# Constructions

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## 1 Introduction

A large amount of Olympiad combinatorics are posed in the form "Find the minimum/maximum $n$ such that..." Constructions are very useful in proving that a minimum/maximum is attained. The hard part is usually in finding such a construction, and this article will attempt to explain the intuition behind finding constructions.

Solving a combinatorics problem is all about looking at the structure behind the problem. Below in a list of some general strategies when searching for constructions. Every one of these strategies is incredibly important, and I expect you to use every single one when solving a problem, so make sure you remember them all!

1. *Always* try the problem for small values of $n$. Not only will getting some small values help you guess the minimum/maximum, but once you find constructions for small values it is usually easy to generalize the constructions.
2. Try things that make sense with the problem. When doing this, ask yourself what you could possibly use and then try things!
3. Be bold with your intuitions. If you find a construction that gives a fairly large maximum to a problem, focus your efforts into proving that it is indeed the maximum. Your intuitions are best verified by looking at small cases.
4. Think simply, think symmetrically. It's rare that a problem has an overly convoluted construction.
5. Be greedy! When all else fails, just eyeball it and start taking the best possible configuration you can think of.
6. Draw a picture/table. They help to reveal the patterns and symmetries of a problem.
7. Do some algebra and prove stuff about the problem to build intuition and to avoid searching for constructions that don't exist.

Finally, here is some advice if you encounter a problem of the form "Prove or disprove..." or "Determine whether there exist...":

1. Often one of the responses is completely unreasonable and you could never expect an Olympiad asking you to prove that, like proving that $22! \cdot 6! + 1$ is a prime number.
2. If you haven't proven the answer is yes within the first half of your time, try proving it is no in the second half of your time, even if it completely goes against your intuition. Do not flip flop between them as you start running out of time!
3. Try very hard to prove that the answer is no, figure out where your argument fails, and exploit it to find a construction.

## 2 Induction

Often induction is an easy way to form constructions: if you have a construction of size $n$, then you may be able to simply add one element to it to get a construction of size $n + 1$.

1. (USAMO 2003) Prove that for every positive integer $n$ there exists an $n$-digit number divisible by $5^n$ all of whose digits are odd.

   *Solution:* Let's make a table of the such numbers we find:

| $n$ | construction |
|---|---|
| 1 | 5 |
| 2 | 75 |
| 3 | 375 |
| 4 | 9375 |

From here it's clear induction will be useful, as we're simply adding one digit before the previous construction. We want to show that $5^n \mid X_n$, where $X_n$ only has odd digits, then there is a digit $Y$ such that $5^{n+1} \mid Y \cdot 10^n + X_n$. From here it suffices to find an odd $Y$ such that $5 \mid Y \cdot 2^n + \frac{X_n}{5^n}$, but we can easily do this by taking $Y \equiv (2^n)^{-1} \frac{-X_n}{5^n} \pmod 5$. This does not guarantee that $Y$ is odd, but if this gives an even $Y$, we can simply add 5 to it and then it will be an odd digit. □

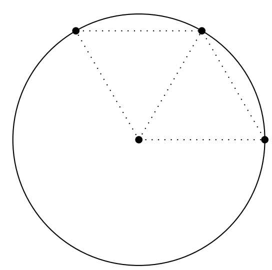
2. (IMO 2015) We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points $A$ and $B$ in $\mathcal{S}$, there is a point $C$ in $S$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points $A$, $B$, and $C$ in $\mathcal{S}$, there is no point $P$ in $\mathcal{S}$ such that $PA = PB = PC$.

   (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of $n$ points.

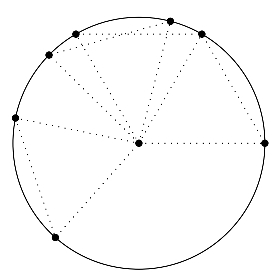
   (b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of $n$ points.

   *Solution:*

   (a) Thinking symmetrically, we should try a regular $n$-gon. It's easy to see that this works for odd $n$. Additionally it is centre-free, a fact which we will use in part (b). If we are bold with this, we should use part (b) as a hint that the construction will not be centre-free. So we start by drawing a circle, and try constructing the $n = 2, 4, 6, 8$ cases using one point as the center and the rest of them on the circle.



   Now since we're breaking it into an odd and even case, it would make sense to add 2 points at a time to the circle, perhaps to balance each other. The best idea is adding them symmetrically, that is, in equilateral triangles:



   (b) Using the regular $n$-gon construction, we can capture all odd $n$. Also, observe that if a configuration is centre-free, then each point can balance at most two other points at the same distance, so each point can balance at most $\frac{n-1}{2}$ pairs. Therefore, the total number of pairs is at most $\frac{n(n-1)}{2}$, with equality if and only if each point balances exactly $\frac{n-1}{2}$ points, which is impossible if $n$ is even.

