Generating Functions

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Useful Identities

For $|x| < 1$, the following identities hold:

1. (Taylor series) $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

   need f to be analytic at 0

2. $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{k+(n-1)}{n-1} x^k$ and $\prod_{k=0}^{\infty} \left(1 + x^{2^k}\right) = \frac{1}{1-x}$

   need $|x| < 1$

3. $\sum_{i \in A, j \in B} (\alpha_i x^i)(\beta_j x^j) = \left(\sum_{i \in A} \alpha_i x^i \right) \left(\sum_{j \in B} \beta_j x^j \right)$

   need absolute convergence

Warmup

1. *How to multiply*: Expand $(1 + x^2 + x^7 + x^{20})(x + x^3 + x^4)$. Expand $(1 + 3x^2 + x^7 + 4x^{20})(x - x^3 + x^4)$.

Things you can do to a GF

1. $(a_0, a_1, a_2, \ldots), (b_0, b_1, b_2, \ldots) \mapsto (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$: $G_1, G_2 \mapsto G_1 + G_2$
2. $(a_0, a_1, a_2, \ldots) \mapsto (\alpha a_0, \alpha a_1, \alpha a_2, \ldots)$: $G \mapsto \alpha G$
3. $(a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, \ldots)$: $G(x) \mapsto xG(x)$
4. $(a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, a_3, \ldots)$: $G(x) \mapsto \frac{G(x) - a_0}{x}$
5. $(a_0, a_1, a_2, \ldots) \mapsto (a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$: $G(x) \mapsto \frac{G(x)}{1-x}$
6. $(a_0, a_1, a_2, \ldots) \mapsto (a_1, 2a_2, 3a_3, \ldots)$: $G \mapsto G'$
7. $(a_0, a_1, a_2, \ldots) \mapsto \left(C, a_0, \frac{a_1}{1}, \frac{a_2}{2}, \ldots\right)$
8. Prove that the GF of H_n is $\frac{-\log(1-x)}{1-x}$.

Recurrences

1. Solve the recurrence $a_0 = 1$, $a_1 = 2$, $a_n = 3a_{n-1} - 2a_{n-2}$.
2. Find the value of $1^2 + 2^2 + \cdots + n^2$.
3. The Catalan numbers are defined by $C_0 = 1$ and

   $$C_n = C_{n-1}C_0 + C_{n-2}C_1 + \cdots + C_0 C_{n-1}$$

   for $n \geq 1$. Find the generating function for C_n, and use it to find an explicit formula for C_n.

Summing

1. Find the value of $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$. What about $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$?
2. Given $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$, find $a_0 + a_4 x^4 + a_8 x^8 + \dots$.
3. Find the value of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{2^n} - 2^{-2^n}}$.

Counting

1. How many solutions are there to $x_1 + x_2 + \cdots + x_m = n$ such that $0 \le x_1, x_2, \dots, x_m \le n$? Such that $0 \le x_1 \le x_2 \le \cdots \le x_m \le n$?
2. Determine the number of k-element subsets of $[n]$ such that the ith largest element of the subset is congruent to $i \mod 2$.

Two fairly hard problems

1. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by
$$a_0 = 1, \quad a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+1}$$
Find the limit
$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{2^k}$$
2. Evaluate
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}$$

Problems

1. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
2. Find the value of $\sum_{k=0}^{n} \binom{n}{k} (-1)^k$. Deduce that the number of subsets of $\{1, 2, \dots, n\}$ with odd size is equal to the number with even size.
3. By comparing the coefficient of x^n in $(x+1)^{a+b}$ and $(x+1)^a(x+1)^b$, prove that
$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k}$$
4. Let $\{1, 1, 2, 3, 5, 8, \dots\}$ be the Fibonacci sequence. Prove that the number
$$\frac{1}{10^3} + \frac{1}{10^6} + \frac{2}{10^9} + \frac{3}{10^{12}} + \cdots = 0.001001002003\dots$$
is a rational number. What is it in reduced form?
5. Find an explicit formula for the nth Fibonacci number. Hint: this formula will likely involve the number $\frac{1+\sqrt{5}}{2}$, a root of $x^2 - x - 1$.
6. How many n-digit numbers, whose digits are in the set $\{2, 3, 7, 9\}$ are divisible by 3?