m_k,n_k,p_k(s+\eta_k^{-1/3}\sigma_r^{-1}l)}\right| \leqslant C\eta_k^{-1/3}e^{-c(s+\eta_k^{-1/3}\sigma_r^{-1}l)}|F_{m_k,n_k,p_k(s)+l}(\zeta_r)| \quad (5.19)$$

for $s \geqslant -T$ for sufficiently large k. Observe the identities $\gamma_{1/r} = \gamma_r/r$, $\sigma_{1/r} = \sigma_r/r^{1/3}$ and $\zeta_r = 1/\zeta_r$ from (1.6), (1.7) and (4.11). Hence, we have

$$n_k = \lfloor \eta_k'/r \rfloor, \quad m_k = \lfloor \eta_k' \rfloor$$
$$p_k(s) = \lfloor \eta_k'\gamma_{1/r} + \eta_k'^{1/3}\sigma_{1/r}s \rfloor$$

for $k \in \mathbb{N}$ and $s \in \mathbb{R}$, where $\eta_k' = r\eta_k$. Then, another application of Lemma 5.1(b) with r and sequence $(\eta_k')_{k\in\mathbb{N}}$ yields

$$|I_{n_k,m_k,p_k(t)+l}| = \left|I_{n_k,m_k,p_k(t+\eta_k^{-1/3}\sigma_r^{-1}l)}\right| \leqslant C'\eta_k^{-1/3}e^{-c'(t+\eta_k^{-1/3}\sigma_r^{-1}l)}|F_{n_k,m_k,p_k(t)+l}(\zeta_{1/r})| \quad (5.20)$$

for $t \geqslant -T$, for sufficiently large k and some constants $C', c' > 0$. In fact, because $s, t \geqslant -T$, we can assume that $c' = c$ at the expense of having larger constants C, C'. Also, it can be verified from (4.1) that

$$F_{m_k,n_k,p_k(s)+l}(\zeta_r)F_{n_k,m_k,p_k(t)+l}(\zeta_{1/r}) = \zeta_r^{p_k(s)-p_k(t)}. \quad (5.21)$$

Combining (5.19), (5.20) and (5.21), we obtain

$$|I_{m_k,n_k,p_k(s)+l}I_{n_k,m_k,p_k(t)+l}| \leqslant Ce^{-c(s+t)}\eta_k{}^{-2/3}e^{-c\sigma_r^{-1}\eta_k^{-1/3}l}\zeta_r^{p_k(s)-p_k(t)} \quad (5.22)$$

for all integer $l \geqslant 0$. In view of (3.19), summing over l yields

$$|K_{m_k,n_k}(p_k(s), p_k(t))| \leqslant \frac{C\eta_k^{-1/3}}{1 - e^{-c\sigma_r^{-1}\eta_k^{-1/3}}}\eta_k^{-1/3}\zeta^{p_k(s)-p_k(t)}e^{-c(s+t)}. \quad (5.23)$$

The first factor on the right-hand side is bounded; thus,

$$|K_{m_k,n_k}(p_k(s), p_k(t))| \leqslant C\eta_k{}^{-1/3}e^{-c(s+t)}\zeta_r^{p_k(s)-p_k(t)} \quad (5.24)$$

for $s, t \geqslant -T$, sufficiently large $k \in \mathbb{N}$ and some constant $C > 0$. Now, using (5.24), properties of the determinant and Hadamard's inequality gives

$$\begin{aligned}\det\left[K_{m_k,n_k}(p_k(s_i), p_k(s_j))\right]_{i,j\in[l]} &= \det\left[\zeta_r^{p_k(s_j)-p_k(s_i)}K_{m_k,n_k}(p_k(s_i), p_k(s_j))\right]_{i,j\in[l]} \quad (5.25)\\ &\leqslant C^l\eta_k^{-l/3}\prod_{i=1}^{l}e^{-cs_i}\left(\sum_{j=1}^{l}e^{-2cs_j}\right)^{1/2}\\ &= (e^{cT}C)^l\eta_k^{-l/3}\prod_{i=1}^{l}e^{-cs_i}\left(\sum_{j=1}^{l}e^{-2c(s_j+T)}\right)^{1/2}\\ &\leqslant (e^{cT}C)^l l^{l/2}\eta_k^{-l/3}\prod_{i=1}^{l}e^{-cs_i},\end{aligned}$$

which is (b) after redefining the constant C.

Each entry of the matrix on the right-hand side of (5.25) is bounded uniformly in $s_1, \dots, s_l \in [-T, T]$ by (5.24). Therefore, to prove (a), it suffices to show that

$$\lim_{k\to+\infty}\sigma_r\eta_k^{1/3}\zeta_r^{p_k(t)-p_k(s)}K_{m_k,n_k}(p_k(s), p_k(t)) = \mathrm{A}(s,t) \quad (5.26)$$

uniformly in $s,t \in [-T,T]$. Let $L > 0$ and $\epsilon > 0$. By Lemma 5.1a, there exist $N \in \mathbb{N}$ such that

$$\left| \frac{\sigma_r {\eta_k}^{1/3} I_{m_k,n_k,p_k(s)+l}}{\zeta_r F_{m_k,n_k,p_k(s)+l}(\zeta_r)} - \operatorname{Ai}(s + \eta_k^{-1/3}\sigma_r^{-1} l) \right| < \frac{\epsilon}{L} \tag{5.27}$$

$$\left| \frac{\sigma_r \eta_k^{1/3} I_{n_k,m_k,p_k(t)+l}}{\zeta_{1/r} F_{n_k,m_k,p_k(t)+l}(\zeta_{1/r})} - \operatorname{Ai}(t + \eta_k^{-1/3}\sigma_r^{-1} l) \right| < \frac{\epsilon}{L}, \tag{5.28}$$

