Generating Functions

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1 Introduction

The *generating function* of a sequence $a_0, a_1, a_2, \dots$ is defined as

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k \geq 0} a_k x^k$$

The generating function of a set S is defined as

$$G(x) = \sum_{r \in S} x^r$$

If we allow sets to have repeats – a *multiset* is a set that allows repeats – then we must count the number of times each element occurs as the coefficient:

$$G(x) = \sum_{r \in S} (\text{\# occurrences of } r) \cdot x^r$$

Let $[x^k]G(x)$ denote the coefficient of x^k in $G(x)$. Generating functions are useful because they allow us to work with sets algebraically. We can manipulate generating functions without worrying about convergence (unless of course you're evaluating it at a point).

2 Useful Facts

1. (Generating function of $\mathbb{N}$) For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n = \prod_{n \geq 0} (1 + x^{2^n})$$

2. (Generalized Binomial Theorem) For any $\alpha \in \mathbb{R}$, let

$$\binom{\alpha}{k} := \frac{\alpha(\alpha - 1) \dots (\alpha - k + 1)}{k!}$$

Then

$$(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n$$

3. For two sequences of the same length $a_1, \dots, a_n$ and $b_1, \dots, b_n$,

$$\left(\sum_{k=1}^{n} a_k \right) \left(\sum_{k=1}^{n} b_k \right) = \sum_{k=1}^{2n} \sum_{i+j=k} a_i b_j$$

Also,

$$\left(\sum_{k=1}^{n} a_i \right)^2 = \sum_{k=1}^{n} a_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j$$

4. The Maclaurin series of f is equal to

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

This is a way of forcibly extracting coefficients if necessary/possible.

3 Problems

1. (Logan Dymond) If x_k, y_k are integers such that $0 \le x_k, y_k \le k$ for all k, prove that for all $n > 2$, the number of solutions to

$$x_1+2x_2+3x_3+\cdots+nx_n=n(n+1)$$

is equal to the number of solutions to

$$0<3y_1+4y_2+5y_3+\cdots+ny_{n-2}\le n(n+1)$$

2. (PFTB) Suppose that the set of natural numbers (including 0) is partitioned into a finite number of infinite arithmetic progressions of ratios $r_1, r_2, \dots, r_n$ and first term $a_1, a_2, \dots, a_n$. Then the following relation is satisfied:

$$\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_n}=1$$

3. (USAMO 1996, weakened) Determine (with proof) whether there is a subset X of the nonnegative integers with the following property: for any integer n there is exactly one solution of $a+2b=n$ with $a, b \in X$.

4. (USAMO 1996) Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a+2b=n$ with $a, b \in X$.

5. (IMOSL 1998) Let $a_0, a_1, \dots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i+2a_j+4a_k$, where i, j, k are not necessarily distinct. Determine a_{1998}.

6. (Putnam 2003) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1, s_2 \in S$, $s_1 \ne s_2$, and $s_1+s_2=n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n)=r_B(n)$ for all n?

7. (Putnam 2000) Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \dots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there are infinitely many integers N for which $S_N = S_0 \cup \{N+a : a \in S_0\}$.

8. (China 2002) Find all natural numbers $n \ge 2$ such that there exists real numbers $a_1, \dots, a_n$ that satisfy

$$\{|a_i-a_j| \mid 1 \le i < j \le n\} = \{1, 2, \dots, n(n-1)/2\}$$

9. (IMC 2015) Consider all 26^{26} words of length 26 in the Latin alphabet. Define the *weight* of the word as $\frac{1}{k+1}$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.