□

3. (IMO 2013) Assume that $k$ and $n$ are two positive integers. Prove that there exist positive integers $m_1, \dots, m_k$ such that
$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \dots \left(1 + \frac{1}{m_k}\right)$$

*Solution:* We construct by induction on $k$. For $k = 1$, it is just $m_1 = n$. Now assume it's true for all $k \le K$ and all $n$. In an ideal world, we would like $a, b$ such that
$$1 + \frac{2^{K+1} - 1}{n} = \left(1 + \frac{2^K - 1}{a}\right)\left(1 + \frac{1}{b}\right) \implies b = \frac{n(a + 2^K - 1)}{2^K(2a - n) + n - a}$$
We guess $a = \frac{n}{2}$, which does indeed solve the even case. Since this was so profitable, try $a = \frac{n+1}{2}$ for the odd case, which also works. Our final proof is:

When $n = 2r - 1$, we have
$$1 + \frac{2^{K+1} - 1}{2r - 1} = \left(1 + \frac{2^K - 1}{r}\right)\left(1 + \frac{1}{2r - 1}\right)$$
When $n = 2r$, we have
$$1 + \frac{2^{K+1} - 1}{2r} = \left(1 + \frac{2^K - 1}{r}\right)\left(1 + \frac{1}{2r + 2^{K+1} - 2}\right)$$
□

4. (IMO 2012) Find all positive integers $n$ for which there exist non-negative integers $a_1, a_2, \dots, a_n$ such that
$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1$$

*Solution:* Before playing with a construction we should look at mods to see if we can narrow it down. If we clear denominators in the second equation by multiplying by $m := \max\{a_1, \dots, a_n\}$, we find that $3^m = 3^{m-a_1} + 3^{m-a_2} \cdot 2 + +3^{m-a_3} \cdot 3 + \cdots + 3^{m-a_n} \cdot n$. Immediately by parity considerations we get $1 \equiv 1 + 2 + \cdots + n$, so $n \equiv 1, 2 \pmod 4$.

Now we look at small cases. We come up with the following constructions:

$$\begin{aligned}
n = 1 &: \quad \tfrac{1}{1} = \tfrac{1}{1} = 1 \\
n = 2 &: \quad \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{1}{3} + \tfrac{2}{3} = 1 \\
n = 5 &: \quad \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{9} + \tfrac{2}{9} + \tfrac{3}{9} + \tfrac{4}{27} + \tfrac{5}{27} = 1 \\
n = 6 &: \quad \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{9} + \tfrac{2}{9} + \tfrac{3}{27} + \tfrac{4}{27} + \tfrac{5}{27} + \tfrac{6}{27} = 1
\end{aligned}$$

Noticing the central symmetry in our constructions, we try taking the middle term and splitting it up. For odd $n$, we find
$$\frac{1}{2^{a_{(n+1)/2}}} = \frac{1}{2^{a_{(n+1)/2}+1}} + \frac{1}{2^{a_{(n+1)/2}+1}} \text{ and } \frac{(n+1)/2}{3^{a_{(n+1)/2}}} = \frac{(n+1)/2}{3^{a_{(n+1)/2}+1}} + \frac{n+1}{3^{a_{(n+1)/2}+1}}$$
In that similar spirit, we seek identities splitting $\frac{1}{2^{a_m}}$ and $\frac{m}{3^{a_m}}$ to get from $4k + 2 \to 4k + 5$. No identities turn out useful, so we try to get from $4k + 2 \to 4k + 9$. Again nothing works (this is where you need faith in your techniques), so we try $4k + 2 \to 4k + 13$. Alas, we find

$$\begin{aligned}
\frac{1}{2^{a_m}} &= \frac{1}{2^{a_m+2}} + \frac{1}{2^{a_m+3}} + \frac{1}{2^{a_m+3}} + \frac{1}{2^{a_m+3}} + \frac{1}{2^{a_m+3}} + \frac{1}{2^{a_m+3}} + \frac{1}{2^{a_m+3}} \\
\frac{m}{3^{a_m}} &= \frac{m}{3^{a_m+2}} + \frac{4m-5}{3^{a_m+3}} + \frac{4m-3}{3^{a_m+3}} + \frac{4m-1}{3^{a_m+3}} + \frac{4m+1}{3^{a_m+3}} + \frac{4m+3}{3^{a_m+3}} + \frac{4m+5}{3^{a_m+3}}
\end{aligned}$$

Applying this with $m = \frac{n+6}{4}$, we almost get what we want, finishing by repetitively applying the identity used in the $4k + 1 \to 4k + 2$ case.

Note: since we only proved $4k + 2 \to 4k + 13$, we would still have to find a construction for $n = 10$, which is easy. □

# 3 Number Theory via Pictures

Pictures are one of the best methods for discovering the structure behind a problem and help to reveal the symmetries. They also help you identify where you want to go with your construction.

1. (RMM 2015) Does there exist an infinite sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_m$ and $a_n$ are coprime if and only if $|m-n|=1$?