for $s,t \in [-T,T]$, $0 \leqslant l < L\sigma_r\eta_k^{1/3}$ and $k \geqslant N$. Using Lemma 5.1b, boundedness of the Airy function on $[-T, L+T]$ and (5.21), we can combine (5.27) and (5.28) via triangle inequality to obtain

$$\begin{aligned}&\left|\sigma_r \eta_k^{1/3} \zeta_r^{p_k(t)-p_k(s)} I_{m_k,n_k,p_k(s)+l} I_{n_k,m_k,p_k(t)+l} - \sigma_r^{-1}\eta_k^{-1/3} \operatorname{Ai}(s+\eta_k^{-1/3}\sigma_r^{-1}l) \operatorname{Ai}(t+\eta_k^{-1/3}\sigma_r^{-1}l)\right| \\ &< \frac{C\epsilon}{L\eta_k^{1/3}}\end{aligned} \tag{5.29}$$

for some constant $C > 0$. By uniform continuity of the Airy function on $[-T, L+T]$, we also have

$$|\operatorname{Ai}(s+\eta_k^{-1/3}\sigma_r^{-1}l)\operatorname{Ai}(t+\eta_k^{-1/3}\sigma_r^{-1}l) - \operatorname{Ai}(s+x)\operatorname{Ai}(t+x)| < \frac{\epsilon}{L} \tag{5.30}$$

whenever $s,t \in [-T,T]$, $0 < l \leqslant L\sigma_r\eta_k^{1/3}$, $x \in [\eta_k^{-1/3}\sigma_r^{-1}(l-1), \eta_k^{-1/3}\sigma_r^{-1}l]$ and $k \geqslant N$, by choosing N larger if necessary. It follows from (5.29) and (5.30) that

$$\left|\sigma_r \eta_k^{1/3} \zeta_r^{p_k(t)-p_k(s)} I_{m_k,n_k,p_k(s)+l} I_{n_k,m_k,p_k(t)+l} - \int\limits_{\eta_k^{-1/3}\sigma_r^{-1}(l-1)}^{\eta_k^{-1/3}\sigma_r^{-1}l} \operatorname{Ai}(s+x)\operatorname{Ai}(t+x)dx\right| < \frac{C\epsilon}{L\eta_k^{1/3}}$$

for some constant $C > 0$. We now sum over $0 < l \leqslant L\sigma_r\eta_k^{1/3}$ and obtain

$$\left|\sigma_r \eta_k^{1/3} \zeta_r^{p_k(t)-p_k(s)} \sum_{0<l\leqslant L\sigma_r\eta_k^{1/3}} I_{m_k,n_k,p_k(s)+l} I_{n_k,m_k,p_k(t)+l} - \int\limits_0^L \operatorname{Ai}(s+x)\operatorname{Ai}(t+x)dx\right| < C\epsilon \tag{5.31}$$

for some constant $C > 0$. Moreover, choosing L large enough, we have

$$\left|\int_L^{+\infty} \operatorname{Ai}(s+x)\operatorname{Ai}(t+x)dx\right| < \epsilon \tag{5.32}$$

Finally, summing (5.22) over $l > L\sigma_r\eta_k^{1/3}$ gives

$$\left|\sigma_r \eta_k^{1/3} \zeta_r^{p_k(t)-p_k(s)} \sum_{l>L\sigma_r\eta_k^{1/3}} I_{m_k,n_k,p_k(s)+l} I_{n_k,m_k,p_k(t)+l}\right| \leqslant Ce^{-c(s+t+2L)} < \epsilon \tag{5.33}$$

for $s,t \in [-T,T]$, $k \geqslant N$ and some constants $C, c > 0$ provided that L is sufficiently large. Then, we conclude (5.26) from (5.31), (5.32) and (5.33). □

6 Proof of Theorem 1.1

In this section, we combine Theorem 4.1 with standard estimates on the Airy kernel to establish

$$\lim_{k\to+\infty} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \sum_{\substack{q_1,\dots,q_l\in\mathbb{N} \\ q_i\geqslant p_k(s)}} \det\left[K_{m_k,n_k}(q_i,q_j)\right]_{i,j\in[l]} = \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \int\limits_{[s,\infty)^l} \det[\mathrm{A}(x_i,x_j)]_{i,j\in[l]} dx_1 \dots dx_l \tag{6.1}$$

for any $s \in \mathbb{R}$. Note that (6.1) implies Theorem 1.1 on account of (2.5) and (4.2).

We first derive a uniform bound in k for the inner sum on the left hand-side of (6.1).

Lemma 6.1. *Let $T > 0$. There exist $N \in \mathbb{N}$ and constants $C, c > 0$ such that*

$$\sum_{\substack{q_1,\ldots,q_l \in \mathbb{N} \\ q_i \geqslant p_k(s_i)}} \det \left[K_{m_k,n_k}(q_i, q_j)\right]_{i,j\in[l]} \leqslant C^l l^{l/2} e^{-c\sum_{i=1}^l s_i}$$

for $l \in \mathbb{N}$, $k \geqslant N$ and $s_1, \ldots, s_l \geqslant -T$.

Proof. Noting that

$$q_i = p_k(s_i) + q_i - p_k(s_i) = p_k(s_i + (q_i - p_k(s_i))\eta_k^{-1/3}\sigma^{-1}),$$

we obtain from Theorem 4.1b that there exists $N \in \mathbb{N}$ and $C, c > 0$ such that

$$\det[K_{m_k,n_k}(q_i, q_j)]_{i,j\in[l]} \leqslant C^l l^{l/2} \eta_k^{-l/3} \exp\left(-c\sum_{i=1}^{l}\left(s_i + (q_i - p_k(s_i))\eta_k^{-1/3}\sigma^{-1}\right)\right)$$

whenever $k \geqslant N$, $s_1, \ldots, s_l \geqslant -T$ and $q_i \geqslant p_k(s_i)$ for $i \in [l]$. Summing over $q_i \geqslant p_k(s_i)$ for $i \in [l]$ yields

$$\sum_{\substack{q_1,\ldots,q_l \in \mathbb{N} \\ q_i \geqslant p_k(s_i)}} \det \left[K_{m_k,n_k}(q_i, q_j)\right]_{i,j\in[l]} \leqslant C^l l^{l/2} e^{-c\sum_{i=1}^l s_i} \frac{\eta_k^{-l/3}}{\left(1 - e^{-c\eta_k^{-1/3}\sigma^{-1}}\right)^l}$$

Since $\eta_k^{-1/3} \leqslant C'(1 - e^{-c\eta_k^{-1/3}\sigma^{-1}})$ for some constant $C' > 0$, the result follows. $\square$

A similar inequality holds for the Airy kernel.

