THE SMOOTH IDEAL

JOHN D. CLEMENS, CLINTON T. CONLEY, AND BENJAMIN D. MILLER

Abstract. We give a classical proof of the generalization of the characterization of smoothness to quotients of Polish spaces by Borel equivalence relations. As an application, we describe the extent to which any given Borel equivalence relation on a Polish space is encoded by the corresponding σ-ideal generated by the family of Borel sets on which it is smooth.

Introduction

A *measurable space* is a set equipped with a σ-algebra of distinguished subsets. Given a pointclass Γ of subsets of such spaces, we say that a function $\pi \colon X \to Y$ between measurable spaces is Γ-*measurable* if the pre-image of every distinguished subset of Y is in Γ. Given equivalence relations E and F on X and Y, we say that the Γ-*measurable cardinality* of X/E is at most that of Y/F, or simply $|X/E|_\Gamma \leq |Y/F|_\Gamma$, if there is a Γ-measurable function $\pi \colon X \to Y$ which factors to an injection of X/E into Y/F. When Γ contains every subset of every measurable space, this yields the usual notion of *cardinality*.

A metric is *Polish* if it is complete and its induced topology is separable. A topological space is *Polish* if it has a compatible Polish metric. A subset of a Polish space is *Borel* if it is in the σ-algebra generated by the underlying topology. A measurable space X is *standard Borel* if its family of distinguished subsets consists precisely of the Borel sets associated with a Polish topology on X, in which case we refer to these distinguished sets as *Borel* sets. A subset Y of a standard Borel space X is then Borel if and only if it is standard Borel when equipped with the *subspace* measurable structure it inherits from X, consisting of all sets of the form $B \cap Y$, where B ranges over all Borel subsets of X (see, for example, [Kec95, Corollary 13.4 and Theorem 15.1]).

2010 *Mathematics Subject Classification.* Primary 03E15; secondary 28A05.

Key words and phrases. Cardinality, definability, dichotomy, equivalence relation, smooth.

The authors were supported in part by SFB Grant 878.

The desire to understand obstacles of definability inherent in mathematical classification problems recently led to the study of Borel cardinals associated with equivalence relations on standard Borel spaces. From this point forward, we will restrict our attention to such relations.

As any two standard Borel spaces of the same cardinality are Borel isomorphic (see, for example, [Kec95, Theorem 15.6]), the notion of Borel cardinality coincides with the classical one when the equivalence relations are trivial. The notions diverge drastically, however, upon passing to quotients by non-trivial Borel equivalence relations. The first notable theorem in this area demonstrates that a major difference occurs just after the countable cardinals: if X is a standard Borel space and E is a Borel equivalence relation on X, then $|X/E|_B \leq |\mathbb{N}|_B$ or $|\mathbb{R}|_B \leq |X/E|_B$ (see [Sil80]). In other words, the Borel analog of the continuum hypothesis holds. Perhaps more surprisingly, there is also an analog of the continuum hypothesis at the next Borel cardinal: if X is a standard Borel space and E is a Borel equivalence relation on X, then $|X/E|_B \leq |\mathbb{R}|_B$ or $|\mathbb{R}/\mathbb{Q}|_B \leq |X/E|_B$ (see [HKL90]).

The former theorem can be seen as a consequence of the latter, which itself hinges on properties of the σ-ideal generated by the family of Borel sets $B \subseteq X$ for which $|B/E|_B \leq |\mathbb{R}|_B$. By generalizing the latter result to equivalence relations on quotient spaces, we will establish rigidity results characterizing the extent to which E is encoded by this σ-ideal.

Earlier results. Suppose that $R \subseteq X \times X$ and $S \subseteq Y \times Y$. A *homomorphism* from R to S is a map $\pi \colon X \to Y$ sending R-related points to S-related points. A *cohomomorphism* from R to S is a homomorphism from the complement of R to the complement of S. A *reduction* of R to S is a function $\pi \colon X \to Y$ which is both a homomorphism and a cohomomorphism from R to S. And an *embedding* of R into S is an injective reduction of R to S.