   *Solution:* Clearly this problem is a combinatorics problem disguised as a number theory problem, since it multiplying a number by $p^2$ or $p^3$ has the same effect as multiplying it by $p$, also it doesn't even matter *which* primes we multiply by. Therefore, let's draw a grid, where a dot represents whether the prime appears in the factorization of the term:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ |
|---|---|---|---|---|---|---|---|
| $p_1$ | | | | | | | |
| $p_2$ | | | | | | | |
| $p_3$ | | | | | | | |
| $p_4$ | | | | | | | |
| $p_5$ | | | | | | | |
| $p_6$ | | | | | | | |
| $\vdots$ | | | | | | | |

   We know that $a_3$ and $a_4$ both share prime factors with $a_1$, but it cannot be the same prime factor. The same holds for $a_4$ and $a_5$, $a_5$ and $a_6$, etc. The simplest idea is:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ |
|---|---|---|---|---|---|---|---|
| $p_1$ | • | | • | | • | | • |
| $p_2$ | • | | | • | | • | |
| $p_3$ | | | | | | | |
| $p_4$ | | | | | | | |
| $p_5$ | | | | | | | |
| $p_6$ | | | | | | | |
| $\vdots$ | | | | | | | |

   Well, that settles everything with $a_1$. All we have to do is continue to play the same game with the rest of the terms. As we fill in dots, we get a picture like this:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ |
|---|---|---|---|---|---|---|---|
| $p_1$ | • | | • | | • | | • |
| $p_2$ | • | | | • | | • | |
| $p_3$ | | • | | • | | • | |
| $p_4$ | | • | | | • | | • |
| $p_5$ | | | • | | • | | • |
| $p_6$ | | | • | | | • | |
| $\vdots$ | | | | | | | |

   □

2. (USAMTS 2015) Nine distinct positive integers are arranged in a circle such that the product of any two non-adjacent numbers in the circle is a multiple of $n$ and the product of any two adjacent numbers in the circle is not a multiple of $n$, where $n$ is a fixed positive integer. Find the smallest possible value for $n$.

   *Solution:* Upon some initial investigations, we see that if $p \mid n$, then there can't be three numbers on the circle that aren't multiples of $p$. Therefore, for all $p \mid n$, we can make a table of those that *aren't* multiples of $p$.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | | | | | | | | | |
| $p_2$ | | | | | | | | | |
| $p_3$ | | | | | | | | | |
| $p_4$ | | | | | | | | | |
| $p_5$ | | | | | | | | | |
| $p_6$ | | | | | | | | | |
| $\vdots$ | | | | | | | | | |

As we play with putting dots in this table, keep in mind that we probably want to minimize the number of prime factors to keep $n$ minimal. Our goal is to get the most out of each prime factor. Thinking symmetrically, it's easy to get this diagram:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | • | | | | | | | | • |
| $p_2$ | • | • | | | | | | | |
| $p_3$ | | • | • | | | | | | |
| $p_4$ | | | • | • | | | | | |
| $p_5$ | | | | • | • | | | | |
| $p_6$ | | | | | • | • | | | |
| $p_7$ | | | | | | • | • | | |
| $p_8$ | | | | | | | • | • | |
| $p_9$ | | | | | | | | • | • |

Therefore, the set $\frac{n}{2\cdot 3}, \frac{n}{3\cdot 5}, \frac{n}{5\cdot 7}, \ldots$ with $n = 2\cdot 3\cdot\cdots\cdot 23$ satisfies the problem. However, look at what each of these products becomes: $\frac{n}{2\cdot 23}\cdot\frac{n}{3\cdot 5}$ for example has a factor of 2 which is undesirable because it feels rather loose. It suggests that we were being wasteful in our construction, and that we can get more out of each prime factor. Indeed, by allowing ourselves to double up on each prime factor, we come up with the new construction:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | •• | • | | | | | | | • |
| $p_2$ | | • | •• | • | | | | | |
| $p_3$ | | | | • | •• | • | | | |
| $p_4$ | | | | | | • | •• | • | |
| $p_5$ | | | | | | | | • | • |

This does significantly better, giving the set $\frac{n}{2\cdot 2}, \frac{n}{2\cdot 3}, \frac{n}{3\cdot 3}, \frac{n}{3\cdot 5}, \ldots$ with $n = 2^2\cdot 3^2\cdot 5^2\cdot 7^2\cdot 11$.

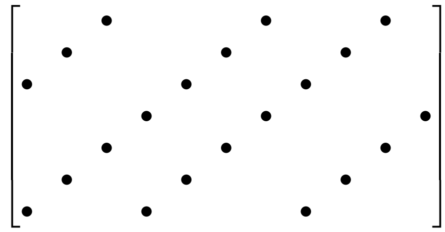
Proving this $n$ is minimal involves formalizing all of our intuitions in the problem. As a brief sketch, prove that $\gcd(a_1, \ldots, a_9) = 1$ by dividing it out, then that $n$ has at least 5 prime factors by a pigeonhole argument on the fact that each $p \mid n$ must appear in at least 7 terms of the circle, and that if $p^3 \mid n$ we can also find a construction for $\frac{n}{p}$. This will prove that $n = 2^2\cdot 3^2\cdot 5^2\cdot 7^2\cdot 11$ is minimal. □