Lemma 6.2. *Let $T > 0$. There exists a constant $C > 0$ such that*

$$\int_{\substack{x_1,\ldots,x_l \in \mathbb{R} \\ x_i \geqslant s_i}} \det \left[\mathrm{A}(x_i, x_j)\right]_{i,j\in[l]} dx_1 \ldots dx_l \leqslant C^l e^{-\sum_{i=1}^l s_i}$$

for $l \in \mathbb{N}$ and $s_1, \ldots, s_l \geqslant -T$.

Proof. It follows from (2.3) that A is symmetric and

$$\begin{aligned}
\sum_{i,j\in[l]} v_i \,\mathrm{A}(x_i, x_j) v_j &= \sum_{i,j\in[l]} \int_0^\infty v_i \,\mathrm{Ai}(x_i + t)\,\mathrm{Ai}(x_j + t) v_j dt \\
&= \int_0^\infty \sum_{i,j\in[l]} v_i \,\mathrm{Ai}(x_i + t)\,\mathrm{Ai}(x_j + t) v_j dt \\
&= \int_0^\infty \left(\sum_{i=1}^{l} \mathrm{Ai}(x_i + t) v_i\right)^2 dt \geqslant 0
\end{aligned}$$

for any $v_1, \ldots, v_l \in \mathbb{R}$. That is, $[\mathrm{A}(x_i, x_j)]_{i,j\in[l]}$ is a nonnegative-definite matrix. Therefore, by (2.4) and Hadamard's inequality, there exists a constant $C > 0$ such that

$$\det[\mathrm{A}(x_i, x_j)]_{i,j\in[l]} \leqslant \prod_{i=1}^{l} \mathrm{A}(x_i, x_i) \leqslant C^l e^{-\sum_{i=1}^l x_i}$$

whenever $x_i \geqslant -T$ for $i \in [l]$. Integrating over $x_i \geqslant s_i$ for $i \in [l]$, where $s_i \geqslant -T$, completes the proof. $\square$

Proof of Theorem 1.1. Introduce $\epsilon > 0$ and $S > s$. By Lemma 6.1,

$$\sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\substack{q_1,\ldots,q_l \in \mathbb{N} \\ q_i \geqslant p_k(s) \\ \max q_i \geqslant p_k(S)}} \det\left[K_{m_k,n_k}(q_i,q_j)\right]_{i,j\in[l]} \leqslant Ce^{-cS} \sum_{l=1}^{\infty} \frac{l^{l/2} e^{-cs(l-1)} C^{l-1}}{(l-1)!}$$

for $k \geqslant N$ for some constants $C, c > 0$ and $N \in \mathbb{N}$. The right-hand side is finite by the root test and can be made less than ϵ choosing S sufficiently large. For such S, we similarly obtain from Lemma 6.2 that

$$\sum_{l=1}^{\infty} \frac{1}{l!} \int\limits_{\substack{x_1,\ldots,x_l \geqslant s \\ \max x_i \geqslant S}} \det[\mathrm{A}(x_i,x_j)]_{i,j\in[l]} dx_1 \ldots dx_l < \epsilon.$$

Hence, it suffices to prove the following truncated version of (6.1).

$$\begin{aligned} &\lim_{k\to\infty} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \sum_{\substack{q_1,\ldots,q_l \in \mathbb{N} \\ p_k(s) \leqslant q_i < p_k(S)}} \det\left[K_{m_k,n_k}(q_i,q_j)\right]_{i,j\in[l]} \\ &= \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \int\limits_{[s,S)^l} \det[\mathrm{A}(x_i,x_j)]_{i,j\in[l]} dx_1 \ldots dx_l. \end{aligned} \tag{6.2}$$

By Lemma 6.1 and finiteness of $\sum_{l=1}^{\infty} l^{l/2} C^l e^{-csl}/l!$, we can conclude (6.2) from dominated convergence if we show that

$$\lim_{k\to\infty} \sum_{\substack{q_1,\ldots,q_l \in \mathbb{N} \\ p_k(s) \leqslant q_i < p_k(S)}} \det\left[K_{m_k,n_k}(q_i,q_j)\right]_{i,j\in[l]} = \int\limits_{[s,S]^l} \det\left[\mathrm{A}(x_i,x_j)\right]_{i,j\in[l]} dx_1 \ldots dx_l. \tag{6.3}$$

for each $l \in \mathbb{N}$.

