MEASURE REDUCIBILITY OF COUNTABLE BOREL EQUIVALENCE RELATIONS

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ABSTRACT. We show that every basis for the countable Borel equivalence relations strictly above $\mathbb{E}_0$ under measure reducibility is uncountable, thereby ruling out natural generalizations of the Glimm-Effros dichotomy. We also push many known results concerning the abstract structure of the measure reducibility hierarchy to its base, using arguments substantially simpler than those previously employed.

Introduction

Over the last few decades, the notion of Borel reducibility of equivalence relations has been used to identify obstacles of definability inherent in classification problems throughout mathematics. While there are far too many such applications to provide an exhaustive list here, a few notable examples include the classifications of torsion-free abelian groups [Hjo99, AK00, Tho03, Tho06], ergodic measure-preserving transformations [Hjo01, FW04, FRW06, FRW11], separable Banach spaces [FLR09, Ros11], and separable C^*-algebras [FTT13a, FTT13b, Sab]. In order to better understand such results, one must obtain insight into the abstract structure of the Borel reducibility hierarchy. Unfortunately, this has turned out to be a very difficult task.

The first of the two main lines of research into the abstract structure of the Borel reducibility hierarchy concerns its base. The first such result appeared in [Sil80], where it was shown that equality on $\mathbb{R}$ is the immediate successor of equality on $\mathbb{N}$ within the co-analytic equivalence relations. Building upon this and operator-algebraic work in [Gli61, Eff65], it was shown in [HKL90] that the relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$, given by $x \mathbb{E}_0 y \iff \exists n \in \mathbb{N} \forall m \geq n \; x(m) = y(m)$, is the immediate

successor of equality on $\mathbb{R}$ within the Borel equivalence relations. Work in this direction stalled shortly thereafter, with [KL97, Theorem 2] ruling out further such results within the Borel equivalence relations. However, the question of whether there are further such results within the countable Borel equivalence relations remains open.

The first of the two main goals of this paper is to show that every basis for the countable Borel equivalence relations strictly above $\mathbb{E}_0$ under measure reducibility is uncountable.

The second of the two main lines of research into the abstract structure of the Borel reducibility hierarchy concerns exotic properties appearing beyond its base. The first such result, due originally to Woodin and later refined in [LV94], was the existence of uncountable families of pairwise incomparable Borel equivalence relations. However, the underlying arguments depended heavily upon Baire category techniques, and [HK96, Theorem 6.2] ensures that such an approach cannot yield incomparability of countable Borel equivalence relations.

This difficulty was eventually overcome in [AK00], yielding the existence of uncountable families of pairwise incomparable countable Borel equivalence relations, in addition to myriad further results concerning the complexity of the Borel reducibility hierarchy. The arguments behind these theorems marked a sharp departure from earlier approaches, relying upon sophisticated supperrigidity machinery for actions of linear algebraic groups.

Soon thereafter, similar techniques were used in [Ada02, Tho02] to obtain many striking new properties of the Borel reducibility hierarchy, such as the existence of countable Borel equivalence relations E to which the disjoint union of two copies of E is not Borel reducible. While many of the underlying arguments were later simplified in [HK05], even these refinements depended upon complex rigidity phenomena. And while the still simpler arguments of [Hjo12] gave rise to pairwise incomparable treeable countable Borel equivalence relations, they still gave little sense of how far one must travel beyond the base of the Borel reducibility hierarchy before encountering such extraordinary behavior.

The second of the two main goals of this paper is to show that such phenomena appear just beyond $\mathbb{E}_0$ under measure reducibility.

We obtain our results by introducing a measureless notion of rigidity, which we establish directly for the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$. In the presence of a measure, this yields strong separability properties of the induced orbit equivalence relation. Many of our results follow rather easily from the latter, while others require an additional graph-theoretic stratification theorem, also established via elementary methods.

Basic notions

A set X is *countable* if there is an injection $\phi\colon X \to \mathbb{N}$. A sequence $(X_r)_{r \in \mathbb{R}}$ of sets is *increasing* if $X_r \subseteq X_s$ for all real numbers $r \le s$.

Suppose that X and Y are standard Borel spaces. We say that a sequence $(x_y)_{y \in Y}$ of points of X is *Borel* if $\{(x_y, y) \mid y \in Y\}$ is a Borel subset of $X \times Y$, and more generally, a sequence $(X_y)_{y \in Y}$ of subsets of X is *Borel* if $\{(x, y) \mid x \in X_y \text{ and } y \in Y\}$ is a Borel subset of $X \times Y$.

Suppose that E is a Borel equivalence relation on X. We say that E is *aperiodic* if all of its classes are infinite, E is *countable* if all of its classes are countable, and E is *finite* if all of its classes are finite. A *subequivalence relation* of E is a subset of E that is an equivalence relation on X. The *E-saturation* of a set $W \subseteq X$, or $[W]_E$, is the smallest E-invariant set containing W. The *orbit equivalence relation* induced by an action of a group Γ on X is the equivalence relation on X given by $x \mathrel{E_\Gamma^X} y \iff \exists \gamma \in \Gamma \; \gamma \cdot x = y$.

Suppose that F is a Borel equivalence relation on Y. A *homomorphism* from E to F is a function $\phi\colon X \to Y$ sending E-equivalent points to F-equivalent points, a *reduction* of E to F is a homomorphism sending E-inequivalent points to F-inequivalent points, and an *embedding* of E into F is an injective reduction.

A *graph* on X is an irreflexive symmetric set $G \subseteq X \times X$. A *path* through G is a sequence $(x_i)_{i \le n}$ with the property that $\forall i < n \; x_i \mathrel{G} x_{i+1}$, in which case n is the *length* of the path. A graph is *acyclic* if there is at most one injective path between any two points. We say that G is a *graphing* of E if E is the smallest equivalence relation on X containing G. When G is acyclic, we also say that G is a *treeing* of E. We say that E is *treeable* if there is a Borel treeing of E.

Suppose that μ is a Borel measure on X. We say that μ is *E-ergodic* if every E-invariant Borel set is μ-null or μ-conull, μ is *E-invariant* if $\mu(B) = \mu(T(B))$ for all Borel sets $B \subseteq X$ and all Borel injections $T\colon B \to X$ whose graphs are contained in E, and μ is *E-quasi-invariant* if the E-saturation of every μ-null set is μ-null.

We say that E is *μ-nowhere reducible* to F if there is no μ-positive Borel set $B \subseteq X$ for which $E \restriction B$ is Borel reducible to F, E is *μ-reducible* to F if there is a μ-conull Borel set $C \subseteq X$ for which $E \restriction C$ is Borel reducible to F, E is *invariant-measure reducible* to F if $E \restriction B$ is μ-reducible to F for every Borel set $B \subseteq X$ and every $(E \restriction B)$-invariant Borel probability measure μ on B, and E is *measure reducible* to F if E is μ-reducible to F for every Borel probability measure μ on X. The corresponding notions of *invariant-measure embeddability* and *measure embeddability* are defined analogously. It is straightforward

to check that invariant-measure embeddability, measure embeddability, and measure reducibility are transitive (and only marginally more difficult to check that invariant-measure reducibility is transitive).

We say that E is *hyperfinite* if it is a union of an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of finite Borel subequivalence relations, E is *μ-nowhere hyperfinite* if there is no μ-positive Borel set $B \subseteq X$ for which $E \restriction B$ is hyperfinite, E is *μ-hyperfinite* if there is a μ-conull Borel set $C \subseteq X$ for which $E \restriction C$ is hyperfinite, E is *invariant-measure hyperfinite* if $E \restriction B$ is μ-hyperfinite for every Borel set $B \subseteq X$ and every $(E \restriction B)$-invariant Borel probability measure μ on B, and E is *measure hyperfinite* if E is μ-hyperfinite for every Borel probability measure μ on X. As a countable Borel equivalence relation is hyperfinite if and only if it is Borel reducible to $\mathbb{E}_0$ (see Theorem 1.3.8), it immediately follows that a countable Borel equivalence relation is invariant-measure hyperfinite if and only if it is invariant-measure reducible to $\mathbb{E}_0$, and measure hyperfinite if and only if it is measure reducible to $\mathbb{E}_0$.

Bases

A *quasi-order* on Q is a reflexive transitive binary relation $\le$ on Q. A *basis* for Q under $\le$ is a set $B \subseteq Q$ such that $\forall q \in Q \exists b \in B\ b \le q$.

Here we seek to elucidate the extent to which measure theory can shed light on the structure of the Borel reducibility hierarchy just beyond $\mathbb{E}_0$. But given our limited knowledge of the structure of the hierarchy, the appropriate meaning of "just beyond" is not entirely clear. We will focus on properties that hold of some relation in every basis for the non-measure-hyperfinite countable Borel equivalence relations under measure reducibility. One should first strive to understand the structure of such bases, the original motivation for this paper.

Theorem A. *Every basis for the non-measure-hyperfinite countable Borel equivalence relations under measure reducibility is uncountable.*

Separability

Although we will later give a somewhat different definition, for the sake of the introduction we will say that F is *projectively separable* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X for which E is μ-nowhere hyperfinite, there is a Borel set $R \subseteq X \times Y$, whose vertical sections are countable, such that $\mu(\{x \in B \mid \neg x\ R\ \phi(x)\}) = 0$ for every Borel set $B \subseteq X$ and every countable-to-one Borel homomorphism $\phi\colon B \to Y$ from $E \restriction B$ to F. It is easy to see that measure-hyperfinite Borel equivalence relations are projectively separable.

Recall that $\mathrm{SL}_2(\mathbb{Z})$ is the group of all two-by-two matrices with integer entries and determinant one. The natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2$ factors over $\mathbb{Z}^2$ to an action of $\mathrm{SL}_2(\mathbb{Z})$ on the quotient space $\mathbb{T}^2$. It is well-known that the orbit equivalence relation induced by this action is not measure hyperfinite, although it is treeable (see Propositions 1.8.2 and 1.8.3). Our primary new tool here is the following.

Theorem B. *The orbit equivalence relation induced by the action of* $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathbb{T}^2$ *is projectively separable.*

We obtain Theorem A by showing that if E is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation, then every basis for the non-measure-hyperfinite Borel subequivalence relations of E under measure reducibility is uncountable.

Ultimately, one would like to have the analogous result for bases for the non-measure-hyperfinite countable Borel equivalence relations measure reducible to E. We show that E is a counterexample if and only if it is a countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility. While the existence of such successors remains open, we show that if there are any at all, then there are uncountably many.

As projective separability and treeability are closed downward under Borel reducibility, every basis for the non-measure-hyperfinite countable Borel equivalence relations under measure reducibility contains a relation whose restriction to some Borel set is not measure hyperfinite, but is projectively separable and treeable. In particular, if we wish to prove that every such basis contains a relation whose restriction to some Borel set has a given property, then it is sufficient to show that the property holds of every non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation.

ANTICHAINS

The existence of a Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible treeable countable Borel equivalence relations was first established in [Hjo12, Theorem 1.1]. In light of the above observations, the following yields a simple new proof of this result, while simultaneously pushing it to the base of the reducibility hierarchy.

Theorem C. *Suppose that* X *is a standard Borel space and* E *is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on* X. *Then there is a Borel sequence* $(E_r)_{r \in \mathbb{R}}$ *of pairwise non-measure-reducible subequivalence relations of* E.

As with our anti-basis theorem, one would like to have the analogous result in which each E_r is measure reducible to E, rather than contained

in E. We show that E is a counterexample if and only if there is a finite family $\mathscr{F}$ of successors of $\mathbb{E}_0$ under measure reducibility for which E is a countable disjoint union of countable Borel equivalence relations measure bi-reducible with those in $\mathscr{F}$.

In particular, it follows that the existence of a sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise non-measure-reducible countable Borel equivalence relations measure reducible to E is equivalent to the existence of a Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible countable equivalence relations measure reducible to E. Moreover, the nonexistence of such sequences implies the stronger fact that every sequence $(E_n)_{n \in \mathbb{N}}$ of countable Borel equivalence relations measure reducible to E contains an infinite subsequence that is increasing under measure reducibility.

Complexity

In [AK00], the existence of perfect families of pairwise incomparable countable Borel equivalence relations with distinguished ergodic Borel probability measures was used to establish a host of complexity results. We obtain simple new proofs of these results, while simultaneously pushing them to the base of the reducibility hierarchy, by establishing the following strengthening of Theorem C.

Theorem D. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X. Then there are Borel sequences $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of E and $(\mu_r)_{r \in \mathbb{R}}$ of Borel probability measures on X such that:*

(1) *Each μ_r is E_r-ergodic and E_r-quasi-invariant.*
(2) *For all distinct $r, s \in \mathbb{R}$, the relation E_r is μ_r-nowhere reducible to the relation E_s.*

While this result is somewhat technical, the complexity results of [AK00] are all obtained as abstract consequences of its conclusion.

Again, one would like to have the analog for which each E_r is measure reducible to E, rather than contained in E. We show that E is a counterexample if and only if it is a countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility.

Products

The existence of non-measure-hyperfinite treeable countable Borel equivalence relations which do not measure reduce every treeable countable Borel equivalence relation was originally established in [Hjo08, Theorem 1.6]. Identify $E \times F$ with the equivalence relation on $X \times Y$

given by $(x_1, y_1)\ (E \times F)\ (x_2, y_2) \iff (x_1\ E\ x_2$ and $y_1\ F\ y_2)$, and let $\Delta(X)$ denote the diagonal on $X \times X$. The following yields a simple new proof of the aforementioned result.

Theorem E. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable countable Borel equivalence relation on X. Then $E \times \Delta(\mathbb{R})$ is not measure reducible to a Borel subequivalence relation of E.*

In [Tho02, Theorem 3.3a], the rigidity results behind [AK00] were used to establish the existence of countable Borel equivalence relations E with the property that for no $n \in \mathbb{N}$ is $E \times \Delta(n+1)$ measure reducible to $E \times \Delta(n)$. While there are non-measure-hyperfinite projectively-separable countable Borel equivalence relations that do not have this property, in light of our observations on bases, the following yields a simple new proof of this result, while simultaneously pushing it to the base of the reducibility hierarchy.

Theorem F. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable countable Borel equivalence relation on X. Then there is a Borel set $B \subseteq X$ such that for no $n \in \mathbb{N}$ is $(E \restriction B) \times \Delta(n+1)$ measure reducible to $(E \restriction B) \times \Delta(n)$.*

CONTAINMENT VERSUS REDUCIBILITY

In [Ada02], the rigidity results behind [AK00] were used to establish the existence of countable Borel equivalence relations $E \subseteq F$ on the same space such that E is not measure reducible to F. This was strengthened by the proof of [Hjo12, Theorem 1.1], which actually provided an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible treeable countable equivalence relations on the same space. In light of our observations on bases, the following yields a simple new proof of this fact, while simultaneously pushing it to the base of the reducibility hierarchy.

Theorem G. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X. Then there is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E.*

EMBEDDABILITY VERSUS REDUCIBILITY

In [Tho02, Theorem 3.3b], the rigidity results behind [AK00] were used to establish the existence of aperiodic countable Borel equivalence

relations E and F for which E is Borel reducible to F, but E is not measure embeddable into F. In fact, such examples were produced with $E = F \times I(2)$, where $I(X) = X \times X$.

If E is invariant-measure hyperfinite and F is aperiodic and countable, then E is measure reducible to F if and only if E is measure embeddable into F (see Proposition 3.2.1). In particular, if E is aperiodic and invariant-measure hyperfinite, then $E \times I(\mathbb{N})$ is measure embeddable into E. In light of our observations on bases, the following yields a simple new proof of the aforementioned result, while simultaneously pushing it to the base of the reducibility hierarchy.

Theorem H. *Suppose that X is a standard Borel space and E is an aperiodic non-invariant-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X. Then there is an aperiodic Borel subequivalence relation F of E with the property that for no $n \in \mathbb{N}$ is $F \times I(n+1)$ measure embeddable into $F \times I(n)$.*

REFINEMENTS

We have taken great care to state our results in forms which make both the theorems and the underlying arguments as clear as possible. Nevertheless, by utilizing several additional ideas, one can obtain many generalizations and strengthenings.

In particular, by establishing analogs of our results for orbit equivalence relations induced by free Borel actions of countable discrete non-abelian free groups, one can rule out strong dynamical forms of the von Neumann conjecture, while simultaneously providing an elementary proof of the existence of continuum-many pairwise incomparable such relations, as found, for example, in [GP05]. Moreover, as the notion of comparison we consider is far weaker than those typically appearing in ergodic theory, our results are correspondingly stronger.

One can also obtain similar results for substantial weakenings of measure reducibility, as well as for broader classes of equivalence relations. We plan to explore such developments in future papers.

CONTENTS

Part 1. **Preliminaries**

We assume familiarity with the basic results and terminology of descriptive set theory, as found in [Kec95]. We provide here all additional standard definitions and previously known results utilized throughout the paper. Although we mainly give references to the relevant arguments, we provide proofs when they are particularly short or difficult to find in the literature. For the sake of simplicity, we assume the axiom of choice throughout. However, with only one slight exception (see §3.4), our results go through with only the axiom of dependent choice.

1.1. Borel equivalence relations

A *partial transversal* of an equivalence relation is a subset of its domain intersecting every equivalence class in at most one point.

Theorem 1.1.1 (Silver). *Suppose that X is a Polish space and E is a co-analytic equivalence relation on X. Then exactly one of the following holds:*

(1) *The relation E has only countably-many classes.*

(2) *There is a continuous injection of $2^{\mathbb{N}}$ into a partial transversal of E.*

Proof. See [Sil80]. ⊠

We say that E is *smooth* if it is Borel reducible to equality on a standard Borel space. The following fact ensures that $\mathbb{E}_0$ is the minimal non-smooth Borel equivalence relation under Borel reducibility.

Theorem 1.1.2 (Harrington-Kechris-Louveau). *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then exactly one of the following holds:*

(1) *The relation E is smooth.*
(2) *There is a continuous embedding of $\mathbb{E}_0$ into E.*

Proof. See [HKL90, Theorem 1.1]. ⊠

1.2. Countable Borel equivalence relations

We begin by considering smoothness in the presence of countability.

Proposition 1.2.1. *Suppose that X is a standard Borel space and E is a finite Borel equivalence relation on X. Then E is smooth.*

Proof. By the isomorphism theorem for standard Borel spaces (see, for example, [Kec95, Theorem 15.6]), there is a Borel linear ordering $\leq$ of X. But then the Lusin-Novikov uniformization theorem (see, for example, [Kec95, Theorem 18.10]) ensures that the function $\phi \colon X \to X$, given by $\phi(x) = \min_{\leq}[x]_E$, is a Borel reduction of E to equality. ⊠

Remark 1.2.2. We say that a subset of X is *E-complete* if it intersects every E-class. A *selector* for E is a reduction of E to equality on X whose graph is contained in E, and a *transversal* of E is an E-complete partial transversal of E. Although the above argument actually yields the apparently stronger fact that every finite Borel equivalence relation has a Borel selector, the Lusin-Novikov uniformization theorem implies that if E is countable, then smoothness, the existence of a Borel selector, the existence of a Borel transversal, and the existence of a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel partial transversals are all equivalent. Moreover, in the special case that E is aperiodic, they are also equivalent to the existence of a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel transversals.