3. (IMO 1977) In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

   *Solution:* Put the first 17 elements in a table

$$\begin{bmatrix} x_1 & x_2 & x_3 & \ldots & x_{11} \\ x_2 & x_3 & x_4 & \ldots & x_{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_7 & x_8 & x_9 & \ldots & x_{17} \end{bmatrix}$$

   Then we realize that the column sums must be all negative, while the row sums must be all positive. To show that 16 is attainable, we need to work as simply as possible. Exploiting the place where our other argument fails, our goal is to alternate between small positive numbers ($a$) and big negative numbers ($b$). Staring at the matrix, we probably want $b$ to appear twice in each column, so we start coloring it symmetrically:



This gives rise to the construction $a, a, b, a, a, a, b, a, a, b, a, a, a, b, a, a$, where any $a, b$ such that $5a + 2b < 0$ and $8a + 3b > 0$ work, such as $(5, -13)$ or $(7, -18)$ or $(8, -21)$. ☐

4. (IMO 2016) A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible value of the positive integer $b$ such that there exists a non-negative integer $a$ for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

*Solution:* Surely $b$ won't be ridiculously large, else we will be dealing with some large prime numbers and we don't know how to deal with large prime numbers. Thus, we should just start at $b = 2$ and go up, hoping we eventually hit the desired $b$.

We will begin by considering what possible prime factors $P(n)$ and $P(n+k)$ can have in common for some small $k$. If $n^2 + n + 1 \equiv 0 \pmod{p}$, then $(n+k)^2 + (n+k) + 1 \equiv 2kn + k^2 + k = k(2n + k + 1) \pmod{p}$. We can either have $k \equiv 0 \pmod{p}$, or $2n + k + 1 \equiv 0 \pmod{p}$. Since $n^2 + n + 1$ is odd, we can write

$$0 \equiv 2n^2 + 2n + 2 = n(2n + k + 1) + (1 - k)n + 2 \equiv (1 - k)n + 2 \pmod{p}$$
$$\implies 0 \equiv (1 - k)(2n) + 4 = (1 - k)(2n + k + 1) + (k - 1)(k + 1) + 4 \equiv k^2 + 3 \pmod{p}$$

Thus, $\gcd(P(n), P(n+1)) = 1$, $\gcd(P(n), P(n+2)) \mid 7$, $\gcd(P(n), P(n+3)) \mid 3$, $\gcd(P(n), P(n+4)) \mid 19$, $\gcd(P(n), P(n+5)) \mid 5 \cdot 7$, etc.

Now time for the pictures. We know it's impossible for $b = 2$ because they are coprime. For $b = 3$, $P(a+2)$ is coprime with both $P(a+1)$ and $P(a+3)$. For $b = 4$, we can construct a diagram where we indicate what numbers have what prime factors, and mark off when they cannot have the factor. For example, it is forced that $7 \mid P(a+2)$ and $P(a+4)$, which means $7 \nmid P(a+1), P(a+3)$, so we can draw this diagram:

| | $P(a+1)$ | $P(a+2)$ | $P(a+3)$ | $P(a+4)$ |
|---|---|---|---|---|
| 3 | | | | |
| 7 | X | O | X | O |

For $b = 5$, we can do the same thing, starting in the center column. It can only be paired with $P(a+1)$ and $P(a+5)$, so we can symmetrically consider just one case.

| | $P(a+1)$ | $P(a+2)$ | $P(a+3)$ | $P(a+4)$ | $P(a+5)$ |
|---|---|---|---|---|---|
| 3 | | | | | |
| 7 | O | X | O | X | X |
| 19 | | | | | |

This forces $P(a+4)$ to be paired with $P(a+1)$.

| | $P(a+1)$ | $P(a+2)$ | $P(a+3)$ | $P(a+4)$ | $P(a+5)$ |
|---|---|---|---|---|---|
| 3 | O | X | X | O | X |
| 7 | O | X | O | X | X |
| 19 | | | | | |

Since we need an $O$ in every column, we need $19 \mid P(a+2)$, which is impossible to pair with anything else. Thus $b \geq 6$. For $b = 6$, we do the same thing, and eventually we will find that the following table works:

| | $P(a+1)$ | $P(a+2)$ | $P(a+3)$ | $P(a+4)$ | $P(a+5)$ | $P(a+6)$ |
|---|---|---|---|---|---|---|
| 3 | O | X | X | O | X | X |
| 5 | | | | | | |
| 7 | X | X | O | X | O | X |
| 19 | X | O | X | X | X | O |

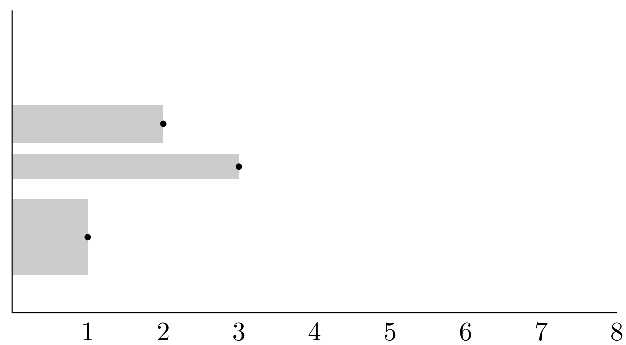
Now that every column has an O, we can use this to find a construction. We need the following congruences to be satisfied:

$$\begin{cases} (a+1)^2 + (a+1) + 1 \equiv 0 \pmod{3} \\ (a+2)^2 + (a+2) + 1 \equiv 0 \pmod{19} \\ (a+3)^2 + (a+3) + 1 \equiv 0 \pmod{7} \\ (a+4)^2 + (a+4) + 1 \equiv 0 \pmod{3} \\ (a+5)^2 + (a+5) + 1 \equiv 0 \pmod{7} \\ (a+6)^2 + (a+6) + 1 \equiv 0 \pmod{19} \end{cases}$$