10. (Putnam 2005) Let S_n denote the set of all permutations of the numbers $1, 2, \dots, n$. For $\pi \in S_n$, let $\sigma(\pi)=1$ if π is an even permutation and $\sigma(\pi)=-1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n}\frac{\sigma(\pi)}{\nu(\pi)+1}=(-1)^{n+1}\frac{n}{n+1}$$

11. (IMC 2016) Let S_n denote the set of permutations of the sequence $(1, 2, \ldots, n)$. For every permutation $\pi = (\pi_1, \ldots, \pi_n)$, let $\text{inv}(\pi)$ be the number of pairs $1 \le i < j \le n$ with $\pi_i > \pi_j$; i.e., the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\text{inv}(\pi)$ is divisible by $n+1$.

    Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

4 Solutions

1. (Logan Dymond) If x_k, y_k are integers such that $0 \le x_k, y_k \le k$ for all k, prove that for all $n > 2$, the number of solutions to

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n(n+1)$$

is equal to the number of solutions to

$$0 < 3y_1 + 4y_2 + 5y_3 + \cdots + ny_{n-2} \le n(n+1)$$

Solution: We want the $x^{n(n+1)}$ coefficient of

$$\prod_{i=1}^{n}\sum_{j=0}^{i} x^{ij} = \prod_{i=1}^{n} \frac{x^{i(i+1)}-1}{x^i - 1}$$

to be equal to the $x^{n(n+1)}$ coefficient of

$$(1 + x + x^2 + \dot{+} x^{n^2+n-1}) \prod_{i=1}^{n-2}\sum_{j=0}^{i} x^{(i+2)j} = \frac{x^{n(n+1)}-1}{x-1}\prod_{i=1}^{n-2}\frac{x^{(i+1)(i+2)}-1}{x^{i+2}-1} = \prod_{i=1}^{n}\frac{x^{i(i+1)}-1}{x^i-1}$$

□

2. (PFTB) Suppose that the set of natural numbers (including 0) is partitioned into a finite number of infinite arithmetic progressions of ratios $r_1, r_2, \ldots, r_n$ and first term $a_1, a_2, \ldots, a_n$. Then the following relation is satisfied:

$$\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} = 1$$

Solution: We have

$$\frac{1}{1-x} = \sum_{k\ge 0} x^k = \sum_{k=1}^{n}\sum_{i\ge 0} x^{a_k + ir_k} = \sum_{k=1}^{n}\frac{x^{a_k}}{1-x^{r_k}}$$

or

$$1 = \sum_{k=1}^{n} x^{a_k} \cdot \frac{1-x}{1-x^{r_k}}$$

$\lim_{x\to 1^{-1}} \frac{1-x}{1-x^{r_k}} = \frac{1}{r_k}$ concludes the proof.

Note: Be cautious when plugging in values of x (or taking limits)! Here it is a finite series so it's okay.

3. (USAMO 1996, weakened) Determine (with proof) whether there is a subset X of the nonnegative integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

   Solution: Let $f(x) := \sum_{s\in X} x^s$. Then

$$\sum_{n\geq 0} x^n = \sum_{a,b\in X} x^{a+2b} = \sum_{a\in X} x^a \sum_{b\in X} x^{2b} = f(x)f(x^2)$$

   Now remember the identity $\sum_{n\geq 0} x^n = (1+x)(1+x^2)(1+x^4)(1+x^8)\dots$. It suggests we can take $f(x) = (1+x)(1+x^4)(1+x^{16})\dots$. It remains to show that $[x^k]f \in \{0,1\}$ for all k, which is clear. □

   Note: From generating functions, we can derive a combinatorial solution! If we expand $f(x) = (1+x)(1+x^4)(1+x^{16})\cdots = 1 + x + x^4 + x^5 + x^{16} + \dots$, note that each x^k must be the sum of distinct powers of 4, i.e., X is the set of all numbers whose base 4 representations have just 0s and 1s as digits. Can you prove combinatorially that this set works?