Fix $l \in \mathbb{N}$. For $k \in \mathbb{N}$, consider the partition of the interval $[s,\infty)$ into intervals of length $\sigma^{-1}\eta_k^{-1/3}$ with endpoints at

$$t(q,k) = s + (q - p_k(s))\sigma^{-1}\eta_k^{-1/3}$$

for $q \geqslant p_k(s)$. Observe that $p_k(t(q,k)) = q$. Also, for $p_k(s) \leqslant q_1,\ldots,q_l < p_k(S)$, we have $s \leqslant t(q_i,k) < s + (p_k(S) - p_k(s))\sigma^{-1}\eta_k^{-1/3} \leqslant S+1$ for each $i \in [l]$. Therefore, by Theorem 4.1a, there exists $N \in \mathbb{N}$ such that

$$\left| \det\left[K_{m_k,n_k}(q_i,q_j)\right]_{i,j\in[l]} - \sigma^{-l}\eta_k^{-l/3} \det\left[\mathrm{A}(t(q_i,k),t(q_j,k))\right]_{i,j\in[l]} \right| < \epsilon\sigma^{-l}\eta_k^{-l/3} \tag{6.4}$$

whenever $k \geqslant N$ and $p_k(s) \leqslant q_i < p_k(S)$ for $i \in [l]$. By uniform continuity of $\det[\mathrm{A}(x_i,x_j)]_{i,j\in[l]}$ on $[s,S+1]^l$, choosing N larger if necessary, we obtain

$$\left| \det\left[K_{m_k,n_k}(q_i,q_j)\right]_{i,j\in[l]} - \int_{R_{q_1,\cdots,q_l,k}} \det[\mathrm{A}(x_i,x_j)]_{i,j\in[l]}\ dx_1 \cdots dx_l \right| < 2\epsilon\sigma^{-l}\eta_k^{-l/3}, \tag{6.5}$$

where $R_{q_1,\ldots,q_l,k}$ denotes the product of the intervals $[t(q_i,k), t(q_i+1,k)]$ for $i \in [l]$. The pairwise intersections of $\{R_{q_1,\ldots,q_l,k} : p_k(s) \leqslant q_i < p_k(S)\}$ are Lebesgue null-sets and their union is

$$[t(p_k(s),k), t(p_k(S),k)]^l = [s, t(p_k(S),k)]^l. \tag{6.6}$$

Hence, by the triangle inequality and (6.5),

$$\left| \sum_{\substack{q_1,\ldots,q_l \in \mathbb{N} \\ p_k(s) \leqslant q_i < p_k(S)}} \det\left[K_{m_k,n_k}(q_i,q_j)\right]_{i,j\in[k]} - \int_{[s,t(p_k(S),k)]^l} \det[\mathrm{A}(x_i,x_j)]_{i,j\in[l]} dx_1 \cdots dx_l \right| \tag{6.7}$$

$$< 2\epsilon\sigma^{-l}\eta_k^{-l/3}(p_k(S) - p_k(s))^l \leqslant 2\epsilon(S - s + 1)^l, \tag{6.8}$$

The set in (6.6) differs from $[s, S]^l$ by a set of measure

$$|t(p_k(S), k)^l - S^l| \leqslant \sigma^{-1}\eta_k^{-1/3} l(S+1)^{l-1}, \tag{6.9}$$

where the inequality follows from $|t(p_k(S), k) - S| \leqslant \sigma^{-1}\eta_k^{-1/3}$ and the mean value theorem. Because (6.9) can be made arbitrarily small and the Airy kernel is bounded on $[s, S+1]$, we have

$$\left| \int_{[s,t(p_k(S),k)]^l} \det[\mathrm{A}(x_i, x_j)]_{i,j\in[l]} dx_1 \cdots dx_l - \int_{[s,S]^l} \det[\mathrm{A}(x_i, x_j)]_{i,j\in[l]} dx_1 \cdots dx_l \right| < \epsilon, \tag{6.10}$$

for $k \geqslant N$ by choosing N large enough. Now (6.3) follows from combining (6.7) and (6.10). $\square$

A Steepest-descent curves of harmonic functions

In this section, we provide a more rigorous justification for Figure 4.1 than indicated in the main text. Our argument relies on some well-known facts from the theory of ODEs, which we briefly recall here. Let $d \in \mathbb{N}$, U be an open subset of $\mathbb{R}^d$, $x_0 \in U$ and $F : U \to \mathbb{R}^d$ be a continuously differentiable function. A solution of the initial value problem

$$x'(t) = F(x(t)) \quad x(t_0) = x_0 \tag{A.1}$$

is a (necessarily twice continuously) differentiable function $\varphi : I \to U$, where I is an open interval containing t_0, such that $\varphi'(t) = F(\varphi(t))$ for all $t \in I$ and $\varphi(t_0) = x_0$. There exists a unique solution $\Phi : J \to U$ of (A.1) that is maximal in the sense that any other solution $\varphi : I \to U$ of (A.1) is a restriction of Φ to I. Let α and β denote the left and the right endpoints of J, respectively, and K be a compact subset of U. Then, either $\alpha = -\infty$ or $\Phi(t) \notin K$ for some $t \in (\alpha, t_0]$. Similarly, either $\beta = +\infty$ or $\Phi(t) \notin K$ for some $t \in [t_0, \beta)$. The proof of these assertions can be found in standard texts on differential equations; see, for example, [1, Chapter 2], [15, Section 7.2].

Let $u : U \to \mathbb{R}$ be a twice continuously differentiable function and $\varphi : I \to U$ be a continuously differentiable curve parametrized by arclength, that is, $|\varphi'(t)| = 1$ for all $t \in I$. We call φ, respectively, a stationary, steepest-descent and steepest-ascent curve of u if

$$\frac{d}{dt}(u(\varphi(t))) = 0, -|\nabla u(\varphi(t))| \text{ and } |\nabla u(\varphi(t))|, \tag{A.2}$$

respectively, for all $t \in I$. Since the directional derivative $\nabla u \cdot \eta$ along any direction η is bounded by $|\nabla u|$, along a steepest-descent curve, u decreases most rapidly and along a steepest-ascent curve, u increases most rapidly. Note also that φ is a steepest-descent curve of u if and only if the reversed curve $t \mapsto \varphi(-t)$ is a steepest-ascent curve of u.