We say that E is Γ-*reducible* to F, or simply $E \leq_\Gamma F$, if there is a Γ-measurable reduction of E to F. Although $|X/E|_\Gamma \leq |Y/F|_\Gamma$ if and only if $E \leq_\Gamma F$, we will use the language of reducibility so as to simplify the statements of results and their proofs.

The former theorem follows from the stronger result, commonly referred to as *Silver's Theorem*, that equality on $2^\mathbb{N}$ is continuously embeddable into every Borel (or even co-analytic) equivalence relation on a Polish space with uncountably many classes (see [Sil80]). The latter theorem follows from the stronger result, commonly referred to as the *Harrington-Kechris-Louveau Theorem*, that the equivalence relation $\mathbb{E}_0$ on $2^\mathbb{N}$, given by $c \mathbin{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n \ c(m) = d(m)$, is continuously embeddable into every Borel equivalence relation on a Polish space which is not smooth (see [HKL90, Theorem 1.1]). Here, a Borel equivalence relation is *smooth* if it is Borel reducible to equality on $2^{\mathbb{N}}$.

Although Silver's Theorem can be viewed as a generalization of the classical theorem asserting the continuum hypothesis for Borel (or even analytic) subsets of Polish spaces (see [Sou17]), its original proof was quite sophisticated, utilizing many tools from mathematical logic. While a simpler proof was later discovered (see [Har76]), it nevertheless relied upon subtle recursion-theoretic features of the real numbers.

The Harrington-Kechris-Louveau Theorem can be viewed as a generalization of earlier results in operator algebras (see [Gli61, Eff65]) and ergodic theory (see [SW82, Wei84]). In addition to utilizing ideas from the arguments underlying these precursors, its proof also builds upon that of Silver's Theorem, and consequently, it too depends heavily upon recursion-theoretic techniques. As the special case of Silver's Theorem for Borel equivalence relations can be seen as a consequence of the Harrington-Kechris-Louveau Theorem, we will focus upon the latter from this point forward.

Generalizations. We say that a set $Y \subseteq X$ is a *partial transversal* of E if it intersects every E-class in at most one point. We say that a set $Y \subseteq X$ is *E-smooth* if the restriction of E to Y is smooth.

We say that E has *bounded finite index* over F if for some $n \in \mathbb{N}$, every class of E is the union of at most n classes of F. We say that E has *σ-bounded finite index* over F if X is the union of countably many Borel sets on which E has bounded finite index over F.

After reviewing the necessary descriptive set-theoretic preliminaries in §1, we generalize the Harrington-Kechris-Louveau Theorem in §2.

Theorem 1. *Suppose that X is a Polish space and E and F are Borel equivalence relations on X. Then exactly one of the following holds:*

(1) *There is an E-smooth Borel set $B \subseteq X$ off of which E has σ-bounded-finite-index over $E \cap F$.*
(2) *There is a continuous embedding of $\mathbb{E}_0$ into the restriction of E to a partial transversal of F.*

When $F \subseteq E$, any reduction of $\mathbb{E}_0$ to the restriction of E to a partial transversal of F factors to an injection of $2^{\mathbb{N}} / \mathbb{E}_0$ into E/F. The Harrington-Kechris-Louveau Theorem is essentially the special case of Theorem 1 in which F is equality on X. As our arguments are entirely classical, this eliminates the need for recursion-theoretic techniques in the proofs of the earlier theorems.

Rigidity. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are families of subsets of X and Y. A *homomorphism* from $\mathcal{I}$ to $\mathcal{J}$ is a function $\pi \colon X \to Y$ sending sets in

$\mathcal{I}$ to sets in $\mathcal{J}$. A *cohomomorphism* from $\mathcal{I}$ to $\mathcal{J}$ is a homomorphism from the complement of $\mathcal{I}$ to the complement of $\mathcal{J}$. And a *reduction* of $\mathcal{I}$ to $\mathcal{J}$ is a function $\pi \colon X \to Y$ which is both a homomorphism and a cohomomorphism from $\mathcal{I}$ to $\mathcal{J}$.

We use $\mathcal{I}_E$ to denote the σ-ideal on X generated by the family of all E-smooth Borel sets. It is easy to see that if $\pi \colon X \to Y$ is a Borel reduction of E to F, then it is also a Borel reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$. While measure-theoretic rigidity arguments can be used to rule out the converse (see, for example, [CM14, Proposition 6.8] and Theorem 5), it is nevertheless not far from true, and leads to Borel rigidity results.