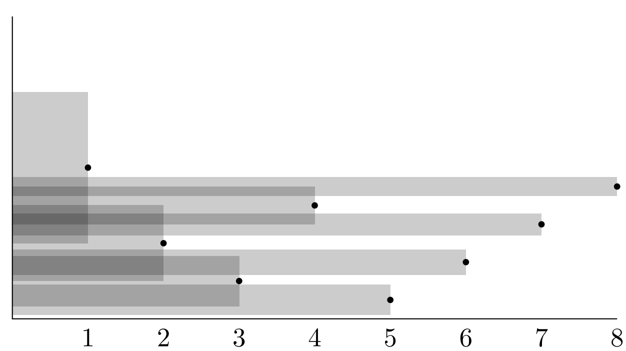
From this we recover $(a, b) = (196, 6)$ as a construction. □

5. (USAMTS 2013) An infinite sequence of positive real numbers $a_1, a_2, a_3, \ldots$ is called *territorial* if for all positive integers $i, j$ with $i < j$, we have $|a_i - a_j| \geq \frac{1}{j}$. Can we find a territorial sequence $a_1, a_2, a_3, \ldots$ for which there exists a real number $c$ with $a_i < c$ for all $i$?

*Solution:* Here we draw our sequence horizontally and represent the $|a_i - a_j| \geq \frac{1}{j}$ condition as a gray bar that we cannot allow new terms to enter, something that looks like this:




Thinking symmetrically, we would ideally want something like this:




□

where we iteratively pack the next terms of the sequence into the cracks in the sequence that we left behind. Indeed, all that remains is a formalization: our territorial sequence will be

$$2,\, 1,\, \frac{1}{2},\, \frac{3}{2},\, \frac{1}{4},\, \frac{3}{4},\, \frac{5}{4},\, \frac{7}{4},\, \frac{1}{8},\, \frac{3}{8},\, \frac{5}{8},\, \frac{7}{8},\, \frac{9}{8},\, \frac{11}{8},\, \frac{13}{8},\, \frac{15}{8},\, \ldots$$

# 4 Greedy Algorithms

Greedy algorithms are useful in examining small cases, since they are the natural idea when playing with a problem.

1. (USAMO 2012) Find all integers $n \geq 3$ such that among any $n$ positive real numbers $a_1, a_2, \ldots, a_n$ with $\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n)$, there exist three that are the side lengths of an acute triangle.

   *Solution:* Add structure by ordering $a_1 \leq a_2 \leq \cdots \leq a_n$. We need to find the biggest possible counter-example, one only containing non-acute angles, i.e., $a_i^2 + a_j^2 \geq a_k^2$ for all $1 \leq i < j < k \leq n$. Everything in this problem is homogeneous so start off with $a_1 = 1$ and $a_2 = x$, so the condition gives $a_n \leq n$.

   Then the smallest we can get for $a_3$ is $\sqrt{a_1^2 + a_2^2} = \sqrt{1 + x^2}$, then for $a_4$ is $\sqrt{a_2^2 + a_3^2} = \sqrt{1 + 2x^2}$, then for $a_5$ is $\sqrt{2 + 3x^2}$, etc. In fact, to minimize everything, $a_2 = 1$ is the best option. So the sequence $\sqrt{F_1}, \sqrt{F_2}, \ldots, \sqrt{F_n}$ is a counterexample as long as $\sqrt{F_n} \leq n$, or $n \leq 12$.

   For $n > 12$, we simply use a pigeonhole argument to show that there must exist some $1 \leq i \leq n - 2$ such that $a_i^2 + a_{i+1}^2 > a_{i+2}^2$. □

2. (IMO 1983) Is it possible to choose 1983 distinct positive integers, all less than or equal to 100,000, no three of which are consecutive terms of an arithmetic progression?

   *Solution:* We greedily choose numbers and find a pattern. After small cases we see $\{1, 2, 4, 5, 10, 11, 13, 14, 28, 29\}$. The sequence jumps at 4, 10, and 28, all of which are 1 more than a power of 3, so perhaps we should look at $\{0, 1, 3, 4, 9, 10, 12, 13, 27, 28\}$ in base 3. Indeed, these all consist of just 0s and 1s in ternary.