We specialize the setting to the case of $d = 2$ from now on and identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual manner $(x, y) \leftrightarrow x + \mathbf{i}y$. The dot product of $z, w \in \mathbb{C}$ is defined as $z \cdot w = \Re(z\overline{w})$. Let V be a nonempty open subset of U such that the gradient $\nabla u \neq 0$ on V. Then the stationary, steepest-descent and steepest-ascent curves of u that lie in V satisfy the ODEs

$$z'(t) = \mathbf{i}\frac{\nabla u(z(t))}{|\nabla u(z(t))|}, \quad z'(t) = -\frac{\nabla u(z(t))}{|\nabla u(z(t))|} \quad \text{and} \quad z'(t) = \frac{\nabla u(z(t))}{|\nabla u(z(t))|},$$

respectively. Since $\dfrac{\nabla u}{|\nabla u|}$ is continuously differentiable on V, if we choose an initial value $z(t_0) = z_0 \in V$ in any of these ODEs, the resulting initial value problem admits a unique maximal solution $\Phi : J \to V$.

We now focus attention to the steepest-descent curves of harmonic functions; these functions are locally the real and imaginary parts of holomorphic functions. To fix some notation, let f denote a nonconstant, holomorphic function on U, and put $u = \Re f$ and $v = \Im f$. The Cauchy-Riemann equations assert that $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, which imply

$$\nabla u(z) = (\partial_x u(z), -\partial_x v(z)) = \overline{f'(z)} \qquad \nabla v(z) = (\partial_x v(z), \partial_x u(z)) = \mathbf{i}\overline{f'(z)}. \tag{A.3}$$

Lemma A.1. *φ is a steepest-descent or -ascent curve of u if and only if φ is a stationary curve of v.*

Proof. Using the chain rule and (A.3), we compute

$$\begin{aligned}\left(\frac{d}{dt}u(\varphi(t))\right)^2 + \left(\frac{d}{dt}v(\varphi(t))\right)^2 &= \big(\nabla u(\varphi(t)) \cdot \varphi'(t)\big)^2 + \big(\nabla v(\varphi(t)) \cdot \varphi'(t)\big)^2 \\ &= \big(\Re\left(f'(\varphi(t))\varphi'(t)\right)\big)^2 + \big(\Im\left(f'(\varphi(t))\varphi'(t)\right)\big)^2 \\ &= |f'(\varphi(t))|^2 = |\nabla u(\varphi(t))|^2.\end{aligned}$$

Then the conclusion readily follows from definition (A.2). $\square$

Let $z_0 \in U$ and n denote the smallest positive integer such that the derivative $f^{(n)}(z_0) \neq 0$. We now show that, in a neighborhood of z_0, the level set $v = v(z_0)$ consists of n distinct steepest descent curves of u passing through z_0. Hence, by Lemma A.1, there are exactly n of these curves.

Proposition A.2. *Let ξ denote the direction of an nth root of $f^{(n)}(z_0)$. For each $k \in [n]$, there exist an open interval I_k containing 0, an open set $U_k \subset \mathbb{C}$ containing z_0 and a continuously differentiable injective curve $\varphi_k : I_k \to U_k$ such that*

(a) $\varphi_k(0) = z_0$.

(b) $\Im f(z) = \Im f(z_0)$ *and* $z \in \bigcup_{k\in[n]} U_k$ *if and only if* $z = \varphi_k(t)$ *for some* $k \in [n]$ *and* $t \in I_k$.

(c) $\varphi_k'(t) = (-1)^k \operatorname{sgn}(t)^{n-1} \dfrac{\overline{f'(\varphi_k(t))}}{|f'(\varphi_k(t))|} \neq 0$ *for* $t \in I_k \setminus \{0\}$ *and* $\varphi_k'(0) = \overline{\xi}\exp(\mathbf{i}\pi k/n)$.

(d) For $k, l \in [n]$ *and* $k \neq l$*, the images of* φ_k *and* φ_l *do not intersect except at* z_0.

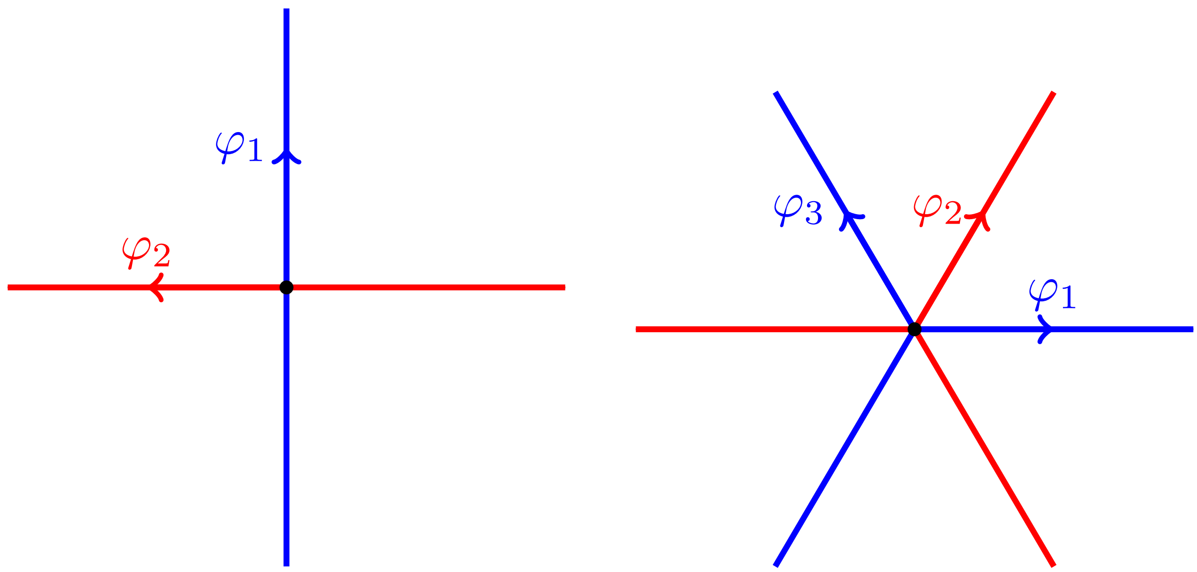
Hence, the steepest-descent curves of u from z_0 are given by the following n parametrizations. For odd k, $t \mapsto \varphi_k(t)$ for $t \geqslant 0, t \in I_k$ and for odd $k+n$, $t \mapsto \varphi_k(-t)$ for $t \geqslant 0, -t \in I_k$. Likewise, the steepest-ascent curves of u from z_0 are given by $t \mapsto \varphi_k(t), t \geqslant 0, t \in I_k$ for even k and $t \mapsto \varphi_k(-t), t \geqslant 0, -t \in I_k$ for even $k+n$. For an illustration of the preceding lemma, see Figure A.1 below.