Proposition 1.2.3. *Suppose that X is a standard Borel space and E is a smooth countable Borel equivalence relation on X. Then every Borel subequivalence relation of E is smooth.*

Proof. By Remark 1.2.2, there is a partition of X into countably-many Borel partial transversals of E. As every partial transversal of E is a partial transversal of all of its subequivalence relations, one more application of Remark 1.2.2 yields the desired result. ⊠

A function $I \colon X \to X$ is an *involution* if $I^2 = \mathrm{id}$.

Theorem 1.2.4 (Feldman-Moore). *Suppose that X is a standard Borel space and $R \subseteq X \times X$ is a reflexive symmetric Borel set whose vertical sections are all countable. Then there are Borel involutions $I_n \colon X \to X$ with the property that $R = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(I_n)$.*

Proof. This follows from the proof of [FM77, Theorem 1]. ⊠

The following can be viewed as generalizations of Rokhlin's Lemma.

Proposition 1.2.5 (Slaman-Steel). *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X. Then there is a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of E-complete Borel subsets of X with empty intersection.*

Proof. By the isomorphism theorem for standard Borel spaces, we can assume that $X = 2^{\mathbb{N}}$. For each $n \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$, let $s_n(x)$ be the lexicographically least $s \in 2^n$ for which $\mathcal{N}_s \cap [x]_E$ is infinite. The Lusin-Novikov uniformization theorem ensures that each of the functions s_n is Borel, thus so too is each of the sets $A_n = \{x \in 2^{\mathbb{N}} \mid s_n(x) \sqsubseteq x\}$. It follows that the sets $B_n = A_n \setminus \bigcap_{n \in \mathbb{N}} A_n$ are as desired. ⊠

Proposition 1.2.6. *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X. Then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint E-complete Borel subsets of X.*

Proof. By Proposition 1.2.5, there is a decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of E-complete Borel subsets of X with empty intersection. Recursively define functions $k_n \colon X \to \mathbb{N}$ by first setting $k_0(x) = 0$, and then defining $k_{n+1}(x) = \min\{k \in \mathbb{N} \mid (A_{k_n(x)} \setminus A_k) \cap [x]_E \neq \emptyset\}$. The Lusin-Novikov uniformization theorem ensures that these functions are Borel, so the sets $B_n = \{x \in X \mid x \in A_{k_n(x)} \setminus A_{k_{n+1}(x)}\}$ are as desired. ⊠

1.3. Hyperfiniteness

We begin with the most basic properties of hyperfiniteness.

Proposition 1.3.1 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and E is a Borel equivalence relation on X. Then the family of Borel sets on which E is hyperfinite is closed under countable unions.*

Proof. See, for example, [DJK94, Proposition 5.2].   ⊠

Proposition 1.3.2 (Dougherty-Jackson-Kechris). *The family of hyperfinite Borel equivalence relations is closed downward under countable-to-one Borel homomorphism.*

Proof. This follows, for example, from [DJK94, Proposition 5.2].   ⊠

Proposition 1.3.3 (Jackson-Kechris-Louveau). *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X. Then there is an aperiodic hyperfinite Borel subequivalence relation F of E.*

Proof. See, for example, [JKL02, Lemma 3.25].   ⊠

We say that a countable discrete group Γ is *hyperfinite* if whenever X is a standard Borel space and $\Gamma \curvearrowright X$ is a Borel action, the induced orbit equivalence relation E_Γ^X is hyperfinite.

Proposition 1.3.4 (Slaman-Steel, Weiss). *The group $\mathbb{Z}$ is hyperfinite.*

Proof. See, for example, [SS88, Lemma 1].   ⊠

We say that E is *hypersmooth* if it is the union of an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of smooth Borel subequivalence relations.

Theorem 1.3.5 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and E is a hypersmooth countable Borel equivalence relation on X. Then E is hyperfinite.*

Proof. See, for example, the beginning of [DJK94, §8].   ⊠

The *tail equivalence relation* induced by a function $T \colon X \to X$ is the equivalence relation on X given by

$$x \; E_t(T) \; y \iff \exists m, n \in \mathbb{N} \; T^m(x) = T^n(y).$$

Theorem 1.3.6 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and $T \colon X \to X$ is Borel. Then $E_t(T)$ is hypersmooth.*

Proof. See, for example, [DJK94, Theorem 8.1].   ⊠

We now mention several further facts concerning reducibility.

Theorem 1.3.7 (Dougherty-Jackson-Kechris). *All hyperfinite Borel equivalence relations on standard Borel spaces are Borel embeddable into all non-smooth Borel equivalence relations on standard Borel spaces.*

Proof. This follows from Theorem 1.1.2 and [DJK94, Theorem 1].   ⊠

Theorem 1.3.8 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

    (1) *The relation E is hyperfinite.*
    (2) *The relation E is Borel reducible to $\mathbb{E}_0$.*

Proof. To see (1) $\implies$ (2), note that $\mathbb{E}_0$ is non-smooth, and appeal to Theorem 1.3.7. To see (2) $\implies$ (1), note that $\mathbb{E}_0$ is hyperfinite, so Proposition 1.3.2 ensures that so too is every countable Borel equivalence relation Borel reducible to $\mathbb{E}_0$. ⊠

Theorem 1.3.9 (Dougherty-Jackson-Kechris). *All hyperfinite Borel equivalence relations on standard Borel spaces are comparable under Borel reducibility.*

Proof. As all standard Borel spaces are comparable under Borel embeddability, and Remark 1.2.2 implies that smooth countable Borel equivalence relations have Borel transversals, it follows from the Lusin-Novikov uniformization theorem that all smooth countable Borel equivalence relations are comparable under Borel reducibility. But the desired result then follows from Theorem 1.3.7. ⊠

Proposition 1.3.10. *Suppose that X is a standard Borel space and E is a non-smooth countable Borel equivalence relation on X. Then there is a Borel reduction $\pi\colon X \to X$ of E to E such that E is non-smooth off of $[\pi(X)]_E$.*

Proof. By Theorem 1.1.2, it is sufficient to establish the proposition for $\mathbb{E}_0$. Towards this end, observe that the function $\pi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, given by

$$\pi(x)(n) = \begin{cases} x(m) & \text{if } n = 2m, \text{ and} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

is as desired. ⊠

1.4. Treeability

Here we note the analog of Proposition 1.3.2 for treeability.

Proposition 1.4.1 (Jackson-Kechris-Louveau). *The family of treeable countable Borel equivalence relations is closed downward under countable-to-one Borel homomorphism.*

Proof. See [JKL02, Proposition 3.3]. ⊠

1.5. Measures

Following [Kec95, §17], we use $P(X)$ to denote the standard Borel space of all Borel probability measures on X, and when X is a Polish space, we use the same notation to denote the Polish space of all Borel probability measures on X. Two Borel measures μ and ν are *orthogonal* if there is a Borel set which is μ-null and ν-conull.

Theorem 1.5.1 (Burgess-Mauldin). *Suppose that X is a standard Borel space and $A \subseteq P(X)$ is an uncountable analytic set of pairwise orthogonal measures. Then there are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in A such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. By the isomorphism theorem for standard Borel spaces, we can assume that X is a zero-dimensional Polish space. Fix a countable clopen basis $\mathscr{A}$ for X. By Theorem 1.1.1, there is a continuous injection $\pi \colon 2^{\mathbb{N}} \to A$. Fix real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, and appeal to the regularity of Borel probability measures on Polish spaces (see, for example, [Kec95, Theorem 17.10]) to recursively obtain $k_n \in \mathbb{N}$, $\phi_n \colon 2^n \to 2^{k_n}$, and $A_n \colon 2^n \to \mathscr{A}$ with the following properties:

(1) $\forall n \in \mathbb{N} \forall s \in 2^n \ \phi_{n+1}(s^\frown(0)) \neq \phi_{n+1}(s^\frown(1))$.
(2) $\forall i < 2 \forall n \in \mathbb{N} \forall s \in 2^n \ \phi_n(s) \sqsubseteq \phi_{n+1}(s^\frown(i))$.
(3) $\forall n \in \mathbb{N} \forall s, t \in 2^n \ (s = t \iff A_n(s) \cap A_n(t) \neq \emptyset)$.
(4) $\forall n \in \mathbb{N} \forall s \in 2^n \forall \mu \in \pi(\mathcal{N}_{\phi_n(s)}) \ \mu(A_n(s)) \geq 1 - \epsilon_n$.

Define $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi_n(c \restriction n)$, and for each $c \in 2^{\mathbb{N}}$, define $B_c = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_n(c \restriction n)$ and $\mu_c = (\pi \circ \phi)(c)$. $\boxtimes$

We now describe a means of coding Borel functions, modulo sets which are null with respect to Borel probability measures, which is uniform in both the function and the measure in question. Let $C(X, Y)$ denote the space of continuous functions from X to Y (see, for example, [Kec95, §4.E]). In order to keep our coding as transparent as possible, we will assume that X, Y, $C(X, Y)$, and $C(Y, X)$ are Polish, and that every continuous partial function from X to Y has a continuous total extension. This holds, for example, when $X = Y = 2^{\mathbb{N}}$.

Proposition 1.5.2. *Suppose that X is a compact Polish space and Y is a Polish space. Then the function $\phi \colon C(X, Y) \times X \to Y$ given by $\phi(f, x) = f(x)$ is continuous.*

Proof. It is sufficient to show that if $U \subseteq Y$ is open and $\phi(f, x) \in U$, then there are open neighborhoods V and W of f and x such that $\phi(V \times W) \subseteq U$. Towards this end, fix a Polish metric d on Y compatible

with its underlying topology. As U is open, there exists $\epsilon > 0$ such that $\mathscr{B}(f(x), \epsilon) \subseteq U$. As f is continuous, there is an open neighborhood W of x such that $f(W) \subseteq \mathscr{B}(f(x), \epsilon/2)$. Fix an open neighborhood V of f such that $\forall g \in V \forall x \in X \; d(f(x), g(x)) \leq \epsilon/2$. It only remains to note that if $(g, y) \in V \times W$, then $d(f(x), g(y)) \leq d(f(x), f(y)) + \epsilon/2 < \epsilon$, thus $g(y) \in U$. $\boxtimes$

We refer to elements c of $C(X, Y)^{\mathbb{N}}$ as *codes* for measurable functions. Proposition 1.5.2 ensures that the sets

$$D_n = \{(c, x) \in C(X, Y)^{\mathbb{N}} \times X \mid \forall m \geq n \; c(m)(x) = c(n)(x)\}$$

and $D = \bigcup_{n \in \mathbb{N}} D_n$ are Borel. We associate with each $c \in C(X, Y)^{\mathbb{N}}$ the map $\phi_c \colon D_c \to Y$, where $\phi_c(x)$ is the eventual value of $(c(n)(x))_{n \in \mathbb{N}}$.

Proposition 1.5.3. *Suppose that X and Y are standard Borel spaces. Then the function $\phi \colon D \to Y$ given by $\phi(c, x) = \phi_c(x)$ is Borel.*

Proof. As $\phi(c, x) = y \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m)(x) = y$, the graph of ϕ is Borel, so ϕ is Borel (see, for example, [Kec95, Theorem 14.12]). $\boxtimes$

The *push-forward* of a Borel measure μ on X through a Borel function $\phi \colon X \to Y$ is given by $(\phi_* \mu)(B) = \mu(\phi^{-1}(B))$.

Proposition 1.5.4. *Suppose that X and Y are standard Borel spaces. Then the function $\phi \colon \{(c, \mu) \in C(X, Y)^{\mathbb{N}} \times P(X) \mid \mu(D_c) = 1\} \to P(Y)$ given by $\phi(c, \mu) = (\phi_c)_* \mu$ is Borel.*

Proof. It is sufficient to show that if $B \subseteq Y$ is Borel and $F \subseteq \mathbb{R}$ is of the form $(a, b]$, where $a < b$ are in $\mathbb{R}$, then the intersection of the sets

$$R = \{(c, \mu) \in C(X, Y)^{\mathbb{N}} \times P(X) \mid \mu(D_c) = 1\}$$

and

$$S = \{(c, \mu) \in C(X, Y)^{\mathbb{N}} \times P(X) \mid (\phi_c)_* \mu(B) \in F\}$$

is Borel. But R is clearly Borel (see, for example, [Kec95, Theorem 17.25]), and to see that S is Borel, observe that $(c, \mu) \in S$ if and only if $\exists n \in \mathbb{N} \forall m \geq n \; \mu(c(m)^{-1}(B) \cap (D_n)_c) \in F$. $\boxtimes$

1.6. Measured equivalence relations

Here we consider countable Borel equivalence relations in the presence of measures.

Proposition 1.6.1. *Suppose that X is a standard Borel space, E is a non-smooth Borel equivalence relation on X, and μ is a Borel probability measure on X. Then there is a μ-null Borel set on which E is non-smooth.*

Proof. By Theorem 1.1.2, there is a continuous embedding $\pi\colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E. For each $c \in 2^{\mathbb{N}}$, the function $\pi_c\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, given by

$$\pi_c(d)(n) = \begin{cases} c(m) & \text{if } n = 2m, \text{ and} \\ d(m) & \text{if } n = 2m + 1, \end{cases}$$

is a continuous embedding of $\mathbb{E}_0$ into $\mathbb{E}_0$. As the sets of the form $\pi_c(2^{\mathbb{N}})$ are pairwise disjoint, it follows that for all but countably many $c \in 2^{\mathbb{N}}$, the function $\pi \circ \pi_c$ is as desired. ⊠

Suppose that $\rho\colon E \to (0, \infty)$ is a *cocycle*, in the sense that $\rho(x, z) = \rho(x, y)\rho(y, z)$ whenever $x \mathrel{E} y \mathrel{E} z$. For each set $Y \subseteq [x]_E$, define $\rho(Y, x) = \sum_{y \in Y} \rho(y, x)$. We say that Y is ρ-*finite* or ρ-*infinite* according to whether $\rho(Y, x)$ is finite or infinite. Our assumption that ρ is a cocycle ensures that the ρ-finiteness of Y does not depend on the choice of $x \in [Y]_E$. We say that ρ is *finite* if every equivalence class of E is ρ-finite, and ρ is *aperiodic* if every equivalence class of E is ρ-infinite. Given $Y, Z \subseteq [x]_E$, define $\rho(Y, Z) = \rho(Y, x)/\rho(Z, x)$. Again, our assumption that ρ is a cocycle ensures that $\rho(Y, Z)$ does not depend on the choice of $x \in [Y]_E$.

Proposition 1.6.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and there is a finite Borel cocycle $\rho\colon E \to (0, \infty)$. Then E is smooth.*

Proof. See, for example, [Mil08, Proposition 2.1]. ⊠

Theorem 1.6.3 (Ditzen)**.** *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then the set of E-ergodic E-quasi-invariant Borel probability measures on X is Borel.*

Proof. See [Dit92, Theorem 2 of Chapter 2]. ⊠

We say that μ is ρ-*invariant* if $\mu(T(B)) = \int_B \rho(T(x), x) \, d\mu(x)$, for all Borel sets $B \subseteq X$ and all Borel injections $T\colon B \to X$ whose graphs are contained in E.

Proposition 1.6.4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is an E-quasi-invariant Borel probability measure on X. Then there is a Borel cocycle $\rho\colon E \to (0, \infty)$ with respect to which μ is invariant.*

Proof. See, for example, [KM04, §8]. ⊠

We say that E is μ-*nowhere smooth* if there is no μ-positive Borel set $B \subseteq X$ for which $E \upharpoonright B$ is smooth.

Proposition 1.6.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0,\infty)$ is an aperiodic Borel cocycle, and μ is a ρ-invariant Borel probability measure on X. Then E is μ-nowhere smooth.*

Proof. See, for example, [Mil08, Proposition 2.1]. ⊠

The following fact usually allows us to assume quasi-invariance.

Proposition 1.6.6. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X. Then there is an E-quasi-invariant Borel probability measure ν on X such that $\mu \ll \nu$ and the two measures take the same values on all E-invariant Borel sets.*

Proof. By Theorem 1.2.4, there is a countable group $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of X whose induced orbit equivalence relation is E. Define $\nu = \sum_{n \in \mathbb{N}} (\gamma_n)_* \mu / 2^{n+1}$.

To see that ν is a Borel probability measure, simply note that it is a convex combination of Borel probability measures. Moreover, if $B \subseteq X$ is an E-invariant Borel set, then $\mu(B) = (\gamma_* \mu)(B)$ for all $\gamma \in \Gamma$, thus $\nu(B) = \sum_{n \in \mathbb{N}} \mu(B)/2^{n+1} = \mu(B)$. And if $N \subseteq X$ is a ν-null Borel set, then $\mu(N) \leq \sum_{n \in \mathbb{N}} (\gamma_n)_* \mu(N) = 0$, thus $\mu \ll \nu$. ⊠

Note that any two E-ergodic E-quasi-invariant Borel probability measures are either orthogonal or equivalent; the following gives a sufficient condition to strengthen equivalence to equality.

Proposition 1.6.7. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0,\infty)$ is a Borel cocycle, and $\mu \ll \nu$ are E-ergodic ρ-invariant Borel probability measures on X. Then $\mu = \nu$.*

Proof. The Radon-Nikodým theorem (see, for example, [Kec95, §17.A]) yields a Borel function $\phi\colon X \to [0,\infty)$ such that $\mu(B) = \int \phi(x) \, d\nu(x)$ for all Borel sets $B \subseteq X$. As $\mu(X) = \nu(X) = 1$, to see that $\mu = \nu$, it is sufficient to show that ϕ is constant on a μ-conull Borel set. Suppose, towards a contradiction, that there are μ-positive Borel sets $A, B \subseteq X$ with the property that $\forall x \in A \forall y \in B \; \phi(x) < \phi(y)$. As E is countable, Theorem 1.2.4 yields a countable group Γ of Borel automorphisms of X whose induced orbit equivalence relation is E. As μ is E-ergodic, there exists $\gamma \in \Gamma$ such that the set $A' = A \cap \gamma^{-1}(B)$ is μ-positive, so

$$\mu(\gamma(A')) = \int_{A'} \rho(\gamma \cdot x, x) \, d\mu(x) = \int_{A'} \phi(x) \rho(\gamma \cdot x, x) \, d\nu(x)$$

and

$$\mu(\gamma(A')) = \int_{\gamma(A')} \phi(x) \, d\nu(x) = \int_{A'} \phi(\gamma \cdot x)\rho(\gamma \cdot x, x) \, d\nu(x),$$

the desired contradiction.   ⊠

A *Borel disintegration* of a Borel probability measure μ on X through a Borel function $\phi \colon X \to Y$ is a Borel sequence $(\mu_y)_{y \in Y}$ of Borel probability measures on X with the property that $\mu = \int \mu_y \, d(\phi_*\mu)(y)$ and $\mu_y(\phi^{-1}(y)) = 1$ for all $y \in Y$. The existence of such sequences is noted, for example, in [Kec95, Exercise 17.35].

