

Greedy Algorithms

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1 Introduction

A *greedy algorithm* is an algorithm that chooses the optimal choice in the short run.

2 Examples

1. Prove that every nonnegative integer can be written uniquely as the sum of one or more distinct powers of 2.

Solution: It's true for $N = 0$. Suppose it's true for all $n < N$. Take k such that $2^k \leq N < 2^{k+1}$, and include 2^k in its representation. We know $N - 2^k$ has a representation. Furthermore, the largest term in its representation must be less than 2^k since $N - 2^k < 2^{k+1} - 2^k = 2^k$. Finally, to prove uniqueness, we show that at each step we are forced to take this value of 2^k . Indeed, if we didn't, the largest possible representation we could have is $2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1 < N$, contradiction. \square

2. (Zeckendorf's theorem) Prove that every positive integer can be written uniquely as the sum of one or more Fibonacci numbers, no two of which are consecutive.

Solution: It's true for $N = 0$. Suppose it's true for all $n < N$. Take k such that $F_k \leq N < F_{k+1}$, and include F_k in its representation. If $N = F_k$ then we're done. Otherwise, we know $N - F_k$ has a representation. Furthermore, the largest term in its representation must be less than F_{k-1} since $N - F_k < F_{k+1} - F_k = F_{k-1}$. I'll leave proving uniqueness as an exercise. Start with the assumption that two distinct sets of Fibonacci numbers sum to the same number. \square

3. Let Δ be the maximum degree in a graph. Prove that the chromatic number of that graph is at most $\Delta + 1$.

Solution: Iteratively color vertices with any color that has not been used by any of its neighbors. Such a color will always exist since there are at most Δ neighbors and $\Delta + 1$ colors. \square

Extra: (Brooks' theorem) Assume the graph is connected and simple. Prove that equality holds if and only if it is either a complete graph or an odd cycle.

4. Let A_1, \dots, A_n be subsets of $\{1, 2, \dots, n\}$ of size 3. Prove that $\lfloor \frac{n}{3} \rfloor$ members of $\{1, 2, \dots, n\}$ can be colored such that each A_i has at least one member that is not colored.

Solution: Let's start with all of $\{1, 2, \dots, n\}$ colored and iteratively un-color the one in the most completely colored sets. Formally, let A be the set of completely colored sets at a given moment. Iteratively find the element that reduces A by the largest amount, and un-color it.

Suppose that after x applications of this algorithm, all elements of A become disjoint. Then the algorithm would terminate after $x + |A|$ applications in total. However, note that $x \leq \frac{n}{2}$ because each application removes at least 2 sets, and $|A| \leq \frac{n-x}{3}$ because there are $n - x$ numbers remaining to be partitioned into disjoint sets of size 3. Therefore,

$$x + |A| \leq x + \frac{n-x}{3} = \frac{n}{3} + \frac{2x}{3} \leq \frac{n}{3} + \frac{n}{3} = \frac{2n}{3}$$

which implies the algorithm terminates in at most $\lceil \frac{2n}{3} \rceil$ applications. Therefore, this algorithm colors at most $\lfloor \frac{n}{3} \rfloor$. \square

5. (IMO 1983) Is it possible to choose 1983 distinct positive integers, all less than or equal to 100,000, no three of which are consecutive terms of an arithmetic progression?

Solution: We greedily choose numbers and find a pattern. After small cases we see $\{1, 2, 4, 5, 10, 11, 13, 14, 28, 29\}$. The sequence jumps at 4, 10, and 28, all of which are 1 more than a power of 3, so perhaps we should look at $\{0, 1, 3, 4, 9, 10, 12, 13, 27, 28\}$ in base 3. Indeed, these all consist of just 0s and 1s in ternary.

Consider the set S of all numbers $\leq 100,000$ whose ternary representations consist of just 0s and 1s. If any three $x, y, z \in S$ satisfy $y - x = z - y$, or $x + z = 2y$, then each corresponding digit of x and z must match up, so $x = z$. Thus S has no arithmetic progressions. Since $1 + 1111111111_3 < 100,000$, we have $|S| \geq 2^{11} > 1983$. Finally shift each term up by 1 to get a construction of strictly positive integers. \square

6. (Russia 2005) In a $2 \times n$ array, we have positive reals such that the sum of the numbers in each of the n columns is 1. Show that we can select one number in each column such that the sum of the selected numbers in each row is at most $\frac{n+1}{4}$.