Proof of Proposition A.2. By the assumption on f, there exist $\epsilon > 0$ and a holomorphic function h on $D(z_0, \epsilon)$ with nonzero derivative such that $f(z) = f(z_0) + h(z)^n$ [23, Theorem 10.32]. Because $f^{(n)}(z_0) = n!h'(z_0)^n$ and h is determined only up to multiplication by an nth root of unity, we may assume that $h'(z_0)$ has direction ξ. Since h is nonconstant and holomorphic, by choosing $\epsilon > 0$ small enough, we also have $h(z) \neq 0$ unless $z = z_0$.

The level set $\Im f(z) = \Im f(z_0)$ in $D(z_0, \epsilon)$ is the same as the level set $\Im(h(z)^n) = 0$. The last equation holds if and only if $h(z) = 0$ or $h(z)$ has direction $\omega_k = \exp(\mathbf{i}\pi k/n)$ for some $k \in [2n]$, which is equivalent to

$$\Im(h(z)\overline{\omega_k}) = 0 \tag{A.4}$$

for some $k \in [n]$. The left-hand side of (A.4) is a continuously differentiable function of two real variables with gradient $\mathbf{i}\omega_k\overline{h'(z)}$, which is nonzero at z_0. Hence, for each $k \in [n]$, it follows from the implicit function theorem that there exist an open interval I_k containing 0, an open set $U_k \subset D(z_0, \epsilon)$ containing z_0 and a continuously differentiable injective curve $\varphi_k : I_k \to U_k$ such that $\varphi_k(0) = z_0$, $|\varphi_k'| = 1$ and $z \in U_k$ satisfies (A.4) if and only if $z = \varphi_k(t)$ for some $t \in I_k$.



(a) For $\xi = 1$, we have $\varphi_1(t) = \mathbf{i}t$ and $\varphi_2(t) = -t$. The steepest-descent curves are given by $\varphi_1(t)$ and $\varphi_1(-t)$ for $t \geqslant 0$. The steepest-ascent curves are given by $\varphi_2(t)$ and $\varphi_2(-t)$ for $t \geqslant 0$.

(b) For $\xi = e^{\mathbf{i}\pi/3}$, we have $\varphi_1(t) = t$, $\varphi_2(t) = e^{\mathbf{i}\pi/3}t$ and $\varphi_3(t) = e^{\mathbf{i}2\pi/3}t$. The steepest-descent curves are given by $\varphi_1(t), \varphi_2(-t)$ and $\varphi_3(t)$ for $t \geqslant 0$. The steepest-ascent curves are given by $\varphi_1(-t), \varphi_2(t)$ and $\varphi_3(-t)$ for $t \geqslant 0$.

Figure A.1: The steepest-descent (blue) and -ascent (red) curves at $z_0 = 0$ for the functions z^2 and $-z^3$, respectively.

Setting $z = \varphi_k(t)$ in (A.4) and differentiating with respect to t via the chain rule, we obtain that $\varphi_k'(t)$ is orthogonal to $\mathbf{i}\omega_k\overline{h'(\varphi_k(t))}$. Since h' is nonzero and continuous, replacing φ_k with $t \mapsto \varphi_k(-t)$ if necessary, we can make $\varphi_k'(t)$ have the same direction as $\omega_k\overline{h'(\varphi_k(t))}$ for $t \in I_k$. In particular, $\varphi_k'(0) = \omega_k\overline{\xi}$.

Since $\overline{\omega_k}h(\varphi_k(t))$ is real-valued, we have

$$\frac{d}{dt}(\overline{\omega_k}h(\varphi_k(t))) = \frac{d}{dt}\Re(\overline{\omega_k}h(\varphi_k(t))) = \overline{\omega_k}h'(\varphi_k(t))\varphi_k'(t) = |h'(\varphi_k(t))| > 0. \tag{A.5}$$

from the chain rule. Also, we can write

$$f'(\varphi_k(t)) = \left[(-1)^k n\overline{\omega_k}^{n-1}h(\varphi_k(t))^{n-1}\right]\overline{\omega_k}h'(\varphi_k(t)),$$

where the factor inside the brackets is real and has sign $(-1)^k \operatorname{sgn}(t)^{n-1}$ by (A.5). It follows that

$$\varphi_k'(t) = \omega_k\frac{\overline{h'(\varphi_k(t))}}{|h'(\varphi_k(t))|} = (-1)^k \operatorname{sgn}(t)^{n-1}\frac{\overline{f'(\varphi_k(t))}}{|f'(\varphi_k(t))|}.$$

for $t \in I_k \smallsetminus \{0\}$.