A *Borel ergodic decomposition* of a Borel cocycle $\rho \colon E \to (0, \infty)$ is a Borel sequence $(\mu_x)_{x \in X}$ of Borel probability measures on X such that $\mu_x = \mu_y$ for all $(x, y) \in E$, $\mu(\{x \in X \mid \mu = \mu_x\}) = 1$ for all E-ergodic ρ-invariant Borel probability measures μ, and $\mu = \int \mu_x \, d\mu(x)$ for all ρ-invariant Borel probability measures μ.

Theorem 1.6.8 (Ditzen)**.** *Suppose that X is a standard Borel space, E is a Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocyle. Then there is a Borel ergodic decomposition of ρ.*

Proof. See [Dit92, Theorem 6 of Chapter 2].   ⊠

A *compression* of E is a Borel injection $T \colon X \to X$, whose graph is contained in E, such that the complement of $T(X)$ is E-complete. We say that E is *compressible* if there is a Borel compression of E.

Proposition 1.6.9. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $B \subseteq X$ is a Borel E-complete set for which $E \restriction B$ is compressible. Then there is a Borel injection $T \colon X \to B$ whose graph is contained in E.*

Proof. Fix a Borel compression $\phi \colon B \to B$ of $E \restriction B$. The Lusin-Novikov uniformization theorem yields a Borel function $\psi \colon X \to B \setminus \phi(B)$ whose graph is contained in E, as well as a Borel function $\xi \colon X \to \mathbb{N}$ such that $\psi \times \xi$ is injective. Set $\pi(x) = \phi^{\xi(x)} \circ \psi(x)$.   ⊠

We say that E is *μ-nowhere compressible* if there is no μ-positive Borel set $B \subseteq X$ for which $E \restriction B$ is compressible.

Theorem 1.6.10 (Hopf)**.** *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is an E-quasi-invariant σ-finite Borel measure on X. If E is μ-nowhere compressible, then there is an E-invariant Borel probability measure $\nu \sim \mu$.*

Proof. See, for example, [Nad98, §10].   ⊠

When μ is an E-invariant Borel probability measure, the μ-*cost* of a graphing G of E is given by

$$C_\mu(G) = \frac{1}{2} \int |G_x| \, d\mu(x).$$

The μ-*cost* of E is the infimum of the costs of its Borel graphings.

Proposition 1.6.11 (Gaboriau). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, $B \subseteq X$ is an E-complete Borel set, and μ_B is the Borel probability measure on B given by $\mu_B(D) = \mu(D)/\mu(B)$. Then $C_\mu(E) - 1 = \mu(B)(C_{\mu_B}(E \restriction B) - 1)$. In particular, it follows that $C_\mu(E) \le C_{\mu_B}(E \restriction B)$.*

Proof. See, for example, [KM04, Theorem 21.1]. ⊠

Proposition 1.6.12 (Gaboriau). *Suppose that X is a standard Borel space, E is an aperiodic treeable countable Borel equivalence relation on X, and μ is an E-invariant Borel probability measure on X for which E is not μ-hyperfinite. Then $C_\mu(E) > 1$.*

Proof. See, for example, [KM04, Corollary 27.12]. ⊠

An E-ergodic measure μ is (E, F)-*ergodic* if there is no μ-null-to-one Borel homomorphism from E to F.

Proposition 1.6.13. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an $(E, \mathbb{E}_0)$-ergodic Borel probability measure on X, and $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of countable Borel equivalence relations on X whose union is E. Then for all $\epsilon > 0$, there is a Borel set $B \subseteq X$ of μ-measure at least $1 - \epsilon$ on which μ is E_n-ergodic for all sufficiently large $n \in \mathbb{N}$.*

Proof. See, for example, [Mil12, Proposition 2.2]. ⊠

For the following, recall the definition of codes for measurable functions given just before Proposition 1.5.3.

Proposition 1.6.14. *Suppose that X and Y are compact Polish spaces and E and F are countable Borel equivalence relations on X and Y. Then the set of pairs $(c, \mu) \in C(X, Y)^{\mathbb{N}} \times P(X)$ for which ϕ_c is a reduction of E to F on an E-invariant μ-conull Borel set is analytic.*

Proof. By Theorem 1.2.4, there are countable groups Γ and Δ of Borel automorphisms of X and Y whose induced orbit equivalence relations are E and F. Then (c, μ) has the desired property if and only if there exists $d \in C(Y, X)^{\mathbb{N}}$ such that the following conditions hold:

(1) $\mu(D_c) = 1$.
(2) $\forall_\mu^* x \in X \forall \gamma \in \Gamma \ \phi_c(x) \ F \ \phi_c(\gamma \cdot x)$.
(3) $\forall_\mu^* x \in X \forall \delta \in \Delta \ (\delta \cdot \phi_c(x) \in D_d \implies x \ E \ \phi_d(\delta \cdot \phi_c(x)))$.
(4) $(\phi_c)_* \mu(D_d) = 1$.

Clearly the sets determined by conditions (1) and (2) are Borel, Proposition 1.5.3 ensures that the set determined by condition (3) is Borel, and Proposition 1.5.4 implies that the set determined by condition (4) is Borel. ⊠

1.7. Measure hyperfiniteness

Here we consider connections between hyperfiniteness and measures.

Proposition 1.7.1. *Suppose that Γ is a countable discrete non-amenable group, X is a standard Borel space, $\Gamma \curvearrowright X$ is a free Borel action, and μ is an E_Γ^X-invariant Borel probability measure on X. Then the induced orbit equivalence relation E_Γ^X is not μ-hyperfinite.*

Proof. See, for example, [JKL02, Proposition 2.5]. ⊠

Theorem 1.7.2 (Dye, Krieger)**.** *Suppose that X is a standard Borel space, μ is a Borel probability measure on X, and $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of μ-hyperfinite Borel equivalence relations on X. Then the equivalence relation $E = \bigcup_{n \in \mathbb{N}} E_n$ is also μ-hyperfinite.*

Proof. See, for example, [KM04, Propositon 6.11]. ⊠

Given equivalence relations E and F on X, define

$$e_\mu(E, F) = \mu(\{x \in X \mid [x]_E \neq [x]_F\}).$$

Proposition 1.7.3. *Suppose that X is a standard Borel space and μ is a Borel probability measure on X. Then e_μ is a complete pseudo-metric.*

Proof. To see that e_μ is a pseudo-metric, it is sufficient to check the triangle inequality. Towards this end, suppose that E_1, E_2, and E_3 are Borel equivalence relations on X, and observe that

$e_\mu(E_1, E_3)$
$\quad = 1 - \mu(\{x \in X \mid [x]_{E_1} = [x]_{E_3}\})$
$\quad \leq 1 - \mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\} \cap \{x \in X \mid [x]_{E_2} = [x]_{E_3}\})$
$\quad = 1 + \mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\} \cup \{x \in X \mid [x]_{E_2} = [x]_{E_3}\}) -$
$\quad \quad (\mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\}) + \mu(\{x \in X \mid [x]_{E_2} = [x]_{E_3}\}))$
$\quad \leq 2 - (\mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\}) + \mu(\{x \in X \mid [x]_{E_2} = [x]_{E_3}\}))$
$\quad = e_\mu(E_1, E_2) + e_\mu(E_2, E_3)$.

To see that e_μ is complete, suppose that $(E_n)_{n \in \mathbb{N}}$ is an e_μ-Cauchy sequence, fix a sequence of real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, and fix a strictly increasing sequence of natural numbers k_n such that

$$\forall n \in \mathbb{N} \forall i, j \geq k_n \ e_\mu(E_i, E_j) \leq \epsilon_n.$$

Note that for all $n \in \mathbb{N}$, the set $Y_n = \{x \in X \mid \forall m \geq n \ [x]_{E_{k_m}} = [x]_{E_{k_n}}\}$ has μ-measure at least $1 - \sum_{m \geq n} \epsilon_m$. In particular, it follows that the set $Y = \bigcup_{n \in \mathbb{N}} Y_n$ is μ-conull. Letting E denote the union of the diagonal on X with the equivalence relations of the form $E_{k_n} \upharpoonright Y_n$ for $n \in \mathbb{N}$, it follows that $E_{k_n} \to_{e_\mu} E$ as $n \to \infty$, thus e_μ is indeed complete. ⊠

It is not difficult to see that e_μ is not separable, even when restricted to the family of Borel equivalence relations on X whose classes are all of cardinality two. In contrast, we have the following.

Proposition 1.7.4. *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then there is a countable family $\mathscr{F}$ of finite Borel subequivalence relations of E such that for all Borel probability measures μ on X, the family $\mathscr{F}$ is e_μ-dense in the set of all finite Borel subequivalence relations of E.*

Proof. Fix an enumeration $(U_n)_{n \in \mathbb{N}}$ of a basis, closed under finite unions, for a Polish topology generating the Borel structure of X. By Theorem 1.2.4, there is a sequence $(f_n)_{n \in \mathbb{N}}$ of Borel automorphisms of X such that $E = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$.

For each $n \in \mathbb{N}$ and $s \in \mathbb{N}^n$, let X_s denote the Borel set of $x \in X$ with the property that whenever $i, j, k < n$, $x \in U_{s(i)} \cap U_{s(j)}$, $y \in U_{s(k)}$, and $f_i(x) = f_k(y)$, there exists $\ell < n$ such that $y \in U_{s(\ell)}$ and $f_j(x) = f_\ell(y)$. Let F_s denote the reflexive Borel relation on X in which distinct points x and y related if there exist $i, j < n$ and $z \in U_{s(i)} \cap U_{s(j)} \cap X_s$ such that $x = f_i(z)$ and $y = f_j(z)$.

Lemma 1.7.5. *Each F_s is an equivalence relation.*

Proof. As F_s is clearly reflexive and symmetric, it is sufficient to show that it is transitive. Towards this end, observe that if $x \ F_s \ y \ F_s \ z$ are pairwise distinct, then there exist $i, j < n$ and $v \in U_{s(i)} \cap U_{s(j)} \cap X_s$ with the property that $x = f_i(v)$ and $y = f_j(v)$, as well as $k, \ell < n$ and $w \in U_{s(k)} \cap U_{s(\ell)} \cap X_s$ with the property that $y = f_k(w)$ and $z = f_\ell(w)$. As $v \in X_s$, there exists $m < n$ with $w \in U_{s(m)}$ and $x = f_i(v) = f_m(w)$, in which case the definition of F_s ensures that $x \ F_s \ z$. ⊠

To see that the family $\mathscr{F} = \{F_s \mid s \in \mathbb{N}^{<\mathbb{N}}\}$ is as desired, suppose that $\epsilon > 0$, F is a finite Borel subequivalence relation of E, and μ is a Borel probability measure on X. Fix $n \in \mathbb{N}$ sufficiently large that

the μ-measure of the set $Y = \{x \in X \mid \forall y, z \in [x]_F \exists i < n \ f^i(y) = z\}$ is strictly greater than $1 - \epsilon$. Set $\delta = \mu(Y) - (1 - \epsilon)$, and define $Y_k = \{x \in X \mid x \ F \ f_k(x)\}$ for all $k < n$. As Borel probability measures on Polish spaces are regular, there exists $s \in \mathbb{N}^n$ with the property that the μ-measure of the set

$$Z_{i,j,k} = \{x \in X \mid (f_i^{-1} \circ f_j)(x) \in U_{s(k)} \iff (f_i^{-1} \circ f_j)(x) \in Y_k)\}$$

is at least $1 - \delta/n^3$, for all $i, j, k < n$.

Lemma 1.7.6. *The set $Z = Y \cap \bigcap_{i,j,k<n} Z_{i,j,k}$ is contained in X_s.*

Proof. We must show that if $i, j, k < n$, $z \in U_{s(i)} \cap U_{s(j)} \cap Z$, $y \in U_{s(k)}$, and $f_i(z) = f_k(y)$, then there exists $\ell < n$ such that $y \in U_{s(\ell)}$ and $f_j(z) = f_\ell(y)$. Towards this end, note that $y = (f_k^{-1} \circ f_i)(z)$, so the fact that $z \in Z$ ensures that $z \in Y_i \cap Y_j$ and $y \in Y_k$. In particular, it follows that $f_j(z) \ F \ z \ F \ f_i(z) = f_k(y) \ F \ y$. The fact that $z \in Y$ then yields $\ell < n$ such that $f_j(z) = f_\ell(y)$. As $y \in Y_\ell$, one more appeal to the fact that $z \in Z$ ensures that $y \in U_{s(\ell)}$.   ⊠

Lemma 1.7.7. *Suppose that $z \in Z$. Then $[z]_F = [z]_{F_s}$.*

Proof. Suppose first that $x \in [z]_F$. As $z \in Y$, there exist $i, j < n$ such that $x = f_i(z)$ and $z = f_j(z)$. Then $z \in Y_i \cap Y_j$, so the fact that $z \in Z$ ensures that $z \in U_{s(i)} \cap U_{s(j)}$. As Lemma 1.7.6 implies that $z \in X_s$, the definition of F_s ensures that $x \in [z]_{F_s}$.

Suppose now that $x \in [z]_{F_s}$. The definition of F_s then yields $i, j < n$ and $w \in U_{s(i)} \cap U_{s(j)} \cap X_s$ such that $x = f_i(w)$ and $z = f_j(w)$. As $z \in Y$, there exists $\ell < n$ such that $z = f_\ell(z)$. As $z \in Y_\ell$, the fact that $z \in Z$ ensures that $z \in U_{s(\ell)}$, so the fact that $w \in X_s$ yields $k < n$ such that $z \in U_{s(k)}$ and $x = f_k(z)$. One more appeal to the fact that $z \in Z$ then ensures that $z \in Y_k$, in which case $x = f_k(z) \in [z]_F$.   ⊠

As $\mu(Z) \geq 1 - \epsilon$, it follows that $e_\mu(F, F_s) \leq \epsilon$.   ⊠

We use $\mathcal{H}_E$ denote the space of Borel probability measures μ on X with respect to which E is μ-hyperfinite. The following fact originally appeared in [Seg97].

Theorem 1.7.8 (Segal). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then there is a Borel set $F \subseteq (\mathbb{N} \times (X \times X)) \times P(X)$ such that for all $\mu \in P(X)$, the following conditions hold:*

(1) *The sets $(F^\mu)_n$ form an increasing sequence of finite Borel subequivalence relations of E.*
(2) *The set $B^\mu = \{x \in X \mid [x]_E \neq \bigcup_{n \in \mathbb{N}} [x]_{(F^\mu)_n}\}$ does not contain a μ-positive Borel subset on which E is hyperfinite.*

In particular, it follows that $\mathcal{H}_E$ is Borel.

Proof. Fix real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. By Proposition 1.7.4, there is a family $\mathscr{E} = \{E_k \mid k \in \mathbb{N}\}$ of finite Borel subequivalence relations of E such that for all Borel probability measures μ on X, the family $\mathscr{E}$ is e_μ-dense in the set of all finite Borel subequivalence relations of E. Then the functions $m_n \colon P(X) \to [0,1]$ given by

$$m_n(\mu) = \sup_{k \in \mathbb{N}} \mu(\{x \in X \mid \forall i < n \; x \; E_k \; f_i(x)\})$$

are Borel, as are the functions $k_n \colon P(X) \to \mathbb{N}$ given by

$$k_n(\mu) = \min\{k \in \mathbb{N} \mid \mu(\{x \in X \mid \forall i < n \; x \; E_k \; f_i(x)\}) > m_n(\mu) - \epsilon_n\},$$

thus so too is the set $F \subseteq (\mathbb{N} \times (X \times X)) \times P(X)$ given by

$$x \; (F^\mu)_n \; y \iff \forall m \geq n \; x \; E_{k_m(\mu)} \; y.$$

To see that F is as desired, suppose that $\mu \in P(X)$. As the sets $(F^\mu)_n = \bigcap_{m \geq n} E_{k_m(\mu)}$ form an increasing sequence of finite Borel subequivalence relations of E, it is enough to show that if $A \subseteq B^\mu$ is a Borel set on which E is hyperfinite, then $\mu(A) = 0$. As B^μ is E-invariant and E is countable, the Lusin-Novikov uniformization theorem and Proposition 1.3.2 allow us to assume that A is E-invariant.

Lemma 1.7.9. *Suppose that $n \in \mathbb{N}$. Then*

$$\mu(\{x \in A \mid \forall i < n \; x \; E_{k_n(\mu)} \; f_i(x)\}) \geq \mu(A) - \epsilon_n.$$

Proof. As $E_{k_n(\mu)}$ is finite, Remark 1.2.2 ensures that it has a Borel transversal $C \subseteq X$ from which the quotient $X/E_{k_n(\mu)}$ inherits a standard Borel structure, and moreover, that the map associating each $E_{k_n(\mu)}$-class with the unique point of C it contains is a Borel reduction of $E/E_{k_n(\mu)}$ to E. Proposition 1.3.2 therefore implies that the restriction of $E/E_{k_n(\mu)}$ to $A/E_{k_n(\mu)}$ is hyperfinite.

Given $\epsilon > 0$, observe that all but finitely many relations E' along any sequence witnessing the hyperfiniteness of the restriction of $E/E_{k_n(\mu)}$ to $A/E_{k_n(\mu)}$, when viewed as equivalence relations on A, satisfy the condition that $\mu(\{x \in A \mid \forall i < n \; x \; E' \; f_i(x)\}) > \mu(A) - \epsilon$. The e_μ-density of $\mathscr{E}$ therefore yields $k \in \mathbb{N}$ such that

$$\mu(A) - \mu(\{x \in A \mid \forall i < n \; x \; E_{k_n(\mu)} \; f_i(x)\}) - \epsilon$$

is strictly less than

$$\mu(\{x \in X \mid \forall i < n \; x \; E_k \; f_i(x)\}) - \mu(\{x \in X \mid \forall i < n \; x \; E_{k_n(\mu)} \; f_i(x)\}).$$

As the definition of $k_n(\mu)$ ensures that the latter quantity is itself strictly less than ϵ_n, it follows that

$$\mu(\{x \in A \mid \forall i < n \; x \; E_{k_n(\mu)} \; f_i(x)\}) > \mu(A) - \epsilon_n - \epsilon,$$

thus $\mu(\{x \in A \mid \forall i < n \ x \ E_{k_n(\mu)} \ f_i(x)\}) \geq \mu(A) - \epsilon_n$, as the former inequality holds for all $\epsilon > 0$.   ⊠

Set $A' = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \{x \in A \mid \forall i < m \ x \ E_{k_m(\mu)} \ f_i(x)\}$, and note that $\mu(A) = \mu(A')$, since $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, thus $\mu(A) = 0$, since $A' \cap B^\mu = \emptyset$.