   Consider the set $S$ of all numbers $\leq 100,000$ whose ternary representations consist of just 0s and 1s. If any three $x, y, z \in S$ satisfy $y - x = z - y$, or $x + z = 2y$, then each corresponding digit of $x$ and $z$ must match up, so $x = z$. Thus $S$ has no arithmetic progressions. Since $1 + 11111111111_3 < 100,000$, we have $|S| \geq 2^{11} > 1983$. Finally shift each term up by 1 to get a construction of strictly positive integers. □

3. (USAMTS 2014) Find the smallest positive integer $n$ that satisfies the following: We can color each positive integer with one of $n$ colors such that the equation

   $$w + 6x = 2y + 3z$$

   has no solutions in positive integers with all of $w, x, y, z$ the same color.

   *Solution:* It's easy to see that 3 colors do not suffice by drawing a diagram of the following form:

   | | color 1 | color 2 | color 3 |
   |---|---|---|---|
   | elements | | | |
   | non-elements | | | |

   to keep track what colors we are going to assign to each number. Once an element is a non-element of two colors, add it to the third color.

   We start by putting 1 as an element of color 1. By $(1, 1, 2, 1)$, 2 must have a different color, so put it in color 2. By $(3, 2, 3, 3)$ and $(3, 1, 3, 1)$, 3 must have a different color from both so put it in color 3. After filling out this table using the sequence of solutions $(6, 2, 6, 2)$, $(3, 3, 6, 3)$, $(9, 6, 9, 9)$, $(9, 3, 9, 3)$, $(6, 4, 6, 6)$, $(2, 2, 4, 2)$, $(6, 6, 12, 6)$, and $(12, 4, 12, 4)$, we get the following table:

   | | color 1 | color 2 | color 3 |
   |---|---|---|---|
   | elements | 1, 6 | 2, 9, 12 | 3, 4 |
   | non-elements | ~~3~~, ~~9~~, ~~4~~, ~~12~~ | ~~3~~, ~~6~~, ~~4~~ | ~~6~~, ~~9~~, ~~12~~ |

which is a contradiction since $(12, 2, 9, 2)$ was monochromatically colored. By the way we disproved 3, it seems basically impossible to disprove anything greater than 3, so let's set out to establish a minimum of 4.

We will use another table, this time with a more structured process. Rewriting the equation as $3(2x - z) = 2y - w$, we just want to keep track of all the values $2a - b$ for each color. Note that we disallow $2a - b = 0$ in any box. Finally, we will add elements to the first color that doesn't create a solution:

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| elements | | | | |
| $2a - b$ values | | | | |

For example, we put 1 in $c_1$ and add $2 \cdot 1 - 1 = 1$ to the bottom row. We then cannot add 2 to $c_1$ since $2 \cdot 1 - 2 = 0$, so put 2 in $c_2$ and add $2 \cdot 2 - 2 = 2$ to the bottom row. We cannot add 3 to $c_1$ since $2 \cdot 3 - 3 = 3 = 3 \cdot 1$. We cannot add it to $c_2$ since it would add $2 \cdot 2 - 3 = 1$ and $2 \cdot 3 - 3 = 3$. Thus add it to $c_3$. Repeat this process many times and we will obtain the following table:

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| elements | 1, 4, 7, 9, 10, 13, 16, 19 | 2, 5, 8, 11, 14, 17, 18, 20 | 3, 12 | 6, 15 |
| $2a - b$ values | 1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 29, 31, 34, 37 | 2, 4, 5, 8, 10, 11, 14, 16, 17, 18, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 38 | 3, 12, 21 | 6, 15, 24 |

Now the pattern is clear: $c_1 = \{3^{2a}(3b+1) \mid a, b \geq 0\}$, $c_2 = \{3^{2a}(3b+2) \mid a, b \geq 0\}$, $c_3 = \{3^{2a+1}(3b+1) \mid a, b \geq 0\}$, and $c_4 = \{3^{2a+1}(3b+2) \mid a, b \geq 0\}$. Proving this is an exercise in modular arithmetic that I will leave to you. $\square$

# 5 Miscellaneous

In the following problems, try to build intuition about the problem and try to find symmetries.

1. (USAMO 2006) For a given positive integer $k$ find, in terms of $k$, the minimum value of $N$ for which there is a set of $2k + 1$ distinct positive integers that has sum greater than $N$ but every subset of size $k$ has sum at most $\frac{N}{2}$.

   *Solution:* Let's add structure to the problem by considering the numbers written in ascending order: $a_1 < a_2 < \cdots < a_{2k+1}$. Then we only need to worry about $a_{k+2} + \cdots + a_{2k+1} \leq \frac{N}{2}$ since it has the largest sum among all subsets of size $k$. To make the total sum as large as possible, take $(a_1, \ldots, a_{k+1}) = (a_{k+2} - (k+1), \ldots, a_{k+2} - 2, a_{k+2} - 1)$. To make $N$ as small as possible, take $(a_{k+3}, \ldots, a_{2k+1}) = (a_{k+2} + 1, \ldots, a_{k+2} + (k-1))$. That gives
   $$N + 1 \leq a_1 + \cdots + a_{2k+1} = (2k+1)(a_{k+2} - 1) \text{ and } ka_{k+2} + \frac{k(k-1)}{2} \leq \frac{N}{2}$$
   so we need
   $$(2k+1)(a_{k+2} - 1) - 1 \leq 2ka_{k+2} + k(k-1) \implies a_{k+2} \leq k^2 + k + 2$$
   Finally, we get $N \geq 2k^3 + 3k^2 + 3k$ with equality at the construction $\{k^2 + 1, k^2 + 2, \ldots, k^2 + 2k + 1\}$. $\square$