We have proved (a), (b) and (c). For (d), suppose that the images of φ_k and φ_l intersect at $z \in U$ for some $k \neq l$. Then, by (A.4), $h(z)$ is orthogonal to both $\mathbf{i}\omega_k$ and $\mathbf{i}\omega_l$. This implies $h(z) = 0$ and, hence, $z = z_0$. $\square$

Let f, u, v and ζ be as defined by (4.5), (4.7), (4.8) and (4.11), respectively. We recall that u is harmonic on $U = \mathbb{C} \smallsetminus \{0, \sqrt{q}, 1/\sqrt{q}\}$ and has nonzero gradient on $U \smallsetminus \{\zeta\}$. Also, $f = u + \mathbf{i}v$ is holomorphic on V, which is $\mathbb{C}$ minus the intervals $(-\infty, \sqrt{q}]$ and $[1/\sqrt{q}, +\infty)$. We apply Proposition A.2 with f in a small disk centered at ζ. Then $n = 3$ and, since $f'''(\zeta) = -2\sigma^3/\zeta^3$, we can take $\xi = e^{\mathbf{i}\pi/3}$. Hence, the level set $v = v(\zeta) = 0$ in a small neighborhood of ζ consists of the images of the curves $\varphi_k : I_k \to D(\zeta, \epsilon)$ as described in the lemma. In particular, we have $\varphi_1'(0) = 1$, $\varphi_2'(0) = e^{\mathbf{i}\pi/3}$ and $\varphi_3'(0) = e^{\mathbf{i}2\pi/3}$.

We now study the global behavior of the steepest-descent and ascent-curves of u from ζ. For each $k \in \{1, 2, 3\}$, put $I_k = (t_k^-, t_k^+)$, and let Φ_k^+ and Φ_k^- denote the maximal steepest descent and ascent curves in $U \smallsetminus \{\zeta\}$ that extend φ_k restricted to $(0, t_k^+)$ and $(t_k^-, 0)$, respectively. Since $\varphi_k(0) = \zeta$, the interval of existence for Φ_k^+ and Φ_k^- are of the form $(0, T_k^+)$ and $(T_k^-, 0)$, respectively, for some $T_k^+ \in [t_k^+, +\infty)$ and $T_k^- \in (-\infty, t_k^-]$.

Lemma A.3. *$T_1^+ = 1/\sqrt{q} - \zeta$, $T_1^- = \sqrt{q} - \zeta$ and $\Phi_1^\pm(t) = \zeta + t$.*

Proof. Because f is real-valued on $V \cap \mathbb{R}$, $\Phi(t) = \zeta + t$ defined for $0 < t < 1/\sqrt{q} - \zeta$ is a steepest descent curve of u. Moreover, Φ is maximal since $\Phi(t)$ approaches the boundary of $U \smallsetminus \{\zeta\}$ as $t \downarrow 0$ and $t \uparrow 1/\sqrt{q} - \zeta$. Since $\Phi' = 1$ but $\Im\varphi_2'(0) > 0$ and $\Im\varphi_3'(0) > 0$, for $t > 0$ sufficiently small, $\Phi(t)$ is not contained in the images $\varphi_2((t_2^-, 0))$ and $\varphi_3((0, t_3^+))$. Hence, Φ intersects the image of φ_1 restricted to $(0, t_1^+)$, the only remaining steepest descent curve. This implies, by uniqueness of the maximal solution, that $\Phi = \Phi_1^+$. Assertions about T_1^- and Φ_1^- are proved similarly. $\square$

Lemma A.4.

(a) $T_2^+ = +\infty$, the image $\Phi_2^+((0, +\infty))$ is in $\mathbb{H}$ and $\lim_{t\to+\infty} |\Phi_2^+(t)| = +\infty$.

(b) $T = T_3^+ < +\infty$, the image $\Phi_3^+((0, T))$ is in $\mathbb{H}$ and $\lim_{t\to T} \Phi_3^+(t) = 0$.

(c) $T_2^- = -T$ and $\Phi_2^-(t) = \overline{\Phi_3^+(-t)}$ for all $t \in (-T, 0)$.

(d) $T_3^- = -\infty$ and $\Phi_3^-(t) = \overline{\Phi_2^+(-t)}$ for all $t \in (-\infty, 0)$.

Proof. To simplify notation, let us write Φ for Φ_3^+. We first show that $\Phi(t) \in \mathbb{H}$ for $t \in (0, T)$. Since $\Im\varphi_3'(0) > 0$, there exists $t_0 \in (0, t_3^+)$ such that $\Phi(t) = \varphi_3(t) \in \mathbb{H}$ for $t \in (0, t_0]$. As u is continuous at ζ and is decreasing along Φ, we can choose $\epsilon > 0$ such that $u(\Phi(t)) < u(z)$ whenever $t \geqslant t_0$ and $z \in D(\zeta, \epsilon)$. To get a contradiction, suppose that $\Phi(t_1) = x_1 \in \mathbb{R}$ for some $t_1 \in (t_0, T)$. Then $x_1 \neq \zeta$ and $\nabla u = \overline{f'}$ has direction $+1$ or -1 in an interval around x_1. It follows that Φ, the unique maximal steepest descent curve passing through x_1, satisfies $\Phi(t) = x_1 - \operatorname{sgn}(f'(x_1))(t - t_1)$ for $t \in (0, T)$, which contradicts $\Phi(t_0) \in \mathbb{H}$. Hence, we conclude that $\Phi((0, T)) \subset \mathbb{H}$.