4. (USAMO 1996) Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

   Solution: This one is a little more involved. Take $f(x) = (1+x)(1+x^{-4})(1+x^{16})(1+x^{-64})\dots(1+x^{(-4)^k})$. Then

$$f(x)f(x^2) = \frac{(1+x)(1+x^2)(1+x^4)(1+x^{16})\dots(1+x^{2^{k+1}})}{x^m}$$

   for some big m. We don't actually have to calculate the value of m; just note that this expands into something of the form

$$f(x)f(x^2) = x^{-a} + x^{-a+1} + \dots + x^{b-1} + x^b$$

   and we can make a and b arbitrarily large by taking k large enough. Thus we can capture all integers. □

   Note: There is a combinatorial construction of X using base -4 expansion!

5. (IMOSL 1998) Let $a_0, a_1, \dots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j, k are not necessarily distinct. Determine a_{1998}.

   Solution: Let $S := \{a_0, a_1, \dots\}$ and $f(x) := \sum_{s\in S} x^s$. Then

$$\prod_{n\geq 0}(1+x^{2^n}) = \frac{1}{1-x} = \sum_{a,b,c\in S} x^{a+2b+4c} = f(x)f(x^2)f(x^4)$$

   Take $f(x) = (1+x)(1+x^8)(1+x^{64})\dots$. This is the numbers whose base 8 representations have just 0s and 1s.

6. (Putnam 2003) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1, s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

   Solution: Let $f(x) := \sum_{a\in A} x^a$ and $g(x) := \sum_{b\in B} x^b$. We want

$$f(x) + g(x) = \sum_{n\geq 0} x^n = \frac{1}{1-x}, \text{ and}$$

$$\sum_{a_1,a_2\in A} x^{a_1+a_2} - \sum_{a\in A} x^{2a} = \sum_{b_1,b_2\in B} x^{b_1+b_2} - \sum_{b\in B} x^{2b}$$

   so

$$
\begin{aligned}
f(x)^2 - f(x^2) = g(x)^2 - g(x^2) &= \left(\frac{1}{1-x} - f(x)\right)^2 - \left(\frac{1}{1-x^2} - f(x^2)\right) \\
&= f(x)^2 + f(x^2) - \frac{2f(x)}{1-x} + \frac{1}{(1-x)^2} - \frac{1}{1-x^2} \implies \\
\frac{x}{1-x^2} = f(x) - (1-x)f(x^2) &= \sum_{a\in A}\left(x^a + x^{2a+1} - x^{2a}\right)
\end{aligned}
$$

Expand $\frac{x}{1-x^2} = x+x^3+x^5+\dots$, so the right-hand side cannot have any even powers. Hence, $2a \in A \implies a \in A$. Also we want the odd coefficients to be exactly 1, so $2a+1 \in A \implies a \notin A$. Also since every integer must be in at least one set, we must have $a \in A \implies 2a \in A$ and $a \in A \implies 2a+1 \in B$.

To finish, we make our construction in the following way: put $0 \in A$, and let the above two rules place the rest of the integers. It's easy to check that it places each integer in exactly one set. □

Note: The combinatorial construction for A is the set of numbers with an even number of 1s in its binary representation, and B is odd number of 1s.

7. (Putnam 2000) Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \dots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there are infinitely many integers N for which $S_N = S_0 \cup \{N+a : a \in S_0\}$.

   Solution: Let $f_n(x) := \sum_{s\in S_n} x^s$. Then, we have $f_{n+1}(x) = (1+x)\sum_{s\in S} x^s \pmod 2$, so $f_N \equiv (1+x)^N f_0(x) \pmod 2$. We want it in the form of $f(x) \equiv (1+x^N)f_0(x) \pmod 2$. However, if N is a power of 2, then $(1+x)^N \equiv 1+x^N \pmod 2$ (this is seen from Pascal's triangle), so all $N = 2^k$ work as long as $N > \max\{S_0\}$ (to avoid internal cancellation). □