As $\mathcal{H}_E = \{\mu \in P(X) \mid \mu(B^\mu) = 0\}$ and the Lusin-Novikov uniformization theorem ensures that the set $B = \{(x, \mu) \in X \times P(X) \mid x \in B^\mu\}$ is Borel, it follows that $\mathcal{H}_E$ is Borel as well.   ⊠

Proposition 1.7.10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and there is a ρ-invariant Borel probability measure μ on X for which E is not μ-hyperfinite. Then there is such a measure which is also E-ergodic.*

Proof. This follows from Theorems 1.6.8 and 1.7.8.   ⊠

We use $\mathcal{E}_E$ to denote the family of all E-ergodic Borel probability measures on X, $\mathcal{Q}_E$ to denote the family of all E-quasi-invariant Borel probability measures on X, and $\mathcal{E}\mathcal{Q}_E$ to denote $\mathcal{E}_E \cap \mathcal{Q}_E$.

Theorem 1.7.11. *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then exactly one of the following holds:*

    (1) *The relation E is measure hyperfinite.*
    (2) *The set $\mathcal{E}\mathcal{Q}_E \setminus \mathcal{H}_E$ is non-empty.*

Proof. Suppose that E is not measure hyperfinite. Proposition 1.6.6 then yields an E-quasi-invariant Borel probability measure μ on X with respect to which E is not μ-hyperfinite, and Propositions 1.6.4 and 1.7.10 give rise to an E-ergodic such measure.   ⊠

We close this section by considering preservation of μ-hyperfiniteness under Borel homomorphisms.

Proposition 1.7.12. *Suppose that X and Y are standard Borel spaces, E is a countable Borel equivalence relation on X, $\phi \colon X \to Y$ is a Borel homomorphism from E to equality, μ is a Borel probability measure on X, $(\mu_y)_{y \in Y}$ is a Borel disintegration of μ through ϕ, and $E \restriction \phi^{-1}(y)$ is μ_y-hyperfinite for $(\phi_* \mu)$-almost every $y \in Y$. Then E is μ-hyperfinite.*

Proof. By Theorem 1.7.8, the set $D = \{y \in Y \mid E$ is μ_y-hyperfinite$\}$ is Borel, and there is a hyperfinite Borel equivalence relation F on X for which there is a Borel set $C \subseteq X$ such that $\mu_y(C) = 1$ and $E \restriction C = F \restriction C$ for all $y \in D$. Then $\mu(C) = 1$, so E is μ-hyperfinite.   ⊠

Proposition 1.7.13. *Suppose that X and Y are standard Borel spaces, E is a countable Borel equivalence relation on X, F is a hyperfinite Borel equivalence relation on Y, $\phi\colon X \to Y$ is a Borel homomorphism from E to F, μ is a Borel probability measure on X, $(\mu_y)_{y \in Y}$ is a Borel disintegration of μ through ϕ, and $E \restriction \phi^{-1}(y)$ is μ_y-hyperfinite for $(\phi_*\mu)$-almost every $y \in Y$. Then E is μ-hyperfinite.*

Proof. Fix an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on Y whose union is F. Proposition 1.7.12 then ensures that each of the equivalence relations $E_n = E \cap (\phi \times \phi)^{-1}(F_n)$ is μ-hyperfinite. As $E = \bigcup_{n \in \mathbb{N}} E_n$, Theorem 1.7.2 implies that E is μ-hyperfinite.   ⊠

1.8. Actions of $\mathrm{SL}_2(\mathbb{Z})$

Let $\sim$ denote the equivalence relation on $\mathbb{R}^2 \setminus \{(0,0)\}$ given by

$$v \sim w \iff \exists r \in \mathbb{R} \ (r > 0 \text{ and } rv = w),$$

and let $\mathbb{T}$ denote the quotient. Define $\mathrm{proj}_{\mathbb{T}}\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{T}$ by $\mathrm{proj}_{\mathbb{T}}(v) = [v]_\sim$, and let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

Proposition 1.8.1 (Jackson-Kechris-Louveau)**.** *The action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is hyperfinite.*

Proof. See the remark following the proof of [JKL02, Lemma 3.6].   ⊠

Let $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ denote the group of all functions $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $T(x) = Ax + b$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$, under composition. Define $\mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})}\colon \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ by $\mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})}(Ax + b) = A$.

Proposition 1.8.2. *Suppose that μ is the Borel probability measure on $\mathbb{T}^2$ induced by Lebesgue measure on $\mathbb{R}^2$. Then the orbit equivalence relation $E^{\mathbb{T}^2}_{\mathrm{SL}_2(\mathbb{Z})}$ is not μ-hyperfinite.*

Proof. As $\mathrm{SL}_2(\mathbb{Z})$ is not amenable and [JKL02, Lemma 3.6] ensures that $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is free off of a μ-null set, this is a consequence of Proposition 1.7.1.   ⊠

Proposition 1.8.3 (Jackson-Kechris-Louveau)**.** *The orbit equivalence relation $E^{\mathbb{T}^2}_{\mathrm{SL}_2(\mathbb{Z})}$ is treeable.*

Proof. See [JKL02, Proposition 3.5].   ⊠

1.9. Complexity

The conclusion of the following summarizes the main results of [AK00].

Theorem 1.9.1 (Adams-Kechris). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $(E_r)_{r \in \mathbb{R}}$ is a Borel sequence of subequivalence relations of E, and $(\mu_r)_{r \in \mathbb{R}}$ is a Borel sequence of Borel probability measures on X such that:*

   (1) *Each μ_r is E_r-ergodic and E_r-quasi-invariant.*
   (2) *The relation E_r is μ_r-nowhere reducible to the relation E_s, for all distinct $r, s \in \mathbb{R}$.*

Then the following hold:

   (a) *There is an embedding of containment on Borel subsets of $\mathbb{R}$ into Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E (in the codes).*
   (b) *Borel bi-reducibility and reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E are both $\mathbf{\Sigma}_2^1$-complete (in the codes).*
   (c) *Every Borel quasi-order is Borel reducible to Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E.*
   (d) *Borel and $\sigma(\mathbf{\Sigma}_1^1)$-measurable reducibility do not agree on the countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E.*

Proof. The proof of [AK00, Theorem 4.1] yields (a), the proof of [AK00, Theorem 5.1] yields (b), the final paragraph of [AK00, §7] yields (c), and the proof of [AK00, Theorem 5.5] yields (d). ⊠

Part 2. Tools

Here we introduce the new ideas underlying our arguments. In §2.1, we show that $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ satisfies a measureless strengthening of amenability. In §2.2, we use this to prove that $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ satisfies a measureless local rigidity property. In §2.3, we establish a strong separability property for orbit equivalence relations induced by such actions. In §2.4, we show that the latter yields countability of an appropriate auxiliary equivalence relation on the underlying space of ergodic quasi-invariant Borel probability measures witnessing the failure of hyperfiniteness, and we derive several consequences of this countability. In §2.5, we provide a general stratification theorem for treeable countable Borel equivalence relations.

2.1. Productive hyperfiniteness

Suppose that Γ is a countable discrete group. The *diagonal product* of actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ is the action $\Gamma \curvearrowright X \times Y$ given by $\gamma \cdot (x, y) = (\gamma \cdot x, \gamma \cdot y)$. We say that a Borel action $\Gamma \curvearrowright X$ on a standard Borel space is *productively hyperfinite* if whenever $\Gamma \curvearrowright Y$ is a Borel action on a standard Borel space, the orbit equivalence relation induced by the diagonal product action $\Gamma \curvearrowright X \times Y$ is hyperfinite.

Proposition 2.1.1. *Suppose that Γ is a countable discrete group, X is a standard Borel space, and $\Gamma \curvearrowright X$ is a Borel action such that:*

(1) *The induced orbit equivalence relation is hyperfinite.*
(2) *The stabilizer of every point is hyperfinite.*
(3) *Only countably-many points have infinite stabilizers.*

Then $\Gamma \curvearrowright X$ is productively hyperfinite.

Proof. Let C denote the set of points whose stabilizers are infinite, and fix an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations whose union is E_Γ^X.

Suppose now that Y is a standard Borel space and $\Gamma \curvearrowright Y$ is a Borel action. For each $n \in \mathbb{N}$, let F_n denote the equivalence relation on $(X \setminus C) \times Y$ for which two $E_\Gamma^{(X \setminus C) \times Y}$-equivalent pairs (x, y) and (x', y') are related exactly when $x \, E_n \, x'$. As each F_n is finite and their union is $E_\Gamma^{(X \setminus C) \times Y}$, the latter equivalence relation is hyperfinite.

It only remains to show that $E_\Gamma^{C \times Y}$ is hyperfinite. As C is countable and Proposition 1.3.1 ensures that the family of Borel sets on which a Borel equivalence relation is hyperfinite forms a σ-ideal, we need only show that $E_\Gamma^{X \times Y}$ is hyperfinite on $\{x\} \times Y$, for all $x \in C$. But this follows from the fact that its restriction to such a set is the orbit equivalence relation induced by a Borel action of the stabilizer of x. ⊠

To apply this to $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$, we must first consider its stabilizers.

Proposition 2.1.2. *Suppose that $\theta \in \mathbb{T}$. Then the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is either trivial or infinite cyclic.*

Proof. We consider first the case that $\theta \cap \mathbb{Z}^2 \neq \emptyset$. Let v denote the unique element of $\theta \cap \mathbb{Z}^2$ of minimal length. Note that the stabilizers of θ and v are one and the same, for if A is in the stabilizer of θ, then v is an eigenvector of A, so minimality ensures that $Av = v$. Minimality also ensures that the coordinates of v are relatively prime, so there exists $a \in \mathbb{Z}^2$ such that $a \cdot v = 1$, in which case $B = \left(\begin{smallmatrix} a_1 & a_2 \\ -v_2 & v_1 \end{smallmatrix} \right)$ is a matrix in $\mathrm{SL}_2(\mathbb{Z})$ for which $Bv = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, thus conjugation by B yields an isomorphism of the stabilizer of v with that of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, and the latter is the infinite cyclic group $\{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \mid n \in \mathbb{Z} \}$.

It remains to consider the case that $\theta \cap \mathbb{Z}^2 = \emptyset$. Fix $v \in \theta$. An elementary calculation reveals that the stabilizer of v is trivial. Let Λ denote the set of eigenvalues of matrices in the stabilizer of θ, noting that Λ forms a group under multiplication.

Lemma 2.1.3. *The group Λ is cyclic.*

Proof. It is sufficient to show that 1 is isolated in $\Lambda \cap [1, \infty)$. Towards this end, suppose that A is in the stabilizer of θ and v is an eigenvector of A with eigenvalue $\lambda > 1$. If μ is the other eigenvalue of A, then $\lambda \mu = \det(A) = 1$, so $\mathrm{tr}(A) = \lambda + \mu = \lambda + 1/\lambda$. As $\mathrm{tr}(A) \in \mathbb{Z}$, another elementary calculation reveals that $\lambda \geq (3 + \sqrt{5})/2$. $\boxtimes$

By Lemma 2.1.3, there is a matrix A in the stabilizer of θ which has an eigenvalue λ generating Λ. Note that if B is any matrix in the stabilizer of θ, then there exists $n \in \mathbb{Z}$ for which v is an eigenvector of B with eigenvalue λ^n, in which case $A^n B^{-1}$ is in the stabilizer of v, so $B = A^n$, thus A generates the stabilizer of θ, hence the latter is cyclic.

Observe finally that if A is a non-identity matrix fixing θ, then any two distinct powers of A are themselves distinct, since the eigenvalues corresponding to v are distinct. In particular, it follows that if the stabilizer of θ is non-trivial, then it is infinite. $\boxtimes$

As a consequence, we can now obtain the main result of this section.

Proposition 2.1.4. *The action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is productively hyperfinite.*

Proof. As Proposition 1.8.1 ensures that the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is hyperfinite, Proposition 2.1.2 ensures that the non-trivial stabilizers of $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are infinite cyclic, and Proposition 1.3.4 ensures that infinite cyclic groups are hyperfinite, it is sufficient to show that only countably many $\theta \in \mathbb{T}$ have non-trivial stabilizers, by Proposition 2.1.1. As every such θ is the equivalence class of an eigenvector of some non-trivial matrix in $\mathrm{SL}_2(\mathbb{Z})$, and every such matrix admits at most two such classes of eigenvectors, this follows from the countability of $\mathrm{SL}_2(\mathbb{Z})$. $\boxtimes$

2.2. PROJECTIVE RIGIDITY

Given $R \subseteq X \times X$, $\Delta \curvearrowright Y$, and $\rho \colon R \to \Delta$, we say that a function $\phi \colon X \to Y$ is ρ-*invariant* if $x_1 \mathrel{R} x_2 \implies \phi(x_1) = \rho(x_1, x_2) \cdot \phi(x_2)$ for all $x_1, x_2 \in X$. The *difference set* associated with two functions $\phi \colon A \subseteq X \to Y$ and $\psi \colon B \subseteq X \to Y$ is given by

$$D(\phi, \psi) = \{x \in A \cap B \mid \phi(x) \neq \psi(x)\} \cup (A \triangle B).$$

We say that $\Delta \curvearrowright Y$ is *projectively rigid* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to \Delta$ is a Borel function, there is essentially at most one countable-to-one ρ-invariant Borel function, in the sense that for any two such functions ϕ and ψ, the relation $E \restriction D(\phi, \psi)$ is hyperfinite.

Theorem 2.2.1. *The action $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is projectively rigid.*

Proof. Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a Borel function, $\phi\colon X \to \mathbb{R}^2$ is a countable-to-one ρ-invariant Borel function, and $\psi\colon X \to \mathbb{R}^2$ is a ρ-invariant Borel function.

Define $\pi\colon D(\phi, \psi) \to \mathbb{T}$ by $\pi(x) = \mathrm{proj}_{\mathbb{T}}(\phi(x) - \psi(x))$, and define $\sigma\colon E \restriction D(\phi, \psi) \to \mathrm{SL}_2(\mathbb{Z})$ by $\sigma(x_1, x_2) = \mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})}(\rho(x_1, x_2))$.

Lemma 2.2.2. *The function π is σ-invariant.*

Proof. Simply observe that if $x_1 \ (E \restriction D(\phi, \psi)) \ x_2$, then

$$
\begin{aligned}
\pi(x_1) &= \mathrm{proj}_{\mathbb{T}}(\phi(x_1) - \psi(x_1)) \\
&= \mathrm{proj}_{\mathbb{T}}(\rho(x_1, x_2) \cdot \phi(x_2) - \rho(x_1, x_2) \cdot \psi(x_2)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot \phi(x_2) - \sigma(x_1, x_2) \cdot \psi(x_2)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot (\phi(x_2) - \psi(x_2))) \\
&= \sigma(x_1, x_2) \cdot \mathrm{proj}_{\mathbb{T}}(\phi(x_2) - \psi(x_2)) \\
&= \sigma(x_1, x_2) \cdot \pi(x_2),
\end{aligned}
$$

thus π is σ-invariant. $\boxtimes$

As $(\mathrm{proj}_{\mathbb{T}^2} \circ \phi) \restriction D(\phi, \psi)$ is also σ-invariant, it follows that the product $\pi \times (\mathrm{proj}_{\mathbb{T}^2} \circ \phi)$ is a countable-to-one homomorphism from $E \restriction D(\phi, \psi)$ to the orbit equivalence relation induced by the diagonal product action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T} \times \mathbb{T}^2$. As Proposition 2.1.4 ensures that $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is productively hyperfinite, it follows that the latter relation is hyperfinite. As Proposition 1.3.2 ensures that the family of hyperfinite Borel equivalence relations is closed downward under countable-to-one Borel homomorphism, it follows that the former relation is also hyperfinite. $\boxtimes$

Remark 2.2.3. As noted by both Manuel Inselmann and one of the anonymous referees, the productive hyperfiniteness of $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ can also be used to show that the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is hyperfinite. To see this, observe that the function $\pi\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{T} \times \mathbb{R}^2$ given by $\pi(x) = (\mathrm{proj}_{\mathbb{T}}(x), x)$ is a reduction of the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright (\mathbb{R}^2 \setminus \{0\})$ to the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T} \times \mathbb{R}^2$.

2.3. Projective separability

Let $L(X, Y)$ denote the set of Borel functions $\phi \colon B \to Y$, where B varies over Borel subsets of X. Let $L(X, \mu, Y)$ denote $L(X, Y)$ equipped with the pseudo-metric $d_\mu(\phi, \psi) = \mu(D(\phi, \psi))$.

Proposition 2.3.1. *Suppose that X and Y are standard Borel spaces, μ is a finite Borel measure on X, and $\mathscr{L} \subseteq L(X, \mu, Y)$. Then the following are equivalent:*

(1) *The space $\mathscr{L}$ is separable.*
(2) *There is a Borel set $R \subseteq X \times Y$, whose vertical sections are countable, with the property that*

$$\forall \phi \in \mathscr{L} \ \mu(\{x \in \mathrm{dom}(\phi) \mid \neg x \ R \ \phi(x)\}) = 0.$$

Proof. To see (1) $\implies$ (2), note that if $\mathscr{D}$ is a countable dense subset of $\mathscr{L}$, then the set $R = \bigcup_{\phi \in \mathscr{D}} \mathrm{graph}(\phi)$ is as desired, since graphs of Borel functions are Borel. To see (2) $\implies$ (1), it is sufficient to show that if condition (2) holds, then there is a countable subset of $L(X, \mu, Y)$ whose closure contains $\mathscr{L}$. As the vertical sections of R are countable, the Lusin-Novikov uniformization theorem yields a countable family $\mathscr{F}$ of Borel partial functions, the union of whose graphs is R. Fix a countable algebra $\mathscr{B}$ of Borel subsets of X, containing the domain of every $\phi \in \mathscr{F}$, such that for all Borel sets $A \subseteq X$ and all $\epsilon > 0$, there exists $B \in \mathscr{B}$ with $\mu(A \triangle B) \le \epsilon$. We then obtain the desired countable dense family by considering those $\psi \colon B \to Y$, where B ranges over $\mathscr{B}$, for which there is a finite partition $\mathscr{A} \subseteq \mathscr{B}$ of B such that $\forall A \in \mathscr{A} \exists \phi \in \mathscr{F} \ \phi \restriction A = \psi \restriction A$. ⊠

We say that a function $\phi \colon Y \to Y'$ is a *homomorphism* from a set $\mathscr{L} \subseteq L(X, \mu, Y)$ to a set $\mathscr{L}' \subseteq L(X, \mu, Y')$ if $\forall \psi \in \mathscr{L} \ \phi \circ \psi \in \mathscr{L}'$.

Proposition 2.3.2. *Suppose that X, Y, and Y' are standard Borel spaces, μ is a Borel probability measure on X, $\mathscr{L} \subseteq L(X, \mu, Y)$ and $\mathscr{L}' \subseteq L(X, \mu, Y')$, there is a countable-to-one Borel homomorphism $\phi \colon Y \to Y'$ from $\mathscr{L}$ to $\mathscr{L}'$, and $\mathscr{L}'$ is separable. Then $\mathscr{L}$ is separable.*

Proof. Fix a Borel set $R' \subseteq X \times Y'$ satisfying the analog of condition (2) of Proposition 2.3.1 for $\mathscr{L}'$, and observe that the set $R = (\mathrm{id} \times \phi)^{-1}(R')$ satisfies condition (2) of Proposition 2.3.1 for $\mathscr{L}$. ⊠

Let $\mathrm{Hom}(E, \mu, F)$ denote the subspace of $L(X, \mu, Y)$ consisting of all countable-to-one partial homomorphisms $\phi \in L(X, \mu, Y)$ from E to F.