2. (USAMO 2011) Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing $2^{2^n}$ by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

   *Solution:* Let's see what happens when we try to prove the assertion. The first thing to do is to start reducing $2^{2^n} \pmod{2^n - 1}$:
   $$2^{2^n} \equiv 2^{2^n} - (2^n - 1)2^{2^n - n} = 2^{2^n - n} \equiv 2^{2^n - 2n} \equiv \cdots \equiv 2^{2^n \pmod{n}} \pmod{2^n - 1}$$
   Furthermore, $2^{2^n \pmod{n}} < 2^n$ so it is in fact the remainder that we're looking for.

   To finish the problem, we want to show that $2^n \pmod{n}$ is always even. We start by trying some small values of $n$. Observation 1: if $n$ is even, then of course $2^n \pmod{n}$ is even. Observation 2: if $p$ is prime, then $2^p \equiv 2$

(mod $p$), which is even. Now let's see what happens with the other numbers, like $n = 9, 15, 25, 27, 33, \ldots$. $2^9 \equiv 8 \pmod 9$, which is even; $2^{15} \equiv 8 \pmod{15}$, which is even; but alas $2^{25} \equiv 7 \pmod{25}$. We have found our counterexample, $n = 25$, as $2^{2^{25}} \equiv 2^7 \pmod{2^{25}-1}$. □

3. (USAMO 2014) Prove that there exists an infinite set of points

$$\ldots, P_{-3}, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, \ldots$$

in the plane with the following property: For any three distinct integers $a$, $b$, and $c$, points $P_a$, $P_b$, and $P_c$ are collinear if and only if $a + b + c = 2014$.

*Solution:* One idea is to parametrize these points: let $P_t = (f(t), g(t))$, and we must have

$$\frac{g(b)-g(a)}{f(b)-f(a)} = \frac{g(c)-g(b)}{f(c)-f(b)} \iff (a-b)(b-c)(c-a) \neq 0 \text{ and } a+b+c = 2014$$

by comparing slopes. Also, when $(a-b)(b-c)(c-a) = 0$, it would make sense that $P_a, P_b, P_c$ are collinear in the degenerate sense. Therefore, we want

$$\begin{aligned}(a-b)(b-c)(c-a)(a+b+c-2014) \mid\ & (g(b)-g(a))(f(c)-f(b)) - (g(c)-g(b))(f(b)-f(a)) \\ &= g(a)(f(b)-f(c)) + g(b)(f(c)-f(a)) + g(c)(f(a)-f(b))\end{aligned}$$

To find suitable $f, g$, all we need to do is expand the left-hand side. Indeed, upon realizing

$$(a-b)(b-c)(c-a)(a+b+c-2014) = \sum_{cyc}(2014a^2 - a^3)(b-c)$$

we find that $P_t = (t, 2014t^2 - t^3)$ is a construction for the problem. □

4. (RMM 2016) A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where $b$, $c$, and $d$ are integer constants and $n$ ranges over all integers, including negative integers.

   (a) Show that there exists a cubic sequence such that the only terms of the sequence which are squares of integers are $a_{2015}$ and $a_{2016}$.

   (b) Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence satisfying the condition in part (a).

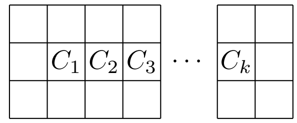
*Solution:* Only the first part is relevant to this article. We solve the problem instead with $a_0$ and $a_1$ because we can just translate it back (plus, this makes our lives easier). One idea is to make $a_0 = 0$, so $d = 0$. We now need $1 + b + c$ to be a square, and $n(n^2 + bn + c)$ not a square for all $n \neq 0, 1$. Judging by the second part of the problem, they probably only care that one of the two is 0, so it's likely that many values of $a_1$ work.

Let's start with $a_1 = 1$, so $b = -c$ and look at $n(n^2 - cn + c)$. There is likely a non-absurd value of $c$ that makes this work. Immediately when we take $c = 1$, we can see that $n(n^2 - n + 1)$ will work since $\gcd(n, n^2 - n + 1) = 1$ and the $n^2 - n + 1$ factor is never a square when $n \neq 0, 1$. This translates to the construction, for $a_{2015}$ and $a_{2016}$, $a_n = (n-2015)^3 - (n-2015)^2 + (n-2015)$.