8. (China 2002) Find all natural numbers $n \geq 2$ such that there exists real numbers $a_1, \dots, a_n$ that satisfy

$$
\{|a_i - a_j| \mid 1 \leq i < j \leq n\} = \{1, 2, \dots, n(n-1)/2\}
$$

   Solution: Let $S = \{a_1, \dots, a_n\}$ and $f(x) = \sum_{s\in S} x^s$. Then

$$
\begin{aligned}
f(x)f(1/x) &= (x^{a_1} + \dots + x^{a_n})(x^{-a_1} + \dots + x^{-a_n}) \\
&= n - 1 + x^{-n(n-1)/2} + \dots + x^{n(n-1)/2} \\
&= n - 1 + \frac{x^{n(n-1)/2+1} - x^{-n(n-1)/2}}{x-1} \\
&= n - 1 + \frac{x^{(n^2-n+1)/2} - x^{-(n^2-n+1)/2}}{x^{1/2} - x^{-1/2}}
\end{aligned}
$$

   Take $x = \exp \frac{3\pi i}{n^2-n+1} =: \exp 2i\theta$ to get

$$
|f(x)|^2 = f(x)f(\overline{x}) = n - 1 + \frac{\sin\frac{3\pi}{2}}{\sin\theta} = n - 1 - \frac{1}{\sin\theta} < n - 1 - \frac{1}{\theta} = n - 1 - \frac{2}{3\pi}(n^2 - n + 1)
$$

   This quantity is negative for all n except $n = 2, 3, 4$. Then take the following constructions: $\{0,1\}$, $\{0,1,3\}$, and $\{0,1,4,6\}$. □

9. (IMC 2015) Consider all 26^{26} words of length 26 in the Latin alphabet. Define the *weight* of the word as $\frac{1}{k+1}$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.

   Solution: Let a_{ni} be the number of n-letter words with $26-i$ distinct letters, and let $f_n(x) := \sum_i a_{ni}x^i$. Since $a_{ni} = (26-i)a_{(n-1)i} + (i+1)a_{(n-1)(i+1)}$, we have

$$f_n(x) = f'_{n-1}(x)(1-x) + 26 f_{n-1}(x)$$

Now let $I_n := \int_0^1 f_n(x)dx$. Integrating by parts, we get

$$I_n = [f_{n-1}(x)(1-x)]_0^1 + 27I_{n-1} = -f_{n-1}(0) + 27I_{n-1} = 27I_{n-1}$$

since $f_{n-1}(0) = a_{(n-1)0} = 0$. Since $I_1 = 1$, we get $I_{26} = 27^{25} = 3^{75}$. □

Note: The weight function $\frac{1}{k+1}$ motivated this solution, since it looks like $\int_0^1 x^k dx$.

10. (Putnam 2005) Let S_n denote the set of all permutations of the numbers $1, 2, \dots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = (-1)^{n+1}\frac{n}{n+1}$$

Solution: Let $f_n(x) := \sum_{\pi\in S_n} \sigma(\pi)x^{\nu(\pi)}$. Then we can either have $\pi(n+1)$ or $n+1$ gets sent somewhere in a cycle. The number of cycles of length ℓ for which $n+1$ can get sent to is $\frac{n!}{(n-\ell+1)!}$, and depending on the cycle length the parity alternates,. Furthermore, none of the elements in this cycle are fixed points, so we have the recurrence

$$\begin{aligned}
f_{n+1}(x) = \sum_{\pi\in S_{n+1}} \sigma(\pi)x^{\nu(\pi)} &= \sum_{\pi\in S_n} \sigma(\pi)x^{\nu(\pi)+1} - \frac{n!}{(n-1)!}\sum_{\pi\in S_{n-1}} \sigma(\pi)x^{\nu(\pi)} \\
&+ \frac{n!}{(n-2)!}\sum_{\pi\in S_{n-2}} \sigma(\pi)x^{\nu(\pi)} - \frac{n!}{(n-3)!}\sum_{\pi\in S_{n-3}} \sigma(\pi)x^{\nu(\pi)} + \dots \\
&= xf_n(x) + n\cdot[f_n(x) - xf_{n-1}(x)]
\end{aligned}$$