Solution: Re-order the columns so that the top row is in increasing order:

a_1	a_2	a_3	\dots	a_n
$1 - a_1$	$1 - a_2$	$1 - a_3$	\dots	$1 - a_n$

Choose the value of k such that $a_1 + \dots + a_k \leq \frac{n+1}{4} < a_1 + \dots + a_{k+1}$. We claim that

$$\frac{n+1}{4} \geq (1 - a_{k+1}) + \dots + (1 - a_n) = (n - k) - (a_{k+1} + \dots + a_n)$$

or

$$a_{k+1} + \dots + a_n \geq \frac{3n-1}{4} - k$$

Comparing averages, we have

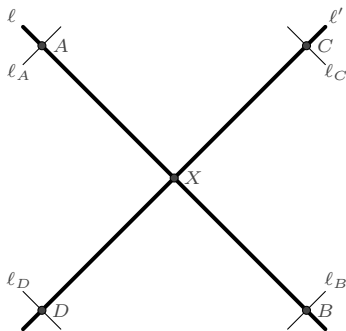
$$\frac{a_{k+1} + \dots + a_n}{n - k} \geq \frac{a_1 + \dots + a_{k+1}}{k + 1} > \frac{n + 1}{4(k + 1)}$$

Finally, it's easy to show that $\frac{n-k}{k+1} \frac{n+1}{4} \geq \frac{3n-1}{4} - k$ by AM-GM. \square

7. (IMO 2014) A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n , in any set of n lines in general position it is possible to colour at least \sqrt{n} lines blue in such a way that none of its finite regions has a completely blue boundary.

Solution: We start with all of the lines uncolored and color them blue iteratively, while keeping track of bad lines by marking certain intersection points.

At each step, color any line that does not pass through any marked intersection points. For each other blue line it intersects, consider this following diagram:



Above, A, B, C, D are chosen to be the closest to X possible. If A and B are unmarked, mark them. Else if C and D are unmarked, mark them. Else, at least two of $\ell_A, \ell_B, \ell_C, \ell_D$ are blue, so mark any of them that are not blue-blue (there are at most 2 points to mark). It's easy to see this process guarantees that no blue region is created. At the end of the process, suppose we have colored k lines. Then the number of marked vertices, at least $2\binom{k}{2}$, is at least the number of uncolored lines, $n - k$, so

$$2\binom{k}{2} = k(k-1) \geq n - k \implies k \geq \sqrt{n}$$

□

3 Problems

- (Germany 2000) There are stones with a total mass of 9 tons that should be transported by trucks. None of the stones is heavier than 1 ton and each vehicle has a capacity of 3 tons. Determine the minimum number of necessary trucks such that the stones can certainly be transported at the same time.
- Prove that for each positive integer n , there exists a unique set of integers a_1, a_2, \dots such that for $a_i \leq i$ for all i and

$$n = a_1 \cdot 1! + a_2 \cdot 2! + a_3 \cdot 3! + \dots$$

- (Netherlands 2014) Let n be a positive integer. Daniel and Merlijn are playing a game. Daniel has k sheets of paper lying next to each other on a table, where k is a positive integer. On each of the sheets, he writes some of the numbers from 1 up to n (he is allowed to write no number at all, or all numbers). On the back of each of the sheets, he writes down the remaining numbers. Once Daniel is finished, Merlijn can flip some of the sheets of paper (he is allowed to flip no sheet at all, or all sheets). If Merlijn succeeds in making all of the numbers from 1 up to n visible at least once, then he wins. Determine the smallest k for which Merlijn can always win, regardless of Daniel's actions.
- (VTRMC 2014) Consider the harmonic series $\sum_{n \geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$. Prove that every positive rational number can be obtained as an *unordered* partial sum of this series. (An unordered partial sum may skip some of the terms $\frac{1}{k}$.)
- (Gabriel Dospinescu) Which rational numbers can be written as the sum of the inverses of finitely many pairwise distinct triangular numbers?
- (IMC 1997) Let $1 < \alpha \leq 2$ be a real number.
 - Show that α has a unique representation as an infinite product

$$\alpha = \left(1 + \frac{1}{n_1}\right) \left(1 + \frac{1}{n_2}\right) \dots$$

where each n_i is a positive integer satisfying $n_{i+1} \geq n_i^2$.

- Show that α is rational if and only if its infinite product has the following property: For some m and all $k \geq m$, $n_{k+1} = n_k^2$.
- (IMOSL 2001, generalized) A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z - y, y - x\} = \{a, b\}$ for $0 < a < b$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.
 - (Math Prize for Girls 2010) Let S be a set of n points in the coordinate plane. Say that a pair of points is *aligned* if the two points have the same x -coordinate or y -coordinate. Prove that S can be partitioned into disjoint subsets such that (a) each of these subsets is a collinear set of points, and (b) at most $n^{3/2}$ unordered pairs of distinct points in S are aligned but not in the same subset.