We note from (4.7) that $u(z)/\log|z| \to 1 + \gamma$ as $|z| \to +\infty$, which implies that $\Phi((0, T)) \subset D(0, R)$ for some $R > 0$. Since $\nabla u = \overline{f'}$ has a zero only at ζ and has no singularities other than the poles at $\{0, \sqrt{q}, 1/\sqrt{q}\}$, there exists $c > 0$ such that $|\nabla u(z)| \geqslant c$ for $z \in D(0, R) \smallsetminus D(\zeta, \epsilon)$ and, hence,

$$\frac{d}{dt}\left(u(\Phi(t))\right) = -|\nabla u(\Phi(t))| \leqslant -c \quad \text{for } t \in [t_0, T). \tag{A.6}$$

We next show that $T < \infty$ and $\Phi(t)$ converges to either 0 or $1/\sqrt{q}$ as $t \to T$. Let $\delta > 0$ and K denote the set obtained by removing disks $D(x, \delta)$ for $x \in \{0, \sqrt{q}, \zeta, 1/\sqrt{q}\}$ from the closed disk $\overline{D(0, R)}$. Since K is compact, if $\Phi((t_0, T)) \subset K$ then $T = +\infty$. However, then (A.6) forces $u(\Phi(t)) \to -\infty$, which is not possible as u is bounded on K. Hence, there exists $T_1 \in [t_0, T)$ such that $\Phi(T_1) \notin K$. If $\delta > 0$ is sufficiently small, $\Phi(T_1) \notin D(\zeta, \delta)$ as $\Phi([t_0, T))$ does not intersect $D(\zeta, \epsilon)$, and $\Phi(T_1) \notin D(\sqrt{q}, \delta)$ since $u(z) \to +\infty$ as $z \to \sqrt{q}$. Also because $\Phi((0, T)) \subset D(0, R)$, we conclude that $\Phi(T_1) \in D(0, \delta)$ or $\Phi(T_1) \in D(1/\sqrt{q}, \delta)$. Let us first consider the case $\Phi(T_1) \in D(0, \delta)$. It follows from (4.9) that $zf'(z) = r + \gamma + O(\delta)$ for all $z \in D(0, \delta)$. This implies existence of $\eta > 0$ such that $\Re(zf'(z)) \geqslant \eta|zf'(z)|$ for all $z \in D(0, \delta)$. Hence, if $\Phi(t) \in D(0, \delta)$, we have

$$\frac{d}{dt}(|\Phi(t)|) = -\frac{\Phi(t)}{|\Phi(t)|} \cdot \frac{\nabla u(\Phi(t))}{|\nabla u(\Phi(t))|} = -\frac{\Re\left(\Phi(t) f'(\Phi(t))\right)}{|\Phi(t) f'(\Phi(t))|} \leqslant -\eta. \tag{A.7}$$

Then, by the (converse of the) mean value theorem, there is no $t \in (T_1, T)$ such that $\Phi(t) \in D(0,\delta)$ and $|\Phi(t)| > |\Phi(T_1)|$. Hence, $\Phi(t) \in D(0,\delta)$ for $t \in (T_1, T)$ and, by (A.7), $T < \infty$ and $\Phi(t) \to 0$ as $t \to T$. A similar argument shows that, in the case $\Phi(T_1) \in D(1/\sqrt{q}, \delta)$, we have $T < \infty$ and $\lim_{t\to T} \Phi(t) = 1/\sqrt{q}$.

We now argue that, in fact, $\Phi(t)$ cannot converge to $1/\sqrt{q}$ as $t \to T$. Let us write Ψ for the steepest ascent curve Φ_2^+ and T' for T_2^+. As in the argument for Φ, one can observe that Ψ does not intersect the real line, and there exist $t_0' \in (0, T')$ and $\epsilon' > 0$ such that $\Psi(t) \notin D(\zeta, \epsilon')$ for $t \in [t_0', T')$. For a contradiction, suppose that $\lim_{t\to T} \Phi(t) = 1/\sqrt{q}$ and let O denote the interior of the Jordan curve that consists of the interval $[\zeta, 1/\sqrt{q}]$ and the image of Φ. Since $\Re\phi_2'(0) > \Re\phi_3'(0)$, the image of Ψ contains points in O. Then, because Ψ does not intersect the boundary of O, the image of Ψ lies in O. Since O is bounded, it follows that $T' = +\infty$. Using that $\Psi([t_0', +\infty)) \subset O \smallsetminus D(\zeta, \epsilon')$, we note the inequality

$$\frac{d}{dt}\left(u(\Psi(t))\right) = |\nabla u(\Psi(t))| \geqslant c' \quad \text{ for } t \in [t_0', +\infty) \tag{A.8}$$

for some constant $c' > 0$ as in (A.6). This implies that $u(\Psi(t)) \to +\infty$ as $t \to +\infty$, which is not possible as u is bounded from above on O. This completes the proof of (b).

To prove (a), it remains to show that $T' = +\infty$ and $|\Psi(t)| \to +\infty$ as $t \to +\infty$. Since Ψ is parametrized with arclength,

$$t - t_0' = \int_{t_0'}^{t} |\Psi'(s)| ds \geqslant \left| \int_{t_0'}^{t} \Psi'(s) ds \right| = |\Psi(t) - \Psi(t_0')| \tag{A.9}$$

for $t \in [t_0', T')$. Let O' denote the interior of the Jordan curve that consists of the interval $[0, \zeta]$ and the image of Φ. Arguing by contradiction, suppose that $\Psi((0, T')) \subset D(0, R')$ for some $R' > 0$. Since Ψ does not intersect the boundary of O', we conclude that $\Psi([t_0', T')) \subset (D(0,R') \smallsetminus O') \smallsetminus D(\zeta, \epsilon')$. Then, $T' = +\infty$ and an inequality of the form (A.8) is in place, which leads to $u(\Psi(t)) \to +\infty$, which is not possible. Hence, $|\Psi(t)| \to +\infty$ and, by (A.9), $T' = +\infty$.

For (c), we observe that the curve $t \mapsto \overline{\Phi(-t)}$ defined on $(-T, 0)$ is a steepest-ascent curve of u because u is symmetric with respect to the real axis. We also have $\left(\overline{\Phi(-t)}\right)' \to e^{\mathbf{i}4\pi/3}$. Hence, by Proposition A.2, $\overline{\Phi(-t)} = \varphi_2(-t)$ for $t > 0$ sufficiently small and (c) follows. The proof of (d) is similar. $\square$

Recall from Section 2 that the contour Γ consists of the curves Φ_3^+ and Φ_2^-. It follows from Lemma A.4 that Γ encloses $\sqrt{q}$ and has finite length.

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