Proposition 2.3.3. *Suppose that X, Y, and Y' are standard Borel spaces, E, F, and F' are countable Borel equivalence relations on X, Y,*

and Y', μ is a Borel probability measure on X, there is a countable-to-one Borel homomorphism $\phi \colon Y \to Y'$ from F to F', and $\mathrm{Hom}(E, \mu, F')$ is separable. Then $\mathrm{Hom}(E, \mu, F)$ is separable.

Proof. As the function ϕ is also a homomorphism from $\mathrm{Hom}(E, \mu, F)$ to $\mathrm{Hom}(E, \mu, F')$, the desired result follows from Proposition 2.3.2. ⊠

We say that F is *projectively separable* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X with respect to which E is μ-nowhere hyperfinite, the space $\mathrm{Hom}(E, \mu, F)$ is separable.

Proposition 2.3.4. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, there is a countable-to-one Borel homomorphism from E to F, and F is projectively separable. Then E is projectively separable.*

Proof. This is a direct consequence of Proposition 2.3.3. ⊠

We next establish the connection between projective rigidity and projective separability.

Theorem 2.3.5. *Suppose that Δ is a countable discrete group, Y is a standard Borel space, and $\Delta \curvearrowright Y$ is a projectively rigid Borel action. Then the orbit equivalence relation $F = E_\Delta^Y$ is projectively separable.*

Proof. Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X with respect to which E is μ-nowhere hyperfinite. Let μ_c denote the counting measure on X. The Lusin-Novikov uniformization theorem yields an increasing sequence $(R_n)_{n \in \mathbb{N}}$ of Borel subsets of $X \times X$ such that $E = \bigcup_{n \in \mathbb{N}} R_n$ and every vertical section of every R_n has cardinality at most n. Set $\nu_n = (\mu \times \mu_c) \restriction R_n$ for all $n \in \mathbb{N}$.

Lemma 2.3.6. *Suppose that $\phi \in \mathrm{Hom}(E, \mu, F)$, $\rho \colon E \restriction \mathrm{dom}(\phi) \to \Delta$ is a Borel function with respect to which ϕ is invariant, $(D_n)_{n \in \mathbb{N}}$ is a sequence of Borel subsets of X with $\sum_{n \in \mathbb{N}} \mu(\mathrm{dom}(\phi) \triangle D_n) < \infty$, $(\rho_n \colon R_n \restriction D_n \to \Delta)_{n \in \mathbb{N}}$ is a sequence of Borel functions such that $\sum_{n \in \mathbb{N}} d_{\nu_n}(\rho \restriction (R_n \restriction \mathrm{dom}(\phi)), \rho_n) < \infty$, and $\phi_n \colon D_n \to Y$ is a ρ_n-invariant Borel function for all $n \in \mathbb{N}$. Then $d_\mu(\phi, \phi_n) \to 0$.*

Proof. For all $n \in \mathbb{N}$, let E_n be the equivalence relation on $\mathrm{dom}(\phi) \cap D_n$ generated by the relation $S_n = (R_n \restriction (\mathrm{dom}(\phi) \cap D_n)) \setminus D(\rho, \rho_n)$.

Sublemma 2.3.7. *For all $n \in \mathbb{N}$, there is a Borel function $\sigma_n \colon E_n \to \Delta$ for which every $(\rho \restriction S_n)$-invariant function is σ_n-invariant.*

Proof. Note that if $x \ E_n \ y$, then there are only countably many $\ell \in \mathbb{N}$ and $(z_i)_{i \leq \ell} \in X^{\ell+1}$ such that $x = z_0$, $\forall i < \ell \ z_i \ S_n \ z_{i+1}$, and $y = z_\ell$, so the Lusin-Novikov uniformization theorem yields Borel functions $\ell \colon E_n \to \mathbb{N}$ and $f \colon E_n \to X^{<\mathbb{N}}$ with the property that

$$\forall (x, y) \in E_n \ x = f_0(x, y) \ S_n \ f_1(x, y) \ S_n \cdots S_n \ f_{\ell(x,y)}(x, y) = y.$$

Define $\sigma_n(x, y) = \prod_{i < \ell(x,y)} \rho(f_i(x, y), f_{i+1}(x, y))$. ⊠

As the restrictions of ϕ and ϕ_n to $\mathrm{dom}(\phi) \cap D_n$ are $(\rho \restriction S_n)$-invariant, they are σ_n-invariant. Note that the set $D = \mathrm{dom}(\phi) \cap \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} D_m$ is $(\mu \restriction \mathrm{dom}(\phi))$-conull.

Sublemma 2.3.8. *There is a $(\mu \restriction D)$-conull Borel set $C \subseteq D$ such that $E \cap (C \times D) \subseteq \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} S_m$.*

Proof. The Lusin-Novikov uniformization theorem ensures that the sets $C_n = \{x \in \mathrm{dom}(\phi) \cap D_n \mid \exists y \in \mathrm{dom}(\phi) \cap D_n \ x \ (R_n \setminus S_n) \ y\}$ are Borel, and Fubini's theorem (see, for example, [Kec95, §17.A]) ensures that $\mu(C_n) \leq d_{\nu_n}(\rho \restriction (R_n \restriction \mathrm{dom}(\phi)), \rho_n)$ for all $n \in \mathbb{N}$. In particular, it follows that the set $C = D \setminus \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} C_m$ is $(\mu \restriction D)$-conull. And if $(x, y) \in E \cap (C \times D)$, then there exists $n \in \mathbb{N}$ for which $x \ R_n \ y$, so the fact that $x \in C$ ensures that $x \ S_m \ y$ for sufficiently large $m \geq n$. ⊠

Suppose now that $\epsilon > 0$. Set $F_n = \bigcap_{m \geq n} E_m$ for all $n \in \mathbb{N}$, and observe that $E \restriction C = \bigcup_{n \in \mathbb{N}} F_n \restriction C$. As Theorem 1.7.2 ensures that the μ-hyperfinite Borel equivalence relations are closed under increasing unions, there are Borel sets $B_n \subseteq C \cap \bigcap_{m \geq n} D_m$ with the property that $\mu(C \setminus B_n) < \epsilon$ and $F_n \restriction B_n$ is $(\mu \restriction B_n)$-nowhere hyperfinite, thus $\phi \restriction B_n = \phi_n \restriction B_n$, for sufficiently large $n \in \mathbb{N}$. ⊠

Fix a countable family $\mathscr{B}$ of Borel subsets of X such that for all Borel sets $A \subseteq X$ and all real numbers $\epsilon > 0$, there exists $B \in \mathscr{B}$ with $\mu(A \triangle B) \leq \epsilon$. Proposition 2.3.1 yields countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Delta)$. For each $n \in \mathbb{N}$, $B \in \mathscr{B}$, $\epsilon \in (0, \infty) \cap \mathbb{Q}$, and $\sigma \in \mathscr{D}_n$ for which it is possible, fix a Borel set $D_{n,B,\epsilon,\sigma} \subseteq X$ with $\mu(B \triangle D_{n,B,\epsilon,\sigma}) \leq \epsilon$, a Borel function $\rho_{n,B,\epsilon,\sigma} \colon R_n \restriction D_{n,B,\epsilon,\sigma} \to \Delta$ such that $d_{\nu_n}(\sigma, \rho_{n,B,\epsilon,\sigma}) \leq \epsilon$, and a $\rho_{n,B,\epsilon,\sigma}$-invariant Borel function $\phi_{n,B,\epsilon,\sigma} \colon D_{n,B,\epsilon,\sigma} \to Y$. It only remains to check that the functions of the form $\phi_{n,B,\epsilon,\sigma}$ are dense in $\mathrm{Hom}(E, \mu, F)$.

Towards this end, suppose that $\phi \in \mathrm{Hom}(E, \mu, F)$, and fix a Borel function $\rho \colon E \restriction \mathrm{dom}(\phi) \to \Delta$ for which ϕ is ρ-invariant. Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive rational numbers for which $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, and for each $n \in \mathbb{N}$, fix $B_n \in \mathscr{B}$ with $\mu(B_n \triangle \mathrm{dom}(\phi)) \leq \epsilon_n$ and $\sigma_n \in \mathscr{D}_n$ such that $d_{\nu_n}(\sigma_n, \rho \restriction (R_n \restriction \mathrm{dom}(\phi))) \leq \epsilon_n$. Then the sets $D_n = D_{n,B_n,\epsilon_n,\sigma_n}$

and the functions $\rho_n = \sigma_{n,B_n,\epsilon_n,\sigma_n}$ and $\phi_n = \phi_{n,B_n,\epsilon_n,\sigma_n}$ are well-defined. As $\mu(\mathrm{dom}(\phi) \triangle D_n) \le 2\epsilon_n$ and $d_{\nu_n}(\rho \restriction (R_n \restriction \mathrm{dom}(\phi)), \rho_n) \le 2\epsilon_n$ for all $n \in \mathbb{N}$, Lemma 2.3.6 ensures that $d_\mu(\phi, \phi_n) \to 0$. $\boxtimes$

In particular, we can now establish the existence of non-trivial projectively-separable countable Borel equivalence relations.

Theorem 2.3.9. *The orbit equivalence relation induced by* $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ *is projectively separable.*

Proof. Note that the orbit equivalence relation in question is Borel reducible to that induced by $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$. As Theorem 2.2.1 ensures that the latter action is projectively rigid, its induced orbit equivalence relation is projectively separable by Theorem 2.3.5. But Proposition 2.3.4 ensures that the projectively-separable countable Borel equivalence relations are closed under Borel reducibility. $\boxtimes$

2.4. THE SPACE OF MEASURES

Here we consider connections between E and $\mathcal{EQ}_E \setminus \mathcal{H}_E$. Theorems 1.6.3 and 1.7.8 ensure that the latter is a Borel subset of $P(X)$.

Proposition 2.4.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and the set $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a single measure-equivalence class. Then E is a successor of $\mathbb{E}_0$ under measure reducibility.*

Proof. Suppose that Y is a standard Borel space and F is a countable Borel equivalence relation on Y which is measure reducible to E, but not to $\mathbb{E}_0$. We must show that E is measure reducible to F.

By Theorem 1.7.11, there exists $\nu \in \mathcal{EQ}_F \setminus \mathcal{H}_F$. By Proposition 1.6.1, there is a ν-null Borel set $N \subseteq Y$ on which F is non-smooth. As F is countable, the Lusin-Novikov uniformization theorem ensures that $[N]_F$ is Borel, so by replacing N with $[N]_F$, we can assume that N is F-invariant. Fix a ν-conull Borel set $C \subseteq {\sim}N$ for which there is a Borel reduction $\phi\colon C \to X$ of $F \restriction C$ to E. As E and F are countable, the Lusin-Novikov uniformization theorem ensures that the set $B = [\phi(C)]_E$ is Borel, and that there is a Borel function $\psi\colon B \to C$ such that $\mathrm{graph}(\phi \circ \psi) \subseteq E$. In particular, it follows that ψ is a Borel reduction of $E \restriction B$ to $F \restriction C$.

Suppose now that μ is a Borel probability measure on X. As Propositon 1.3.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel reducibility, it follows that the push-forward ν' of $\nu \restriction C$ through ϕ is not in $\mathcal{H}_E$. By Proposition 1.6.6, there is an E-quasi-invariant Borel probability measure ν'' on X such

that $\nu' \ll \nu''$ and the two measures have the same E-invariant null Borel sets. Then $\nu'' \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, so $E \upharpoonright {\sim}B$ is measure hyperfinite, thus there is a Borel set $A \subseteq {\sim}B$ such that $E \upharpoonright A$ is hyperfinite and $\mu(A \cup B) = 1$. As Theorem 1.3.7 ensures that every hyperfinite Borel equivalence relation is Borel reducible to every non-smooth Borel equivalence relation, it follows that there is a Borel reduction $\psi' \colon A \to N$ of $E \upharpoonright A$ to $F \upharpoonright N$. As $\psi \cup \psi'$ is a reduction of $E \upharpoonright (A \cup B)$ to F, it follows that E is μ-reducible to F, thus E is measure reducible to F. $\boxtimes$

Proposition 2.4.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a non-empty countable union of measure-equivalence classes. Then E is a countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility.*

Proof. Suppose that N is a non-empty countable set and $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is the disjoint union of the measure-equivalence classes of Borel probability measures μ_n on X, for $n \in N$. Fix a partition $(B_n)_{n \in N}$ of X into E-invariant Borel sets with the property that $\mu_n(B_n) = 1$ for all $n \in N$, and observe that Proposition 2.4.1 ensures that each $E \upharpoonright B_n$ is a successor of $\mathbb{E}_0$ under measure reducibility. $\boxtimes$

On the other hand, we have the following.

Proposition 2.4.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes. Then there are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint E-invariant subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. As measure equivalence is Borel, Theorem 1.1.1 yields a Borel sequence $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of pairwise orthogonal Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$. Theorem 1.5.1 then implies that by thinning down $(\mu_c)_{c \in 2^{\mathbb{N}}}$, we can ensure the existence of a Borel sequence $(A_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint Borel subsets of X such that $\mu_c(A_c) = 1$ for all $c \in 2^{\mathbb{N}}$. Define $B_c = \{x \in X \mid [x]_E \subseteq A_c\}$. $\boxtimes$

Combining the previous two results yields the following.

Proposition 2.4.4. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite countable Borel equivalence relation on X. Then at least one of the following holds:*

  (1) *The relation E is a countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility.*

(2) *There are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint E-invariant subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in $\mathcal{E}\mathcal{Q}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. This follows from Propositions 2.4.2 and 2.4.3. ⊠

Let $\ll_{E,F}$ denote the set of all $(\mu, \nu) \in (\mathcal{E}\mathcal{Q}_E \setminus \mathcal{H}_E) \times (\mathcal{E}\mathcal{Q}_F \setminus \mathcal{H}_F)$ for which there is a μ-conull Borel set $C \subseteq X$ and a Borel reduction $\phi \colon C \to Y$ of $E \upharpoonright C$ to F sending $(\mu \upharpoonright C)$-positive sets to ν-positive sets. Clearly $\ll_{E,F}$ is transitive, and if $C \subseteq X$ is a μ-conull Borel set and $\phi \colon C \to X$ is a Borel reduction of $E \upharpoonright C$ to F, then $\mu \ll_{E,F} \phi_*(\mu \upharpoonright C)$. When $E = F$, we simply write $\ll_E$. It is easy to see this is an equivalence relation, in spite of our adherence to the usual measure-theoretic abuse of notation.

The following fact provides a partial converse to Proposition 2.4.1.

Proposition 2.4.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and some vertical section of $\ll_E$ is a countable union of measure-equivalence classes. Then the following are equivalent:*

(1) *The set $\mathcal{E}\mathcal{Q}_E \setminus \mathcal{H}_E$ is a single measure-equivalence class.*
(2) *The relation E is a successor of $\mathbb{E}_0$ under measure reducibility.*

Proof. By Proposition 2.4.1, it is sufficient to show that if $\mathcal{E}\mathcal{Q}_E \setminus \mathcal{H}_E$ contains multiple measure-equivalence classes, then E is not a successor of $\mathbb{E}_0$ under measure reducibility. Towards this end, fix $\mu \in \mathcal{E}\mathcal{Q}_E \setminus \mathcal{H}_E$ for which the corresponding vertical section of $\ll_E$ is a countable union of measure-equivalence classes, as well as a Borel probability measure ν on X in $\mathcal{E}\mathcal{Q}_E \setminus \mathcal{H}_E$ for which $\mu \not\sim \nu$. Fix an E-invariant ν-conull Borel set $D \subseteq X$ which is null with respect to every measure in the μth vertical section of $\ll_E$ which is not measure equivalent to ν.

Lemma 2.4.6. *Suppose that $A \subseteq X \setminus D$ is a μ-conull Borel set and $B \subseteq D$ is a ν-conull Borel set. Then there is no Borel reduction $\phi \colon A \cup B \to D$ of $E \upharpoonright (A \cup B)$ to $E \upharpoonright D$.*

Proof. Suppose that ϕ is such a reduction. Then our choice of D ensures that $(\phi \upharpoonright A)_*(\mu \upharpoonright A) \ll \nu$, so $\mu \ll_E \nu$. As $\ll_E$ is transitive, it follows that $\mu \ll_E (\phi \upharpoonright B)_*(\nu \upharpoonright B)$, so our choice of D also ensures that $(\phi \upharpoonright B)_*(\nu \upharpoonright B) \ll \nu$. Then there exist $x \in A$ and $y \in B$ such that $\phi(x) \mathrel{E} \phi(y)$, contradicting the fact that ϕ is a reduction. ⊠

In particular, it follows that E is not measure reducible to $E \upharpoonright D$, and therefore cannot be a successor of $\mathbb{E}_0$ under measure reducibility. ⊠

The following provides a partial converse to Proposition 2.4.2.

Proposition 2.4.7. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and every vertical section of $\ll_E$ is a countable union of measure-equivalence classes. Then the following are equivalent:*

(1) *The set $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a non-empty countable union of measure-equivalence classes.*
(2) *The relation E is a non-empty countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility.*

Proof. By Proposition 2.4.2, it is sufficient to show that if N is a non-empty countable set and $(B_n)_{n \in N}$ is a partition of X into E-invariant Borel sets on which E is a successor of $\mathbb{E}_0$ under measure reducibility, then $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a countable union of measure-equivalence classes. Towards this end, note that for all $n \in N$, every vertical section of $\ll_{E \restriction B_n}$ is a countable union of measure-equivalence classes, so Proposition 2.4.5 ensures that $\mathcal{EQ}_{E \restriction B_n} \setminus \mathcal{H}_{E \restriction B_n}$ is the measure-equivalence class of some Borel probability measure μ_n on B_n. Identifying μ_n with the corresponding Borel probability measure on X, it follows that $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is the union of the measure-equivalence classes of μ_n, for $n \in N$.   ⊠

Summarizing these results, we obtain the following.

Theorem 2.4.8. *Suppose that X is a standard Borel space, E is a non-measure-hyperfinite countable Borel equivalence relation on X, and every vertical section of $\ll_E$ is a countable union of measure-equivalence classes. Then exactly one of the following holds:*

(1) *The relation E is a countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility.*
(2) *There are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint E-invariant subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. If $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a non-empty countable union of measure-equivalence classes, then Proposition 2.4.7 ensures that condition (2) holds, and its proof implies that condition (3) fails. If $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes, then Proposition 2.4.7 ensures that condition (2) fails, and Proposition 2.4.3 implies that condition (3) holds.   ⊠

In light of our earlier results, the following yields a criterion for ensuring that Borel subequivalence relations of successors of $\mathbb{E}_0$ under measure reducibility are again successors of $\mathbb{E}_0$ under measure reducibility.