9. (IMOSL 2013) Let n be a positive integer. Find the smallest integer k with the following property; Given any real numbers a_1, \dots, a_d such that $a_1 + a_2 + \dots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \dots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.
10. (IMO 2014) For each positive integer n , the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

4 Solutions

1. (Germany 2000) There are stones with a total mass of 9 tons that should be transported by trucks. None of the stones is heavier than 1 ton and each vehicle has a capacity of 3 tons. Determine the minimum number of necessary trucks such that the stones can certainly be transported at the same time.

Solution: Consider 10 stones of mass 0.9 each; clearly these cannot be packed into 3 trucks, so we need at least 4. To show 4 trucks suffices, place stones arbitrarily (but validly) in three trucks until you cannot place any more. Then each truck must have at least 2 tons in it (else we can pack more). Therefore, we have packed at least 6 tons, which means we can pack the remaining mass, less than $9 - 6 = 3$ tons, into a fourth. \square

2. Prove that for each positive integer n , there exists a unique set of integers a_1, a_2, \dots such that for $a_i \leq i$ for all i and

$$n = a_1 \cdot 1! + a_2 \cdot 2! + a_3 \cdot 3! + \dots$$

Solution: It's true for $N = 0, 1$. Suppose it's true for all $n < N$. Take k such that $k! \leq N < (k+1)!$ and j such that $jk! \leq N < (j+1)k!$. We know that $N - jk!$ has a representation. Furthermore, the largest term in its representation must be less than $k!$ since $N - jk! < (j+1)k! - jk! = k!$. Finally, to prove uniqueness, we show that at each step we are forced to take this value of $jk!$. Indeed, if we didn't, the largest possible representation we could have is

$$(j-1)k! + \sum_{i=1}^k i \cdot i! = (j-1)k! + \sum_{i=1}^k [(i+1)! - i!] = (j-1)k! + k! - 1 = jk! - 1 < N$$

contradiction. \square

3. (Netherlands 2014) Let n be a positive integer. Daniel and Merlijn are playing a game. Daniel has k sheets of paper lying next to each other on a table, where k is a positive integer. On each of the sheets, he writes some of the numbers from 1 up to n (he is allowed to write no number at all, or all numbers). On the back of each of the sheets, he writes down the remaining numbers. Once Daniel is finished, Merlijn can flip some of the sheets of paper (he is allowed to flip no sheet at all, or all sheets). If Merlijn succeeds in making all of the numbers from 1 up to n visible at least once, then he wins. Determine the smallest k for which Merlijn can always win, regardless of Daniel's actions.

Solution: On the first page, one side must have at least $\lceil \frac{n}{2} \rceil$ numbers on it. Then perform this algorithm with $k-1$ pages and $n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$. The sufficient number of applications of this is equal to the number of binary digits of n , $\lfloor \log_2 n \rfloor + 1$. As a construction that we can't do it in less, consider when the i th page consists of those numbers whose i th binary digit is 1 on one side and 0 on the other side. \square

4. (IMOSL 2001, generalized) A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z-y, y-x\} = \{a, b\}$ for $0 < a < b$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

Solution: For each historic set $\{x, y, z\}$ with $x < y < z$ that we construct, color x red, y green, and z blue. Begin with all nonnegative integers uncolored. Then iteratively perform the following algorithm: consider the

smallest uncolored number k ; if $k + a$ is uncolored construct the set $\{k, k + a, k + a + b\}$ and color it accordingly. Otherwise, construct the set $\{k, k + b, k + a + b\}$ and color it accordingly.

To see it works, note that $k + a + b$ is initially uncolored because it's greater than the other colored numbers. Suppose that $k + b$ is already colored. Then it can't be red because it is greater than k . It can't be green because $k + b - a, k + b - b \geq k$. If it were blue, then $k + b - (a + b) = k - a$ must be red. However, since k was uncolored at that moment, our algorithm says that we must color $k - a + a = k$, contradiction. \square

5. (VTRMC 2014) Consider the harmonic series $\sum_{n \geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$. Prove that every positive rational number can be obtained as an *unordered* partial sum of this series. (An unordered partial sum may skip some of the terms $\frac{1}{k}$.)

Solution: \square

6. (Gabriel Dospinescu) Which rational numbers can be written as the sum of the inverses of finitely many pairwise distinct triangular numbers?

Solution: \square

7. (IMC 1997) Let $1 < \alpha \leq 2$ be a real number.