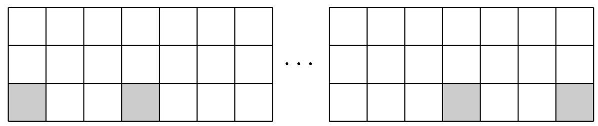
For the second part, research elliptic curves and the group law. This is basically Vieta jumping for cubics rather than quadratics. □

5. (OMMC 2014) Each cell is painted white in an $m \times n$ grid. To *swap the color* of a cell is to paint it black if it is white and paint it white if it is black. For each of the $(m-2)(n-2)$ $3 \times 3$ squares in the grid, we pick two of its sides and swap the colors of each of the three cells on that side, swapping twice if it lies on both of the chosen edges. Find the smallest integer $M$ such that for any $m \times n$ grid, it is possible to end up with at most $M$ black cells.

*Solution:* First we see that for a $4 \times 4$ square, we can make it so that they are all white at the end. Therefore, for even $m, n$ we can always reduce it to 0. If one of $m, n$ is even and the other is odd, then it reduces to just one row:




Call each of the four moves by B (bottom), R (right), T (top), L (left). We can perform BR on $C_1$ and $C_2$, LR on $C_i$ for $3 \le i \le k-2$, and BL on $C_{k-1}$ and $C_k$ to get:



We can do a similar construction for $m, n$ both odd, forming a big L. The answer is $M = 4$. □

# 6 Problems

1. (USAMO 1990) A certain state issues license plates consisting of six digits (from 0 to 9). The state requires that any two license plates differ in at least two places. (For instance, the numbers 027592 and 020592 cannot both be used.) Determine, with proof, the maximum number of distinct license plates that the state can use.

2. (USAMO 1986)

   (a) Do there exist 14 consecutive positive integers each of which is divisible by one or more primes $p$ from the interval $2 \le p \le 11$?

   (b) Do there exist 21 consecutive positive integers each of which is divisible by one or more primes $p$ from the interval $2 \le p \le 13$?

3. (USAMO 1980) Determine the maximum number of three-term arithmetic progressions which can be chosen from a sequence of $n$ real numbers.

4. (USAMO 2008) Prove that for each positive integer $n$, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \ldots k_n - 1$ is the product of two consecutive integers.

5. (USAMO 2000) Find the smallest positive integer $n$ such that if $n$ squares are colored of a $1000 \times 1000$ chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

6. (USAMO 1998) A computer screen shows a $98 \times 98$ chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

7. (IMO 1997) An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the $i$-th row and the $i$-th column together contain all elements of $S$. Show that:

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of $n$.

8. (USAMO 2009) Let $n$ be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \ldots, n-1, n\}$ which does not contain three elements $a, b, c$ (not necessarily distinct) satisfying $a + b + c = 0$.

9. (USAMO 2007) An *animal* with $n$ cells is a connected figure consisting of $n$ equal-sized cells. A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* if it cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

10. (EGMO 2012) A set $A$ of integers is called *sum-full* if $A \subseteq A + A$, i.e. each element $a \in A$ is the sum of some pair of (not necessarily different) elements $b, c \in A$. A set $A$ of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of the elements of a finite nonempty subset of $A$. Does there exist a sum-full zero-sum-free set of integers?

11. (IMO 2014) Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of $n^2$ unit squares. A configuration of $n$ rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer $k$ such that, for each peaceful configuration of $n$ rooks, there is a $k \times k$ square which does not contain a rook on any of its $k^2$ unit squares.

12. (EGMO 2013) Let $n$ be a positive integer.

    (a) Prove that there exists a set $S$ of $6n$ pairwise different positive integers, such that the least common multiple of any two elements of $S$ is no larger than $32n^2$.

    (b) Prove that every set $T$ of $6n$ pairwise different positive integers contains two elements the least common multiple of which is larger than $9n^2$.

13. (RMM 2011) Prove that there exist two functions $f, g : \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

14. (IMO 1993) Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Determine if there exists a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$ with the following properties:

    (a) $f(1) = 2$;

    (b) $f(f(n)) = f(n) + n$, $(n \in \mathbb{N})$.

15. (EGMO 2016) Let $k$ and $n$ be integers such that $k \geq 2$ and $k \leq n \leq 2k - 1$. Place rectangular tiles, each of size $1 \times k$, or $k \times 1$ on a $n \times n$ chessboard so that each tile covers exactly $k$ cells and no two tiles overlap. Do this until no further tile can be placed in this way. For each such $k$ and $n$, determine the minimum number of tiles that such an arrangement may contain.

16. (IMO 1999) Let $n$ be an even positive integer. We say that two different cells of an $n \times n$ board are *neighboring* if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

17. (IMO 1994) For any positive integer $k$, let $f_k$ be the number of elements in the set $\{k+1, k+2, \dots, 2k\}$ whose base 2 representation contains exactly three 1s.

    (a) Prove that for any positive integer $m$, there exists at least one positive integer $k$ such that $f(k) = m$.

    (b) Determine all positive integers $m$ for which there exists *exactly one* $k$ with $f(k) = m$.

18. (EGMO 2016) Let $m$ be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are coloured blue, such that every cell is related to at lest two blue cells. Determine the minimum number of blue cells.

19. (IMC 2013) Does there exist an infinite set $M$ consisting of positive integers such that for any $a, b \in M$, with $a < b$, the sum $a + b$ is square-free?