Furthermore, we can hand-calculate that $f_1(x) = x$, $f_2(x) = x^2-1$, and after a few terms we can guess that $f_n(x) = (x-1)^{n-1}(x+n-1)$. To show by induction, we just need

$$x(x-1)^{n-1}(x+n-1) + n\cdot\left[(x-1)^{n-1}(x+n-1) - x(x-1)^{n-2}(x+n-2)\right] = (x-1)^n(x+n)$$

11. (IMC 2016) Let S_n denote the set of permutations of the sequence $(1, 2, \dots, n)$. For every permutation $\pi = (\pi_1, \dots, \pi_n)$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \le i < j \le n$ with $\pi_i > \pi_j$; i.e., the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n+1$.

Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

Solution: Let $g_n(x) := \sum_{\sigma\in S_n} x^{\mathrm{inv}(\sigma)}$. Let's add $n+1$ in all possible places to all elements of S_n. If we add $n+1$ at the very end of a permutation, it creates no new inversions. If we add it second-to-last, it creates one new inversion. All the way until when we add it to the very beginning where it creates n new inversions. Thus $g_{n+1}(x) = (1+x+\dots+x^n)g_n(x)$. With $g_1(x) = 1$, we have

$$g_n(x) = (1+x)(1+x+x^2)\dots(1+x+\dots+x^{n-1}) = \frac{(x-1)(x^2-1)\dots(x^n-1)}{(x-1)^n}$$

where $x \ne 1$ and $g_n(1) = |S_n| = n!$. Using roots of unity filter with $\omega = e^{2\pi i/p}$ for prime $p \ge 3$,

$$pf(p-1) = (p-1)! + \sum_{k=1}^{p-1} \frac{(\omega^k-1)(\omega^{2k}-1)\dots(\omega^{(p-1)k}-1)}{(\omega^k-1)^{p-1}}$$

The numerator is simplified to p using the identity $(x-\omega^k)(x-\omega^{2k})\dots(x-\omega^{(p-1)k}) = 1+x+\dots+x^{p-1}$. Thus it remains to show that the following sum is positive when $p \equiv 3 \pmod 4$ and negative when $p \equiv 1 \pmod 4$:

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{1}{(\omega^k-1)^{p-1}} &= \sum_{k=1}^{(p-1)/2} \left[\frac{1}{(\omega^k-1)^{p-1}} + \frac{1}{(\omega^{-k}-1)^{p-1}}\right] \\
&= \sum_{k=1}^{(p-1)/2} \frac{1+\omega^{(p-1)k}}{(\omega^k-1)^{p-1}} = \sum_{k=1}^{(p-1)/2} \frac{\frac{\omega^{(p-1)k/2}+\omega^{-(p-1)k/2}}{2}}{\left(\frac{\omega^{k/2}-\omega^{-k/2}}{2i}\right)^{p-1}} \frac{2}{(2i)^{p-1}} \\
&= \frac{1}{2^p(-1)^{(p-1)/2}} \sum_{k=1}^{(p-1)/2} \frac{\cos\frac{k(p-1)\pi}{p}}{\left(\sin\frac{k\pi}{p}\right)^{p-1}} = \frac{1}{2^p(-1)^{(p-1)/2}} \sum_{k=1}^{(p-1)/2} \frac{(-1)^k\cos\frac{k\pi}{p}}{\left(\sin\frac{k\pi}{p}\right)^{p-1}}
\end{aligned}$$

For very large p, the $k=1$ term determines the sign of the whole sum since $\cos\frac{k\pi}{p}$ is decreasing in magnitude and $\sin\frac{k\pi}{p}$ is increasing in magnitude (note that $0 < \frac{k\pi}{p} < \frac{\pi}{2}$). Thus, for very large p we get that $(-1)^{(p+1)/2}$ is the sign of $pf(p-1)-(p-1)!$, the desired result.