- (a) Show that α has a unique representation as an infinite product

$$\alpha = \left(1 + \frac{1}{n_1}\right) \left(1 + \frac{1}{n_2}\right) \dots$$

where each n_i is a positive integer satisfying $n_{i+1} \geq n_i^2$.

- (b) Show that α is rational if and only if its infinite product has the following property: For some m and all $k \geq m$, $n_{k+1} = n_k^2$.

Solution: For $k \geq 1$, let $S_k := (1 + \frac{1}{k+1}, 1 + \frac{1}{k}]$. Note that S_1, S_2, \dots partition $(1, 2]$. Consider the following algorithm to choose n_1, n_2, \dots : find the unique $k \geq 1$ such that $\alpha \in S_k$, take $n_1 = k + 1$, and get n_2, n_3, \dots from this algorithm on $\frac{\alpha}{1 + \frac{1}{n_1}}$. Note the identity $\prod_{i \geq 0} (1 + x^{2^i}) = \frac{1}{1-x}$ for all $0 < x < 1$.

First we show that this algorithm does not terminate and generates a sequence satisfying $n_1 \leq n_2^{1/2} \leq n_3^{1/4} \leq n_4^{1/8} \leq \dots$. It suffices to show that $1 < \frac{\alpha}{1+n_1} \leq 1 + \frac{1}{n_1^2-1}$. But $\frac{\alpha}{1+n_1} > \frac{1 + \frac{1}{k+1}}{1 + \frac{1}{k+1}} = 1$ and $\frac{\alpha}{1+n_1} \leq \frac{1 + \frac{1}{k}}{1 + \frac{1}{k+1}} = 1 + \frac{1}{k^2+2k} = 1 + \frac{1}{n_1^2-1}$.

Now we show that $\alpha = (1 + \frac{1}{n_1})(1 + \frac{1}{n_2}) \dots$. We case on the rationality of α .

Suppose $\alpha \in \mathbb{Q}$, and let $\alpha = \frac{p}{q}$. If $p - q = 1$ then this algorithm will generate the sequence $n_1, n_2, n_3, \dots = p, p^2, p^4, \dots$ where $(1 + \frac{1}{p})(1 + \frac{1}{p^2})(1 + \frac{1}{p^4}) \dots = \frac{1}{1 - \frac{1}{p}} = \frac{p}{p-1} = \alpha$. Otherwise, we show that eventually this algorithm will hit a value where $p - q = 1$. It suffices to show that if $p - q \geq 2$, then $\frac{p'}{q'} = \frac{\alpha}{1 + \frac{1}{n_1}} = \frac{pn_1}{q(n_1+1)}$ where $p' - q' < p - q$. Indeed, we have $p' - q' \leq pn_1 - q(n_1 + 1) = n_1(p - q) - q$. But since $n_1 = k \leq \frac{1}{p/q-1} = \frac{q}{p-q}$, with equality impossible since $p - q \geq 2$, we have $p' - q' < p - q$.

Now suppose $\alpha \notin \mathbb{Q}$. Let $\beta_i := (1 + \frac{1}{n_1}) \dots (1 + \frac{1}{n_i})$. Then b_1, b_2, \dots is an increasing sequence, and it's easy to see that $\beta_i \leq \alpha$. Therefore it converges to some $\beta \leq \alpha$. Now since $n_1 \geq 2$, we have $n_i \geq 2^{2^{i-1}}$. Thus, $\frac{\alpha}{\beta_i} \leq (1 + \frac{1}{n_i})(1 + \frac{1}{n_i}) \dots = \frac{1}{1 - \frac{1}{n_i^2}} \leq \frac{1}{1 - \frac{1}{2^{2^i}}} = 1 + \frac{1}{2^{2^i-1}}$. Thus, $\alpha - \beta_i \leq \frac{\beta_i}{2^{2^i-1}} \leq \frac{\alpha}{2^{2^i-1}} \rightarrow 0$. Thus, $\alpha = \beta$.

Now we show that this is the only representation. It suffices to show that $n_1 = k + 1$ is forced. If $n_1 \leq k$ then $(1 + \frac{1}{n_1})(1 + \frac{1}{n_2}) \dots > 1 + \frac{1}{k} \geq \alpha$. If $n_1 \geq k + 2$ then $(1 + \frac{1}{n_1})(1 + \frac{1}{n_2}) \dots \leq (1 + \frac{1}{n_1})(1 + \frac{1}{n_1}) \dots = \frac{1}{1 - \frac{1}{n_1^2}} \leq \frac{1}{1 - \frac{1}{k+2}} = 1 + \frac{1}{k+1} < \alpha$.