Proposition 2.4.9. *Suppose that X is a standard Borel space, $E \subseteq F$ are countable Borel equivalence relations on X, μ is an E-ergodic*

F-quasi-invariant Borel probability measure on X, and $\mathcal{EQ}_F \setminus \mathcal{H}_F$ is contained in the measure-equivalence class of μ. Then $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is also contained in the measure-equivalence class of μ.

Proof. Suppose that $\nu \in \mathcal{EQ}_E$ but $\mu \not\sim \nu$. Then there is an E-invariant μ-null ν-conull Borel set $C \subseteq X$, in which case Proposition 1.6.6 yields a Borel probability measure $\nu' \gg \nu$ with the same F-invariant null sets. As the F-quasi-invariance of μ ensures that $[C]_F$ is μ-null, it follows that $\nu' \in \mathcal{H}_F$. As Proposition 1.3.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that $\nu' \in \mathcal{H}_E$, thus $\nu \in \mathcal{H}_E$. ⊠

We also have the following criterion for ensuring strong ergodicity.

Proposition 2.4.10. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, F is hyperfinite, and $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$ is not (E, F)-ergodic. Then $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes.*

Proof. Fix a μ-null-to-one Borel homomorphism $\phi\colon X \to Y$ from E to F, as well as a Borel disintegration $(\mu_y)_{y \in Y}$ of μ through ϕ.

Then the set $C = \{y \in Y \mid E \text{ is not } \mu_y\text{-hyperfinite}\}$ is Borel by Theorem 1.7.8. As E is μ-nowhere hyperfinite, Proposition 1.7.13 ensures that C is $(\phi_*\mu)$-conull.

In particular, as ϕ is μ-null-to-one, it follows that C is uncountable, in which case there is an uncountable partial transversal $P \subseteq C$ of F. Theorem 1.7.11 then yields Borel probability measures ν_y on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $[\phi^{-1}(y)]_E$ is ν_y-conull, for all $y \in P$. As the latter sets are pairwise disjoint, it follows that $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes. ⊠

We next compute a bound on the complexity of $\ll_{E,F}$.

Proposition 2.4.11. *Suppose that X and Y are standard Borel spaces and E and F are countable Borel equivalence relations on X and Y. Then $\ll_{E,F}$ is analytic.*

Proof. Note that $\mu \ll_{E,F} \nu$ if and only if there is a code c for a measurable function $\phi_c\colon X \to Y$ such that $(\phi_c)_*(\mu \restriction \mathrm{dom}(\phi_c)) \ll \nu$ and ϕ_c is a reduction of E to F on a μ-conull set. Proposition 1.5.4 ensures that the former relation is Borel, and Proposition 1.6.14 implies that the latter relation is analytic. ⊠

We close this section by noting that our hypothesis on $\ll_E$ holds of all projectively-separable countable Borel equivalence relations.

Proposition 2.4.12. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, and F is projectively separable. Then the vertical sections of $\ll_{E,F}$ are countable unions of measure-equivalence classes.*

Proof. Suppose that $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, and let A denote the vertical section of $\ll_{E,F}$ corresponding to μ. As Proposition 2.4.11 ensures that $\ll_{E,F}$ is analytic, so too is A. As measure equivalence is Borel, Theorem 1.1.1 implies that if A is not a union of countably-many measure-equivalence classes, then there is a Borel sequence $(\nu_c)_{c \in 2^\mathbb{N}}$ of pairwise orthogonal Borel probability measures on Y in A. Theorem 1.5.1 then ensures that by passing to an appropriate subsequence, we can ensure that there is a Borel sequence $(D_c)_{c \in 2^\mathbb{N}}$ of pairwise disjoint subsets of Y such that $\nu_c(D_c) = 1$ for all $c \in 2^\mathbb{N}$. But for each $c \in 2^\mathbb{N}$, there is a μ-conull Borel set $C_c \subseteq X$ for which there is a Borel reduction $\phi_c \colon C_c \to D_c$ from $E \restriction C_c$ to $F \restriction D_c$, contradicting the projective separability of F.   ⊠

2.5. STRATIFICATION

Proposition 1.3.3 ensures that every aperiodic countable Borel equivalence relation has an aperiodic hyperfinite Borel subequivalence relation. This is the special case of the following fact, in which G is the difference of E and equality, and ρ is the constant cocycle.

Proposition 2.5.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, G is a Borel graphing of E, and $\rho \colon E \to (0, \infty)$ is an aperiodic Borel cocycle. Then there is a Borel subgraph H of G generating a hyperfinite Borel equivalence relation on which ρ is aperiodic.*

Proof. As graphs of Borel functions are themselves Borel, the following observation implies that it is sufficient to establish the proposition on an E-complete Borel set.

Lemma 2.5.2. *Suppose that $B \subseteq X$ is an E-complete Borel set and H is a Borel subgraph of $G \restriction B$ generating a hyperfinite Borel equivalence relation on which ρ is aperiodic. Then there is a Borel function $f \colon \sim B \to X$ such that $\mathrm{graph}(f^{\pm 1}) \cup H$ is a subgraph of G generating a hyperfinite Borel equivalence relation on which ρ is aperiodic.*

Proof. As the vertical sections of G are countable, the Lusin-Novikov uniformization theorem yields Borel sets $B_n \subseteq X$ and Borel functions $f_n \colon B_n \to X$ with the property that $G = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$. Let $d_G(x, B)$ denote the length of the shortest G-path from x to an element of B.

Observe that this function is Borel, as it can also be expressed, for $x \notin B$, as the least $n \in \mathbb{N}$ for which there exist $k_1, \ldots, k_n \in \mathbb{N}$ such that $f_{k_1} \circ \cdots \circ f_{k_n}(x) \in B$. Noting that for each $x \in X$, the set of $y \in G_x$ with the property that $d_G(y, B) = d_G(x, B) - 1$ is countable, one more application of the Lusin-Novikov uniformization theorem yields a Borel function $f \colon {\sim}B \to X$, whose graph is contained in G, such that $d_G(f(x), B) = d_G(x, B) - 1$ for all $x \in {\sim}B$. As every connected component of $\mathrm{graph}(f^{\pm 1}) \cup H$ contains a connected component of H, it follows that ρ is aperiodic on the equivalence relation generated by $\mathrm{graph}(f^{\pm 1}) \cup H$. As the function sending x to $f^{d_G(x,B)}(x)$ is a Borel reduction of the latter equivalence relation to that generated by H, and Proposition 1.3.2 ensures that the class of hyperfinite Borel equivalence relations is closed under Borel reducibility, it follows that the equivalence relation generated by $\mathrm{graph}(f^{\pm 1}) \cup H$ is hyperfinite. $\boxtimes$

We will now recursively construct an increasing sequence $(H_n)_{n \in \mathbb{N}}$ of approximations to the desired graph, beginning with $H_0 = \emptyset$. Given H_n, let E_n denote the equivalence relation induced by H_n, and let B_n denote the set of all $x \in X$ for which ρ is finite on $E_n \restriction [x]_E$. As H_n and E are countable, the Lusin-Novikov uniformization theorem ensures that these sets are Borel. As Proposition 1.6.2 implies that countable Borel equivalence relations admitting finite Borel cocycles to $\mathbb{R}$ are smooth, it follows that $E_n \restriction B_n$ is smooth. Remark 1.2.2 therefore yields a Borel transversal $A_n \subseteq B_n$ of $E_n \restriction B_n$. Let R_n be the relation consisting of all $(x, (y, (x', y'))) \in A_n \times (A_n \times (B_n \times B_n))$ for which $x \ E_n \ x' \ (G \setminus E_n) \ y' \ E_n \ y$ and $\rho([x]_{E_n}, [y]_{E_n}) \leq 1$. As the vertical sections of R_n are countable, the Lusin-Novikov uniformization theorem ensures that the set $A_n' = \mathrm{proj}_{A_n}(R_n)$ is Borel, there is a Borel uniformization $f_n' \colon A_n' \to A_n \times (B_n \times B_n)$ of R_n, and both of the sets $S_n = f_n'(A_n')$ and $H_n' = \mathrm{proj}_{B_n \times B_n}(S_n)^{\pm 1}$ are Borel.

Set $H_{n+1} = H_n \cup H_n'$. To see that the equivalence relation E_{n+1} generated by H_{n+1} is hyperfinite, we consider the function $f_n \colon A_n \to A_n$ given by $f_n = (\mathrm{proj}_{A_n} \circ f_n') \cup (\mathrm{id} \restriction (A_n \setminus A_n'))$. As Theorem 1.3.6 ensures that $E_t(f_n)$ is hypersmooth, Theorem 1.3.5 implies that it is hyperfinite. As Proposition 1.3.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel reducibility, and the unique function $\phi_n \colon B_n \to A_n$ such that $\forall x \in B_n \ x \ E_n \ \phi_n(x)$ is a Borel reduction of $E_{n+1} \restriction B_n$ to $E_t(f_n)$, it follows that E_{n+1} is hyperfinite. This completes the recursive construction.

As every equivalence class of $E \restriction {\sim}B_n$ contains a ρ-infinite equivalence class of E_n, it follows from Lemma 2.5.2 that we can construct the desired graph off of the set $B_\infty = \bigcap_{n \in \mathbb{N}} B_n$. In order to construct

the desired graph on B_∞, set $H_\infty = \bigcup_{n \in \mathbb{N}} H_n$ and let E_∞ denote the equivalence relation generated by H_∞. As $E_\infty = \bigcup_{n \in \mathbb{N}} E_n$, it follows that $E_\infty \upharpoonright B_\infty$ is hypersmooth, so Theorem 1.3.5 ensures that it is hyperfinite. By one more application of Lemma 2.5.2, it is therefore sufficient to observe that there do not exist $(G \setminus E_\infty)$-related points $x, y \in B_\infty$ for which the corresponding equivalence classes $[x]_{E_\infty}, [y]_{E_\infty}$ are ρ-finite.

Suppose, towards a contradiction, that there are such points. Then there exists $n \in \mathbb{N}$ such that $\rho([x]_{E_\infty}, [x]_{E_n}), \rho([y]_{E_\infty}, [y]_{E_n}) < 2$. As $\rho([x]_{E_n}, [y]_{E_n}) \leq 1$ or $\rho([y]_{E_n}, [x]_{E_n}) \leq 1$, it follows that $\phi_n(x) \in A'_n$ or $\phi_n(y) \in A'_n$, so $\rho([x]_{E_{n+1}}, [x]_{E_n}) \geq 2$ or $\rho([y]_{E_{n+1}}, [y]_{E_n}) \geq 2$, thus $\rho([x]_{E_\infty}, [x]_{E_{n+1}}) < 1$ or $\rho([y]_{E_\infty}, [y]_{E_{n+1}}) < 1$, which is impossible.   ⊠

In particular, we obtain the following measure-theoretic corollary.

Proposition 2.5.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-quasi-invariant Borel probability measure on X for which E is μ-nowhere smooth, and G is a Borel graphing of E. Then there is a Borel subgraph H of G whose induced equivalence relation is μ-nowhere smooth but hyperfinite.*

Proof. By Proposition 1.6.4, there is a Borel cocycle $\rho \colon E \to (0, \infty)$ with respect to which μ is invariant. As Proposition 1.6.2 ensures that countable Borel equivalence relations admitting finite Borel cocycles to $\mathbb{R}$ are smooth, by throwing away an E-invariant μ-null Borel set on which E is smooth, we can assume that ρ is aperiodic. Proposition 2.5.1 then yields a Borel subgraph H of G generating a hyperfinite equivalence relation on which ρ is aperiodic. As Proposition 1.6.5 ensures that every such relation is μ-nowhere smooth, the result follows.   ⊠

The following yields disjoint Borel sets which, in the measure-theoretic setting, are complete with respect to different equivalence relations.

Proposition 2.5.4. *Suppose that X is a standard Borel space, E and F are aperiodic countable Borel equivalence relations on X, and μ and ν are Borel probability measures on X. Then there are disjoint Borel sets $A, B \subseteq X$ such that $\mu([A]_E) = \nu([B]_F) = 1$.*

Proof. By two applications of Proposition 1.2.5, there are decreasing sequences $(A_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ of Borel subsets of X such that each A_n is E-complete, each B_n is F-complete, and $\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} B_n = \emptyset$. Fix real numbers $\epsilon_n > 0$ such that $\epsilon_n \to 0$ as $n \to \infty$, and recursively construct strictly increasing sequences $(i_n)_{n \in \mathbb{N}}$ and $(j_n)_{n \in \mathbb{N}}$ of natural numbers by setting $i_0 = 0$, and given $n \in \mathbb{N}$ and $i_n \in \mathbb{N}$, choosing $j_n > \max_{m < n} j_m$ sufficiently large that $\mu([A_{i_n} \setminus B_{j_n}]_E) \geq 1 - \epsilon_n$, as well

as $i_{n+1} > i_n$ sufficiently large that $\nu([B_{j_n} \setminus A_{i_{n+1}}]_F) \geq 1 - \epsilon_n$. Define $A = \bigcup_{n \in \mathbb{N}} (A_{i_n} \setminus B_{j_n})$ and $B = \bigcup_{n \in \mathbb{N}} (B_{j_n} \setminus A_{i_{n+1}})$. ⊠

A *directed graph* on X is an irreflexive subset G of $X \times X$. The *domain* of such a relation is the set of x for which G_x is non-empty. An *oriented graph* on X is an irreflexive antisymmetric subset H of $X \times X$. An *orientation* of a graph G is an oriented graph H with $G = H^{\pm 1}$. Although the domain of an orientation H of a graph G can be strictly smaller than the domain of G itself, we do have the following.

Proposition 2.5.5. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, G is a locally countable Borel graph on X, and μ is an E-quasi-invariant Borel probability measure on X for which E is μ-nowhere smooth and the domain of G has μ-conull E-saturation. Then there is a Borel orientation H of G whose domain has μ-conull E-saturation.*

Proof. For each Borel set $B \subseteq X$, put $X_B = \{x \in B \mid [x]_{E \restriction B}$ is finite$\}$. As E is countable, the Lusin-Novikov uniformization theorem ensures that such sets are Borel, as are E-saturations of Borel sets.

Lemma 2.5.6. *Suppose that $B \subseteq X$ is Borel. Then $[X_B]_E$ is μ-null.*

Proof. As E is countable, the Lusin-Novikov uniformization theorem ensures that there is a Borel reduction of $E \restriction [X_B]_E$ to $E \restriction X_B$. As Proposition 1.2.1 ensures that $E \restriction X_B$ is smooth, so too is $E \restriction [X_B]_E$. As E is μ-nowhere smooth, it follows that $[X_B]_E$ is μ-null. ⊠

We consider now the special case that G is of the form $\text{graph}(I)$, where $A \subseteq X$ is a Borel set and $I \colon A \to A$ is a Borel involution. Proposition 1.2.1 and Remark 1.2.2 yield a Borel transversal $B \subseteq A$ of the equivalence relation generated by G. Lemma 2.5.6 ensures that the set $C = [A]_E \setminus [X_B \cup X_{A \setminus B}]_E$ is μ-conull.

We use E_B, $E_{A \setminus B}$, μ_B, and $\mu_{A \setminus B}$ to denote the restrictions of E, $(I \times I)^{-1}(E)$, μ, and $I_* \mu$ to $B \cap C$. As E_B and $E_{A \setminus B}$ are aperiodic, Proposition 2.5.4 yields a Borel set $B' \subseteq B$, an E_B-invariant μ_B-null Borel set $N_B \subseteq C$, and an $E_{A \setminus B}$-invariant $\mu_{A \setminus B}$-null Borel set $N_{A \setminus B} \subseteq C$ such that $B' \cup N_B$ is E_B-complete and $(B \setminus B') \cup N_{A \setminus B}$ is $E_{A \setminus B}$-complete. As μ is E-quasi-invariant, the set $D = C \setminus [N_B \cup N_{A \setminus B}]_E$ is μ-conull. Let H denote the graph of the restriction of I to $B' \cup I(B \setminus B')$.

The fact that B is a transversal of the equivalence relation generated by I ensures that H is an oriented graph. To see that H is an orientation of G, note that if $x \mathrel{G} y$, then $x \in B$ or $y \in B$, from which it follows that $(x \in B'$ or $y \in I(B \setminus B'))$ or $(y \in B'$ or $x \in I(B \setminus B'))$, so $(x \mathrel{H} y$ or $y \mathrel{H} x)$ or $(y \mathrel{H} x$ or $x \mathrel{H} y)$, thus $x \mathrel{H} y$ or $y \mathrel{H} x$. To see

that the E-saturation of the domain of H is μ-conull, it is enough to show that the domain of H intersects the E-class of every $x \in D$. Towards this end, note that $A \cap [x]_E$ is non-empty, thus so too is $B \cap [x]_E$ or $(A \setminus B) \cap [x]_E$, in which case $B' \cap [x]_E$ or $I(B \setminus B') \cap [x]_E$ is non-empty as well, hence the domain of H intersects $[x]_E$.

We now consider the general case. As G is locally countable, Theorem 1.2.4 yields Borel sets $A_n \subseteq X$ and Borel involutions $I_n \colon A_n \to A_n$, with pairwise disjoint graphs, such that $G = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(I_n)$. Setting $G_n = \mathrm{graph}(I_n)$, $X_n = [A_n]_E$, and $\mu_n = \mu \upharpoonright X_n$, the above special case yields Borel orientations H_n of G_n whose domains have μ_n-conull E-saturations. Then $H = \bigcup_{n \in \mathbb{N}} H_n$ is a Borel orientation of G whose domain has μ-conull E-saturation. $\boxtimes$

A μ-*stratification* of E is an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of E whose union is E and which is strictly increasing on every μ-positive Borel set.

Theorem 2.5.7. *Suppose that X is a standard Borel space, E is a treeable countable Borel equivalence relation on X, and μ is an E-quasi-invariant Borel probability measure on X for which E is μ-nowhere hyperfinite. Then there is a Borel μ-stratification of E.*

Proof. Fix a Borel treeing G of E. By Proposition 2.5.3, we can assume that there is Borel subgraph H of G whose induced equivalence relation F is μ-nowhere smooth but hyperfinite. As E is μ-nowhere hyperfinite, the F-saturation of the domain of $G \setminus H$ is μ-conull. As F is μ-nowhere smooth, Proposition 2.5.5 ensures that there is a Borel orientation K of $G \setminus H$ whose domain has μ-conull F-saturation. As μ is E-quasi-invariant, by throwing out an E-invariant μ-null Borel set, we can assume that the domain of K intersects every F-class. By Proposition 1.6.4, there is a Borel cocycle $\rho \colon E \to (0, \infty)$ with respect to which μ is invariant. As F is μ-nowhere smooth and Proposition 1.6.2 ensures that F is smooth on the finite part of $\rho \upharpoonright (F \upharpoonright \mathrm{dom}(K))$, by throwing out another μ-null Borel set, we can assume that $\rho \upharpoonright (F \upharpoonright \mathrm{dom}(K))$ is aperiodic, and therefore that $F \upharpoonright \mathrm{dom}(K)$ is aperiodic. The E-quasi-invariance of μ again allows us to ensure that the set we throw out is E-invariant. Proposition 1.2.6 then yields a partition of the domain of K into a sequence $(B_q)_{q \in \mathbb{Q}}$ of pairwise disjoint F-complete Borel sets. Set $K_r = K \upharpoonright (\bigcup_{q < r} B_q \times X)$ and $G_r = H \cup K_r^{\pm 1}$ for all $r \in \mathbb{R}$. As G_r is locally countable, the Lusin-Novikov uniformization theorem ensures that the equivalence relations E_r induced by the graphs G_r are Borel.