Finally, for part (b), we have shown that if $\alpha \in \mathbb{Q}$, then the sequence has such a tail. Also, if the sequence has such a tail, then $\alpha = \prod_{1 \leq i < m} (1 + \frac{1}{n_i}) \cdot \frac{1}{1 - \frac{1}{n_m}} \in \mathbb{Q}$, so we're done. \square

8. (Math Prize for Girls 2010) Let S be a set of n points in the coordinate plane. Say that a pair of points is *aligned* if the two points have the same x -coordinate or y -coordinate. Prove that S can be partitioned into disjoint subsets such that (a) each of these subsets is a collinear set of points, and (b) at most $n^{3/2}$ unordered pairs of distinct points in S are aligned but not in the same subset.

Solution: We will use induction on n . As a base case, $n = 0$ is trivial. Now assume that there are at most $k^{3/2}$ pairs of aligned points in S is for all $0 \leq k \leq n - 1$, for some positive integer n .

Consider the largest subset T of S whose points either all have the same x -coordinate or all have the same y -coordinate (in fact, assume without loss of generality it is the same x -coordinate). Let $t = |T|$. For each point $P \in T$, it is part of another set of size at most t with the same y -coordinate, so the number of points in $S \setminus T$ aligned with some point in T is at most $t \cdot t = t^2$. But obviously this can only be at most n pairs because that is the number of points in total. Therefore, it suffices to prove that

$$\min\{t^2, n\} + (n - t)^{3/2} \leq n^{3/2}$$

If $n \leq t^2$, then

$$\min\{t^2, n\} + (n - t)^{3/2} \leq n + (n - \sqrt{n})^{3/2} \leq n^{3/2}$$

for $n \geq 4$, or $t \geq 2$. When $t = 1$, no two points are aligned, the problem statement clearly holds. If $n \geq t^2$, then

$$\min\{t^2, n\} + (n - t)^{3/2} \leq t^2 + (n - t^2)^{3/2} \leq n^{3/2}$$

where the rightmost inequality can be seen from the fact that the inequality holds at $n = t^2$ and the right-hand side always grows faster than the left-hand side (since $n^{3/2}$ is convex and the right-hand side is its translation t^2 units to the right and up). \square

9. (IMOSL 2013) Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers a_1, \dots, a_d such that $a_1 + a_2 + \dots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \dots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

Solution: This is a generalization of the Germany stones in trucks problem. Consider the set $a_1 = \dots = a_{2n-1} = \frac{n}{2n-1}$. We see that $a_1 + \dots + a_{2n-1} = (2n - 1) \frac{n}{2n-1} = n$, and if any two are in the same group then it will have sum at least $\frac{2n}{2n-1} > 1$. Therefore, at least $k = 2n - 1$ groups are necessary.

Now suppose that we pack the numbers into as few groups as possible, say $g_1 \leq \dots \leq g_m$. Then $g_i + g_j > 1$ for all i, j , else we could combine the groups. Thus,

$$2n = (g_1 + g_2) + (g_2 + g_3) + \dots + (g_m + g_1) > 1 + 1 + \dots + 1 = m$$

so $m \leq 2n - 1$. \square

10. (IMO 2014) For each positive integer n , the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Solution: Replace 100 with m and $99 + \frac{1}{2}$ with $m - \frac{1}{2}$. Assume there is at most 1 coin of denomination $\frac{1}{2k}$ (else combine them into one coin of denomination $\frac{1}{k}$) and at most $2k$ coins of denomination $\frac{1}{2k+1}$ (else combine them into one coin of denomination 1). We can also assume that there are no coins of denomination 1 because it would reduce the problem to $m - 1$ groups of size $(m - 1) - \frac{1}{2}$, which is the same problem.

For each k group all the coins of the form $\frac{1}{2k+1}$ and $\frac{1}{2k+2}$. There is at most a total value of $\frac{2k}{2k+1} + \frac{1}{2k+2} < 1$ in each of these groups. For $1 \leq k \leq m - 1$, this is $m - 1$ groups that take care of the coins of denomination $\frac{1}{3}$ to $\frac{1}{2m}$. We would also have to place $\frac{1}{2}$ into its own group, which is a total of m groups.

For all coins of denomination smaller than $\frac{1}{2^m}$, place these into whatever groups still have space. Assume that some coin can't be packed. This means all the other groups are packed with at least $1 - \frac{1}{2^{m+1}}$ coins, which is a contradiction because the total value would be at least $m \left(1 - \frac{1}{2^{m+1}}\right) > m - \frac{1}{2}$. \square