Suppose now that $B \subseteq X$ is a Borel set for which there are real numbers $r < s$ with $E_r \upharpoonright B = E_s \upharpoonright B$. Then $B \cap [x]_{E_s} \subseteq [x]_{E_r}$ for all

$x \in B$. As $G_r \subseteq G_s$ and the latter graph is acyclic, it follows that if $x \in B$ and $y \in [x]_{E_s} \setminus [x]_{E_r}$, then there is a unique point of $[y]_{E_r}$ of minimal distance to $[x]_{E_r}$ with respect to the graph metric associated with G_s. Let $\phi \colon [B]_{E_s} \setminus [B]_{E_r} \to [B]_{E_s} \setminus [B]_{E_r}$ be the function sending each point of its domain to the corresponding point of its E_r-class. As E is countable, the Lusin-Novikov uniformization theorem ensures that $[B]_{E_r}$, $[B]_{E_s}$, and ϕ are Borel. As ϕ is a selector for the restriction of E_r to $[B]_{E_s} \setminus [B]_{E_r}$, it follows that this restriction is smooth. As F is μ-nowhere smooth and Proposition 1.2.3 ensures that the class of smooth countable Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that E_r is also μ-nowhere smooth. In particular, this means that the set $[B]_{E_s} \setminus [B]_{E_r}$ is μ-null, and since μ is E_s-quasi-invariant, so too is the E_s-saturation of $[B]_{E_s} \setminus [B]_{E_r}$. As every E_r-class is properly contained in the corresponding E_s-class, it follows that B is contained in this saturation, and is therefore μ-null as well, hence $(E_r)_{r \in \mathbb{R}}$ is indeed a μ-stratification of E.   ⊠

Part 3. Applications

Here we obtain our main results. While our theorems were listed in the introduction in order of importance, we now proceed according to the amount of new machinery behind the arguments, with those requiring the least appearing first. In §3.1, we use the countability of the vertical sections of $\ll_E$ to establish our results on products. In §3.2, we combine the countability of the vertical sections of $\ll_E$ with facts about compressibility and costs of equivalence relations to obtain our results on the distinction between embeddability and reducibility. In §3.3, we combine projective separability, facts about $\ll_E$, and the existence of stratifications to obtain our results on antichains and the distinction between containment and reducibility. In §3.4, we use these tools to obtain our anti-basis theorems. And in §3.5, we combine these tools with Theorem 1.9.1 to obtain our complexity results.

3.1. Products

We begin this section with an observation concerning measurable reducibility of products.

Proposition 3.1.1. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, m is a continuous Borel probability measure on $\mathbb{R}$, $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, and the μth vertical section of $\ll_{E,F}$ is a countable union of measure-equivalence classes. Then $E \times \Delta(\mathbb{R})$ is $(\mu \times m)$-nowhere reducible to F.*

Proof. Suppose, towards a contradiction, that there is a $(\mu \times m)$-positive Borel set $B \subseteq X \times \mathbb{R}$ on which there is a Borel reduction $\phi \colon B \to Y$ of $E \times \Delta(\mathbb{R})$ to F. As E is countable, the Lusin-Novikov uniformization theorem ensures that $[B]_{E \times \Delta(\mathbb{R})}$ is Borel, in addition to yielding a Borel reduction of $(E \times \Delta(\mathbb{R})) \upharpoonright [B]_{E \times \Delta(\mathbb{R})}$ to $(E \times \Delta(\mathbb{R})) \upharpoonright B$. By replacing B with its $(E \times \Delta(\mathbb{R}))$-saturation, we can therefore assume that B is $(E \times \Delta(\mathbb{R}))$-invariant. Note that the set $R = \{r \in \mathbb{R} \mid \mu(B^r) > 0\}$ is m-positive, by Fubini's theorem. As m is continuous, it follows that R is uncountable. For each $r \in R$, Proposition 1.6.6 yields an F-quasi-invariant Borel probability measure ν_r on Y such that $(\phi^r)_*(\mu \upharpoonright B^r) \ll \nu_r$, but the two measures have the same F-invariant null sets. But then the ν_r are pairwise orthogonal elements of the μth vertical section of $\ll_{E,F}$, the desired contradiction.   ⊠

This has the following consequences for measure reducibility.

Proposition 3.1.2. *Suppose that X and Y are standard Borel spaces, E is a non-measure-hyperfinite countable Borel equivalence relation on X, and F is a projectively separable countable Borel equivalence relation on Y. Then $E \times \Delta(\mathbb{R})$ is not measure reducible to F.*

Proof. By Theorem 1.7.11, there exists $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$. Proposition 2.4.12 then implies that the μth vertical section of $\ll_{E,F}$ is a countable union of measure-equivalence classes. Fix a continuous Borel probability measure m on $\mathbb{R}$. As Proposition 3.1.1 ensures that $E \times \Delta(\mathbb{R})$ is $(\mu \times m)$-nowhere reducible to F, the former is not measure reducible to the latter.   ⊠

Theorem 3.1.3 (Hjorth). *There is a non-measure-hyperfinite treeable countable Borel equivalence relation to which some treeable countable Borel equivalence relation is not measure reducible.*

Proof. Proposition 3.1.2 ensures that every non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation has the desired property.   ⊠

We now consider products with smaller equivalence relations.

Proposition 3.1.4. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, m is a strictly positive probability measure on 2, $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, $\nu \in \mathcal{EQ}_F \setminus \mathcal{H}_F$, and $\mu \ll_{E,F} \nu$. If the μth vertical section of $\ll_{E,F}$ is a countable union of measure-equivalence classes, then there is an F-invariant ν-conull Borel set $C \subseteq Y$ for which $E \times \Delta(2)$ is not $(\mu \times m)$-reducible to $F \upharpoonright C$.*

Proof. Fix an F-invariant ν-conull Borel set $C \subseteq Y$ which is ν'-null for every measure ν' in the vertical section of $\ll_{E,F}$ corresponding to μ, other than those which are measure equivalent to ν. Suppose, towards a contradiction, that there is a $(\mu \times m)$-positive Borel set $B \subseteq X \times 2$ on which there is a Borel reduction $\phi \colon B \to Y$ of $E \times \Delta(2)$ to $F \upharpoonright C$. As E is countable, the Lusin-Novikov uniformization theorem ensures that $[B]_{E \times \Delta(2)}$ is Borel, in addition to yielding a Borel reduction of $(E \times \Delta(2)) \upharpoonright [B]_{E \times \Delta(2)}$ to $(E \times \Delta(2)) \upharpoonright B$. By replacing B with its $(E \times \Delta(2))$-saturation, we can therefore assume that B is $(E \times \Delta(2))$-invariant. Proposition 1.6.6 then yields $(F \upharpoonright C)$-quasi-invariant Borel probability measures ν_i on C with the property that $(\phi^i)_*(\mu \upharpoonright B^i) \ll \nu_i$ but the two measures have the same E-invariant null Borel sets, for all $i < 2$. As ν_0 and ν_1 are orthogonal elements of the vertical section of $\ll_{E,F \upharpoonright C}$, this contradicts our choice of C. $\boxtimes$

This has the following consequence for measure reducibility.

Proposition 3.1.5. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable countable Borel equivalence relation on X. Then there is a Borel set $B \subseteq X$ on which E is not measure hyperfinite such that $(E \upharpoonright B) \times \Delta(2)$ is not measure reducible to $E \upharpoonright B$.*

Proof. By Theorem 1.7.11, there exists $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$. Proposition 2.4.12 then implies that the μth vertical section of $\ll_E$ is a countable union of measure-equivalence classes. Fix a strictly positive probability measure m on 2. As Proposition 3.1.4 yields a μ-conull Borel set $C \subseteq X$ for which $E \times \Delta(2)$ is not $(\mu \times m)$-reducible to $E \upharpoonright C$, it follows that $(E \upharpoonright C) \times \Delta(2)$ is not measure reducible to $E \upharpoonright C$. $\boxtimes$

Remark 3.1.6. A similar argument can be used to show that if X and Y are standard Borel spaces, E is a non-measure-hyperfinite countable Borel equivalence relation on X, and F is a non-measure-hyperfinite projectively-separable countable Borel equivalence relation on Y, then there is a Borel set $B \subseteq Y$ on which F is not measure-hyperfinite such that $E \times \Delta(2)$ is not measure reducible to $F \upharpoonright B$.

3.2. REDUCIBILITY WITHOUT EMBEDDABILITY

We begin this section with an observation concerning the relationship between measurable reducibility and measurable embeddability.

Proposition 3.2.1. *Suppose that X and Y are standard Borel spaces, E is an invariant-measure-hyperfinite countable Borel equivalence relation on X, F is an aperiodic countable Borel equivalence relation Y,*

and μ is a Borel probability measure on X. Then E is μ-reducible to F if and only if E is μ-embeddable into F.

Proof. Suppose that E is μ-reducible to F, and fix a μ-conull Borel set $C \subseteq X$ on which there is a Borel reduction $\phi\colon C \to Y$ of E to F. As E is countable, the Lusin-Novikov uniformization theorem ensures that $[C]_E$ is Borel, and there is a Borel reduction of $E \restriction [C]_E$ to $E \restriction C$. By replacing ϕ with its composition with such a function, we can therefore assume that C is itself E-invariant. Proposition 1.6.6 ensures that there is an E-quasi-invariant Borel probability measure on X, with respect to which μ is absolutely continuous, which agrees with μ on all E-invariant Borel sets. By replacing μ with such a measure, we can assume that μ is E-quasi-invariant.

We handle first the case that F is smooth. Then $E \restriction C$ is also smooth. As E is countable, Remark 1.2.2 yields partitions $(C_n)_{n \in \mathbb{N}}$ of C into Borel partial transversals of E, and $(D_n)_{n \in \mathbb{N}}$ of Y into Borel transversals of F. One then obtains an embedding $\pi\colon C \to Y$ of $E \restriction C$ into F by setting

$$\pi(x) = y \iff \exists n \in \mathbb{N} \ (x \in C_n, \ y \in D_n, \text{ and } \phi(x) \ F \ y).$$

As C inherits a standard Borel structure from X and functions between standard Borel spaces are Borel if and only if their graphs are Borel, it follows that π is Borel.

We next turn to the case that F is non-smooth. As Proposition 1.3.10 ensures that there is a Borel reduction of F to the restriction of F to an F-invariant Borel set off of which F is smooth, by composing such a reduction with ϕ, we can assume that the restriction of F to the set $Z = {\sim}[\phi(X)]_E$ is non-smooth. As ϕ is countable-to-one, the Lusin-Novikov uniformization theorem yields an $(E \restriction C)$-complete Borel set $B \subseteq C$ on which ϕ is injective.

Fix a Borel set $A \subseteq B$ of maximal μ-measure on which E is compressible. As E is countable, the Lusin-Novikov uniformization theorem ensures that $[A]_E$ is Borel. As Proposition 1.6.9 ensures that countable Borel equivalence relations can be Borel embedded into their restrictions to complete compressible Borel sets, there is a Borel injection $\psi\colon [A]_E \to A$ whose graph is contained in E. Then the function $\pi = \phi \circ \psi$ is a Borel embedding of $E \restriction [A]_E$ into $F \restriction \phi(C)$.

If $\mu([A]_E) = 1$, then it follows that E is μ-embeddable into F. Otherwise, Theorem 1.6.10 ensures that $\mu \restriction (B \setminus [A]_E)$ is equivalent to an $E \restriction (B \setminus [A]_E)$-invariant Borel probability measure ν on $B \setminus [A]_E$. As E is invariant-measure hyperfinite, there is a ν-conull Borel set $A' \subseteq B \setminus [A]_E$ on which E is hyperfinite. As E is countable, the Lusin-Novikov uniformization theorem ensures that $[A']_E$ is Borel and

there is a Borel reduction of $E \upharpoonright [A']_E$ to $E \upharpoonright A'$. As Proposition 1.3.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel reducibility, it follows that $E \upharpoonright [A']_E$ is also hyperfinite. As Theorem 1.3.7 ensures that every hyperfinite Borel equivalence relation is Borel embeddable into every non-smooth Borel equivalence relation, there is a Borel embedding $\pi' \colon [A']_E \to Z$ of $E \upharpoonright [A']_E$ into $F \upharpoonright Z$. As $\mu([A \cup A']_E) = 1$ and $\pi \cup \pi'$ is an embedding of $E \upharpoonright [A \cup A']_E$ into F, the proposition follows. ⊠

This has the following consequence for the relationship between measure embeddability and measure reducibility.

Proposition 3.2.2. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, E is invariant-measure hyperfinite, and F is aperiodic. Then E is measure reducible to F if and only if E is measure embeddable into F.*

Proof. This is a direct consequence of Proposition 3.2.1. ⊠

In particular, we obtain the following.

Proposition 3.2.3. *Suppose that X is a standard Borel space and E is an aperiodic invariant-measure-hyperfinite countable Borel equivalence relation on X. Then $E \times I(\mathbb{N})$ is measure embeddable into E.*

Proof. This is a direct consequence of Proposition 3.2.2. ⊠

We next record a natural obstacle to measurable embeddability. We use $\mathcal{I}_E$ to denote the family of all E-invariant Borel probability measures on X, and $\mathcal{EI}_E$ to denote $\mathcal{E}_E \cap \mathcal{I}_E$.

Proposition 3.2.4. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, $\mu \in \mathcal{EI}_E \setminus \mathcal{H}_E$, $\nu \in \mathcal{EI}_F \setminus \mathcal{H}_F$, $C_\mu(E) < C_\nu(F)$, and the μth vertical section of $\ll_{E,F}$ is the measure-equivalence class of ν. Then E is not μ-embeddable into F.*

Proof. Suppose, towards a contradiction, that there is a μ-conull Borel set $C \subseteq X$ on which there is a Borel embedding $\pi \colon C \to Y$ of E into F. Then $\pi_*(\mu \upharpoonright C) \ll \nu$, since otherwise Proposition 1.6.6 would yield an F-quasi-invariant Borel probability measure ν' on Y with the same F-invariant Borel sets as $\pi_*(\mu \upharpoonright C)$, in which case the E-ergodicity of μ would ensure that ν' is F-ergodic, and the downward closure of the family of hyperfinite Borel equivalence relations under Borel embeddability (see Proposition 1.3.2) would imply that F is ν'-nowhere hyperfinite, despite the fact that ν and ν' are orthogonal. Let ν_D be the Borel probability measure on the set $D = \pi(C)$ given by

$\nu_D(B) = \nu(B)/\nu(D)$. As $\pi_*(\mu \restriction C) \ll \nu_D$ and both measures are $(F \restriction D)$-ergodic and $(F \restriction D)$-invariant, Proposition 1.6.7 implies that $\pi_*(\mu \restriction C) = \nu_D$. The formula for the cost of Borel restrictions given by Proposition 1.6.11 then ensures that $C_\nu(F) \le C_{\nu_D}(F \restriction D) = C_\mu(E)$, a contradiction.   ⊠

As a special case, we obtain the following.

Proposition 3.2.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\mu \in \mathcal{EI}_E \setminus \mathcal{H}_E$, $1 < C_\mu(E) < \infty$, and the μth vertical section of $\ll_E$ is the measure-equivalence class of μ. Then for no $n \in \mathbb{N}$ is it the case that $E \times I(n+1)$ is μ-embeddable into $E \times I(n)$.*

Proof. Let m_n denote the uniform probability measure on n. Then the formula for the cost of Borel restrictions given by Proposition 1.6.11 ensures that $C_{\mu \times m_{n+1}}(E \times I(n+1)) < C_{\mu \times m_n}(E \times I(n))$ for all $n \in \mathbb{N}$, so Proposition 3.2.4 implies that $E \times I(n+1)$ is not $(\mu \times m_{n+1})$-embeddable into $E \times I(n)$.   ⊠

Putting these observations together, we obtain the following.

Proposition 3.2.6. *Suppose that X is a standard Borel space and E is an aperiodic non-invariant-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X. Then there is an aperiodic Borel subequivalence relation F of E such that for no $n \in \mathbb{N}$ is $F \times I(n+1)$ measure embeddable into $F \times I(n)$.*

Proof. Fix a Borel set $B \subseteq X$ and an $(E \restriction B)$-invariant Borel probability measure μ on B such that $E \restriction B$ is not μ-hyperfinite. Fix a Borel graphing G of $E \restriction B$. As G is locally countable, the Lusin-Novikov uniformization theorem yields an increasing sequence $(G_n)_{n \in \mathbb{N}}$ of Borel subgraphs of G of bounded vertex degree whose union is G. As Theorem 1.7.2 ensures that the increasing union of μ-hyperfinite Borel equivalence relations is μ-hyperfinite, there exists $n \in \mathbb{N}$ sufficiently large for which the equivalence relation F generated by G_n is not μ-hyperfinite. Note that $C_\nu(F) < \infty$ for every F-invariant Borel probability measure ν on B. By Proposition 1.7.10, there exists $\nu \in \mathcal{EI}_F \setminus \mathcal{H}_F$. As Proposition 2.3.4 ensures that the class of projectively-separable countable Borel equivalence relations is closed downward under Borel restrictions and Borel subequivalence relations, it follows that F is projectively separable. As Proposition 2.4.12 ensures that the vertical sections of $\ll_{F'}$ are countable unions of measure-equivalence classes, there is a ν-conull Borel set $C \subseteq B$ which is null with respect to every measure in the νth vertical section of $\ll_{F'}$, with the exception of those in the

measure-equivalence class of ν. By removing a ν-null Borel subset of C, we can assume that the relation $F' = F \restriction C$ is aperiodic. As Proposition 1.4.1 ensures that the family of treeable countable Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that F' is treeable, so $1 < C_\nu(F') < \infty$ by Proposition 1.6.12, thus Proposition 3.2.5 implies that for no $n \in \mathbb{N}$ is it the case that $F' \times I(n+1)$ is ν-embeddable into $F' \times I(n)$. Proposition 1.3.3 then yields an aperiodic hyperfinite Borel subequivalence relation F'' of $E \restriction \sim C$, in which case $F' \cup F''$ is as desired.   ⊠

3.3. Antichains

In this section, we produce perfect sequences of pairwise non-measure reducible Borel subequivalence relations of a given projectively-separable treeable countable Borel equivalence relation.

We begin by noting that hyperfiniteness rules out such sequences.

Proposition 3.3.1. *Suppose that X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X, and E_1 and E_2 are Borel subequivalence relations of E. Then E_1 and E_2 are comparable under Borel reducibility.*

Proof. As Proposition 1.3.2 ensures that the family of hyperfinite Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that E_1 and E_2 are themselves hyperfinite. But Theorem 1.3.9 implies that any two hyperfinite Borel equivalence relations are comparable under Borel reducibility.   ⊠

We next turn our attention to very special sorts of antichains.

Proposition 3.3.2. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable countable Borel equivalence relation on X. Then exactly one of the following holds:*

  (1) *The relation E is a non-empty countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility.*
  (2) *There are Borel sequences $(B_c)_{c \in 2^\mathbb{N}}$ of pairwise disjoint E-invariant subsets of X and $(\mu_c)_{c \in 2^\mathbb{N}}$ of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ with the property that $\mu_c(B_c) = 1$ for all $c \in 2^\mathbb{N}$, and for no distinct $c, d \in 2^\mathbb{N}$ is it the case that $E \restriction B_c$ is μ_c-reducible to $E \restriction B_d$.*

Proof. By Theorem 2.4.8, it is sufficient to show that if $(B_c)_{c \in 2^\mathbb{N}}$ is a Borel sequence of pairwise disjoint E-invariant sets and $(\mu_c)_{c \in 2^\mathbb{N}}$ is a Borel sequence of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^\mathbb{N}$, then by passing to a perfect subsequence,

one can ensure that for no distinct $c, d \in 2^{\mathbb{N}}$ is it the case that $E \upharpoonright B_c$ is μ_c-reducible to $E \upharpoonright B_d$. Towards this end, let R denote the binary relation on $2^{\mathbb{N}}$ in which two sequences $c, d \in 2^{\mathbb{N}}$ are R-related if $E \upharpoonright B_c$ is μ_c-reducible to $E \upharpoonright B_d$. Then Proposition 1.6.14 ensures that R is analytic, and therefore has the Baire property. As the projective separability of E ensures that the vertical sections of R are countable, it follows that the vertical sections of R are meager, so the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) ensures that R is itself meager, in which case Mycielski's theorem (see, for example, [Kec95, Theorem 19.1]) yields the desired perfect subsequence.   ⊠

In particular, this allows us to characterize the circumstances under which there is a perfect sequence of pairwise non-measure reducible countable Borel equivalence relations which are measure reducible to a given projectively-separable countable Borel equivalence relation.

Proposition 3.3.3. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable countable Borel equivalence relation on X. Then exactly one of the following holds:*

(1) *There is a finite family $\mathscr{F}$ of successors of $\mathbb{E}_0$ under measure reducibility for which E is a non-empty countable disjoint union of Borel equivalence relations which are measure bi-reducible with those in $\mathscr{F}$.*

(2) *There is a Borel sequence $(E_c)_{c \in 2^{\mathbb{N}}}$ of pairwise non-measure-reducible countable equivalence relations measure reducible to E.*

Proof. In light of Proposition 3.3.2, we can assume that E is a non-empty countable disjoint union of a sequence $(E_n)_{n \in \mathbb{N}}$ of successors of $\mathbb{E}_0$ under measure reducibility.

To see that at least one of these conditions holds, note that if condition (1) fails, then by passing to an infinite subsequence, we can assume that the relations E_n are pairwise non-measure-reducible. Proposition 2.4.5 then ensures that if $n \in \mathbb{N}$ and $\mu \in \mathcal{EQ}_{E_n} \setminus \mathcal{H}_{E_n}$, then E_n is not μ-reducible to $\bigsqcup_{m \in \mathbb{N} \setminus \{n\}} E_m$. In particular, if $(N_c)_{c \in 2^{\mathbb{N}}}$ is a Borel sequence of subsets of $\mathbb{N}$ such that $N_c \nsubseteq N_d$ for all distinct $c, d \in 2^{\mathbb{N}}$, then the relations $E_c = \bigsqcup_{n \in N_c} E_n$ are pairwise non-measure-reducible.

To see that the conditions are mutually exclusive, we will establish the stronger fact that if condition (1) holds, then every sequence $(F_n)_{n \in \mathbb{N}}$ of countable Borel equivalence relations measure reducible to E has an infinite subsequence that is (not necessarily strictly) increasing under measure reducibility. Towards this end, note that for each $n \in \mathbb{N}$, there is a sequence $(k_{F,n})_{F \in \mathscr{F}}$ of countable cardinals such that F_n is measure bi-reducible with $\bigsqcup_{F \in \mathscr{F}} F \times \Delta(k_{F,n})$. A straightforward

induction shows that, by passing to an infinite subsequence, we can assume that $k_{F,m} \leq k_{F,n}$ for all $F \in \mathscr{F}$ and $m \leq n$ in $\mathbb{N}$. But this implies that $(F_n)_{n \in \mathbb{N}}$ is increasing under measure reducibility.   ⊠

As a corollary, we obtain the following.

Proposition 3.3.4. *Suppose that X is a standard Borel space and E is a projectively-separable countable Borel equivalence relation on X. Then the following are equivalent:*

(1) *There is a sequence $(E_n)_{n \in \mathbb{N}}$ of countable Borel equivalence relations measure reducible to E for which no infinite subsequence is (not necessarily strictly) increasing under measure reducibility.*

(2) *There is a sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise non-measure-reducible countable Borel equivalence relations measure reducible to E.*

(3) *There is a Borel sequence $(E_c)_{c \in 2^{\mathbb{N}}}$ of pairwise non-measure-reducible countable equivalence relations measure reducible to E.*

Proof. This follows from the proof of Proposition 3.3.3.   ⊠

We next turn our attention to subequivalence relations. The main additional tool we will need is the following observation concerning the power of μ-stratifications in the presence of projective separability.

Proposition 3.3.5. *Suppose that X is a standard Borel space, E is a projectively-separable countable Borel equivalence relation on X, μ is a Borel probability measure on X, $(B_n)_{n \in \mathbb{N}}$ is a sequence of μ-positive Borel subsets of X, and $(E_{n,r})_{r \in \mathbb{R}}$ is a Borel $(\mu \restriction B_n)$-stratification of $E \restriction B_n$ such that $\bigcap_{r \in \mathbb{R}} E_{n,r}$ is $(\mu \restriction B_n)$-nowhere hyperfinite, for all $n \in \mathbb{N}$. Then there is a Borel embedding $\pi \colon \mathbb{R} \to \mathbb{R}$ of the usual ordering of $\mathbb{R}$ into itself such that $E_{m,\pi(r)}$ is $(\mu \restriction B_m)$-nowhere reducible to $E_{n,\pi(s)}$ for all distinct $(m,r), (n,s) \in \mathbb{N} \times \mathbb{R}$.*

Proof. Let $R_{m,n}$ denote the relation on $\mathbb{R}$ in which two real numbers r and s are related if $E_{m,r}$ is $(\mu \restriction B_m)$-somewhere reducible to $E_{n,s}$.

Lemma 3.3.6. *Every horizontal section of every $R_{m,n}$ is countable.*

Proof. Suppose, towards a contradiction, that there exist $m, n \in \mathbb{N}$ and $t \in \mathbb{R}$ for which $R_{m,n}^t$ is uncountable. For each $r \in R_{m,n}^t$, fix a μ-positive Borel set $B_{m,r} \subseteq B_m$ on which there is a Borel reduction $\phi_r \colon B_{m,r} \to B_n$ of $E_{m,r}$ to $E_{n,t}$. Then there exists $\epsilon > 0$ such that $\mu(B_{m,r}) \geq \epsilon$ for uncountably many $r \in R_{m,n}^t$. As each ϕ_r is a homomorphism from $(\bigcap_{r \in \mathbb{R}} E_{m,r}) \restriction B_{m,r}$ to E, the $(\mu \restriction B_m)$-nowhere hyperfiniteness of $(\bigcap_{r \in \mathbb{R}} E_{m,r}) \restriction B_m$ coupled with the projective separability of E ensures the existence of distinct $r, s \in R_{m,n}^t$ for which $\mu(B_{m,r}), \mu(B_{m,s}) \geq \epsilon$ and

$d_\mu(\phi_r, \phi_s) < \epsilon$. Then $\{x \in B_{m,r} \cap B_{m,s} \mid \phi_r(x) = \phi_s(x)\}$ is a μ-positive Borel set on which $E_{m,r}$ and $E_{m,s}$ coincide, a contradiction.   ⊠

Proposition 1.6.14 ensures that each $R_{m,n}$ is analytic, and therefore has the Baire property. As the horizontal sections of each $R_{m,n}$ are countable and therefore meager, the Kuratowski-Ulam theorem ensures that each $R_{m,n}$ is meager, thus so too is their union R, in which case Mycielski's theorem yields a continuous injection $\phi \colon 2^\mathbb{N} \to \mathbb{R}$ with respect to which pairs of distinct sequences in $2^\mathbb{N}$ are mapped to R-unrelated pairs of real numbers. Galvin's theorem (see, for example, [Kec95, Theorem 19.7]) ensures that by replacing ϕ with its composition with an appropriate continuous function from $2^\mathbb{N}$ to $2^\mathbb{N}$, we can assume that it is an embedding of the lexicographical ordering of $2^\mathbb{N}$ into the usual ordering of $\mathbb{R}$. Fix a Borel embedding $\psi \colon \mathbb{R} \to 2^\mathbb{N}$ of the usual ordering of $\mathbb{R}$ into the lexicographical ordering of $2^\mathbb{N}$, and observe that the function $\pi = \phi \circ \psi$ is as desired.   ⊠

In particular, this yields the following measure-theoretic result.

Theorem 3.3.7. *Suppose that X is a standard Borel space, E is a projectively-separable treeable countable Borel equivalence relation on X, and μ is a Borel probability measure on X for which E is μ-nowhere hyperfinite. Then there is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise μ-nowhere reducible subequivalence relations of E.*

Proof. As Proposition 1.6.6 yields an E-quasi-invariant Borel probability measure ν for which $\mu \ll \nu$, Theorem 2.5.7 yields a Borel μ-stratification $(F_r)_{r \in \mathbb{R}}$ of E. As Theorem 1.7.2 ensures that the family of μ-hyperfinite countable Borel equivalence relations is closed under increasing unions, there is a partition $(B_n)_{n \in \mathbb{N}}$ of X into μ-positive Borel sets, as well as a sequence $(r_n)_{n \in \mathbb{N}}$ of real numbers, such that $F_{r_n} \upharpoonright B_n$ is $(\mu \upharpoonright B_n)$-nowhere hyperfinite for all $n \in \mathbb{N}$. Fix order-preserving Borel injections $\phi_n \colon \mathbb{R} \to (r_n, \infty)$, and appeal to Proposition 3.3.5 to obtain a Borel embedding $\phi \colon \mathbb{R} \to \mathbb{R}$ of the usual ordering of $\mathbb{R}$ into itself such that $F_{(\phi_m \circ \phi)(r)} \upharpoonright B_m$ is $(\mu \upharpoonright B_m)$-nowhere reducible to $F_{(\phi_n \circ \phi)(s)}$ for all distinct $(m, r), (n, s) \in \mathbb{N} \times \mathbb{R}$. Then the relations $E_r = \bigcup_{n \in \mathbb{N}} (F_{(\phi_n \circ \phi)(r)} \upharpoonright B_n)$ are as desired.   ⊠

In the special case that the equivalence relation in question is a successor of $\mathbb{E}_0$ under measure reducibility, we can ensure that the same holds of the subequivalence relations.

Theorem 3.3.8. *Suppose that X is a standard Borel space, E is a projectively-separable treeable countable Borel equivalence relation on X which is a successor of $\mathbb{E}_0$ under measure reducibility, and μ is*

a Borel probability measure on X for which E is μ-nowhere hyperfinite. Then there is an increasing Borel sequence $(E_r)_{r\in\mathbb{R}}$ of pairwise μ-nowhere reducible subequivalence relations of E consisting of successors of $\mathbb{E}_0$ under measure reducibility with the property that μ is $(\bigcap_{r\in\mathbb{R}} E_r)$-ergodic.

Proof. Proposition 1.6.6 yields an E-quasi-invariant Borel probability measure $\nu \gg \mu$ agreeing with μ on all E-invariant Borel sets. By Proposition 1.6.4, there is a Borel cocycle $\rho\colon E \to (0,\infty)$ with respect to which ν is invariant. As Theorem 1.6.8 ensures the existence of a Borel ergodic decomposition of ρ, Proposition 2.4.5 implies that $\mathcal{EQ}_E \setminus \mathcal{H}_E$ consists of a single measure-equivalence class, and Proposition 1.6.7 implies that E is not almost-everywhere hyperfinite with respect to at most one measure along the ergodic decomposition, it follows from Proposition 1.7.12 that ν is E-ergodic. Proposition 2.4.10 therefore implies that ν is $(E, \mathbb{E}_0)$-ergodic.

Theorem 2.5.7 yields a Borel ν-stratification $(F_r)_{r\in\mathbb{R}}$ of E. Theorem 1.7.2 ensures that not every F_r is ν-hyperfinite, so by passing to a Borel subsequence, we can assume that there is a ν-positive Borel set $B \subseteq X$ on which $\bigcap_{r\in\mathbb{R}} F_r$ is ν-nowhere hyperfinite. Proposition 1.6.13 implies that by passing to a further subsequence, we can also assume that $\nu \restriction B$ is $(\bigcap_{r\in\mathbb{R}} F_r \restriction B)$-ergodic. Proposition 3.3.5 therefore yields a Borel embedding $\phi\colon \mathbb{R} \to \mathbb{R}$ of the usual ordering of $\mathbb{R}$ into itself such that $F_{\phi(r)} \restriction B$ is $(\nu \restriction B)$-nowhere reducible to $F_{\phi(s)}$ for all distinct $r, s \in \mathbb{R}$. As E is countable, the Lusin-Novikov uniformization theorem ensures that the set $[B]_E$ is Borel, and that there is an extension of the identity function on B to a Borel function $\psi\colon [B]_E \to B$ whose graph is contained in E. Let E_r denote the equivalence relation given by $x \mathrel{E_r} y \iff \psi(x) \mathrel{F_{\phi(r)}} \psi(y)$ on $[B]_E$, and which is trivial off of $[B]_E$. As Proposition 2.4.9 ensures that each $\mathcal{EQ}_{E_r} \setminus \mathcal{H}_{E_r}$ consists of a single measure-equivalence class, Proposition 2.4.1 implies that each E_r is a successor of $\mathbb{E}_0$ under measure reducibility. ⊠

We close this section with the Borel analogs of these results.

Theorem 3.3.9. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X. Then there is an increasing Borel sequence $(E_r)_{r\in\mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E.*

Proof. Appeal to Theorem 1.7.11 to obtain a Borel probability measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, and apply Theorem 3.3.7. ⊠

Theorem 3.3.10. *Suppose that X is a standard Borel space and E is a projectively-separable treeable countable Borel equivalence relation on X which is a successor of $\mathbb{E}_0$ under measure reducibility. Then there is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E which are themselves successors of $\mathbb{E}_0$ under measure reducibility.*

Proof. Appeal to Theorem 1.7.11 to obtain a Borel probability measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, and apply Theorem 3.3.8. $\boxtimes$

3.4. Bases

Here we establish the nonexistence of small bases $\mathscr{B} \subseteq \mathscr{E}$ for $\mathscr{E}$ under measure reducibility. We obtain the optimal result in this direction when working below successors of $\mathbb{E}_0$ under measure reducibility.

Theorem 3.4.1. *Suppose that X is a standard Borel space and E is a projectively-separable treeable countable Borel equivalence relation on X that is a successor of $\mathbb{E}_0$ under measure reducibility. Then every basis for the non-measure-hyperfinite Borel subequivalence relations of E has cardinality at least $2^{\aleph_0}$.*

Proof. By Theorem 3.3.10, there is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E, which are also successors of $\mathbb{E}_0$ under measure reducibility. Then each element of $\mathscr{B}$ is measure reducible to at most one E_r, thus $|\mathscr{B}| \geq 2^{\aleph_0}$. $\boxtimes$

While we can nearly obtain the analogous result without the assumption that E is a successor of $\mathbb{E}_0$ under measure reducibility, there is a slight metamathematical wrinkle. Although we have thus far freely used the axiom of choice throughout the paper, it is not difficult to push through all of our arguments under the axiom of dependent choice. While the cardinality restriction appearing below implies only that bases are necessarily uncountable under the axiom of dependent choice, it yields the full result that bases have size continuum under the axiom of choice, as well as in models of the axiom of dependent choice where every subset of the real numbers has the Baire property and there is an injection of the real numbers into every non-well-orderable set, such as $L(\mathbb{R})$ under the axiom of determinacy (see [CK11]).

Theorem 3.4.2. *Suppose that X is a standard Borel space, E is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X, and $\mathscr{B}$ is a basis for the non-measure-hyperfinite Borel subequivalence relations of E under measure reducibility. Then $\mathbb{R}$ is a union of $|\mathscr{B}|$-many countable sets.*

Proof. By Theorem 3.3.9, there is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E. But then each element of $\mathscr{B}$ is measure reducible to only countably-many relations of the form E_r.   ⊠

3.5. COMPLEXITY

In this section, we establish a technical strengthening of Theorem 3.3.7 which gives rise to our complexity results.

Theorem 3.5.1. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X. Then there are Borel sequences $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of E and $(\mu_r)_{r \in \mathbb{R}}$ of Borel probability measures on X such that:*

  (1) *Each μ_r is E_r-quasi-invariant and E_r-ergodic.*
  (2) *The relation E_r is μ_r-nowhere reducible to the relation E_s, for all distinct $r, s \in \mathbb{R}$.*

Proof. Note that if E is not a countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility, then Proposition 3.3.2 yields the desired result. On the other hand, if E is a countable disjoint union of successors of $\mathbb{E}_0$ under measure reducibility, then there is an E-invariant Borel set $B \subseteq X$ on which E is a successor of $\mathbb{E}_0$ under measure reducibility. Proposition 2.4.5 then yields a Borel probability measure μ on B for which $E \restriction B$ is μ-nowhere hyperfinite, in which case one obtains the desired equivalence relations by trivially extending those given by Theorem 3.3.8 from B to X.   ⊠

As a consequence, we obtain the following.

Theorem 3.5.2. *Suppose that X is a standard Borel space and E is a non-measure-hyperfinite projectively-separable treeable countable Borel equivalence relation on X. Then the following hold:*

  (a) *There is an embedding of containment on Borel subsets of $\mathbb{R}$ into Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E (in the codes).*
  (b) *Borel bi-reducibility and reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E are both $\mathbf{\Sigma}_2^1$-complete (in the codes).*
  (c) *Every Borel quasi-order is Borel reducible to Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E.*

(d) *Borel and $\sigma(\mathbf{\Sigma}_1^1)$-measurable reducibility do not agree on the countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E.*

Proof. By Theorem 3.5.1, there are Borel sequences $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of E and $(\mu_r)_{r \in \mathbb{R}}$ of Borel probability measures on X such that:

(1) Each μ_r is E_r-quasi-invariant and E_r-ergodic.
(2) The relation E_r is μ_r-nowhere reducible to the relation E_s, for all distinct $r, s \in \mathbb{R}$.

But then Theorem 1.9.1 yields the desired result. ⊠

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