We say that a function $\pi \colon X \to Y$ is a *quasi-homomorphism* from E to F if there exists $n \in \mathbb{N}$ for which the restriction of F to the image of each E-class has at most n classes. We say that a function $\pi \colon X \to Y$ is a *quasi-cohomomorphism* from E to F if there exists $n \in \mathbb{N}$ for which the restriction of E to the pre-image of each F-class has at most n classes. And we say that a function $\pi \colon X \to Y$ is a *quasi-reduction* of E to F if it is both a quasi-homomorphism and a quasi-cohomomorphism from E to F.

We say that a Borel function $\pi \colon X \to Y$ is a *σ-quasi-homomorphism* from E to F if X is the union of countably many Borel sets on which π is a quasi-homomorphism from E to F. We say that a Borel function $\pi \colon X \to Y$ is a *σ-quasi-reduction* of E to F if X is the union of countably many Borel sets on which π is a quasi-reduction of E to F. And we say that a Borel function $\pi \colon X \to Y$ is *smooth-to-one* (with respect to E and F) if the restriction of E to the pre-image of each F-class is smooth.

In §3, we characterize Borel morphisms between smooth ideals.

Theorem 2. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and $\phi \colon X \to Y$ is Borel. Then ϕ is a cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is an E-smooth Borel set off of which ϕ is a smooth-to-one σ-quasi-homomorphism from E to F.*

Theorem 3. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and $\phi \colon X \to Y$ is Borel. Then ϕ is a reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is an E-smooth Borel set, whose image under ϕ is F-smooth, off of which ϕ is a σ-quasi-reduction of E to F.*

Following the standard abuse of language, we say that an equivalence relation is *countable* if its equivalence classes are all countable. When at least one of the equivalence relations in question is countable, these results yield characterizations of the existence of Borel morphisms between smooth ideals.

Theorem 4. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and E or F is countable. Then there is a Borel cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is a smooth-to-one Borel quasi-homomorphism from E to F.*

Theorem 5. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, E or F is countable, and F has uncountably many classes. Then there is a Borel reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is a Borel quasi-reduction of E to $F \times \Delta(\mathbb{N})$.*

Here, $\Delta(X)$ denotes the equality relation on X, and the *product* of equivalence relations E and F on X and Y is the equivalence relation on $X \times Y$ given by $(x_1, y_1)\ (E \times F)\ (x_2, y_2) \iff (x_1\ E\ x_2 \text{ and } y_1\ F\ y_2)$.

Homogeneity. We say that $\mathcal{I}$ is *cohomorphism homogeneous* if for every Borel set $Y \subseteq X$, either $Y \in \mathcal{I}$ or there is a Borel cohomomorphism from $\mathcal{I}$ to $\mathcal{I} \restriction Y$. We say that $\mathcal{I}$ is *reduction homogeneous* if for every Borel set $Y \subseteq X$, either $Y \in \mathcal{I}$ or there is a Borel reduction of $\mathcal{I}$ to $\mathcal{I} \restriction Y$. In §4, we characterize the smooth ideals with these properties.

Theorem 6. *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then the following are equivalent:*

1. *There is a Borel reduction of E to $\mathbb{E}_0$.*
2. *The ideal $\mathcal{I}_E$ is reduction homogeneous.*
3. *The ideal $\mathcal{I}_E$ is cohomomorphism homogeneous.*

This yields natural examples of σ-ideals which are not cohomomorphism homogeneous, addressing a question from [Zap08, Chapter 2].

1. Preliminaries

Here we list the descriptive set-theoretic preliminaries used throughout the paper. In §1.1, we consider functions. In §1.2, we introduce analytic sets. In §1.3, we discuss Baire category. In §1.4, we state the first reflection theorem. In §1.5, we mention uniformization results. And in §1.6, we review Borel equivalence relations.

1.1. **Functions.** The *graph* of a function $\phi\colon X \to Y$ is the subset of $X \times Y$ given by $\text{graph}(\phi) = \{(x, \phi(x)) \mid x \in X\}$. The following is one half of a characterization of Borel functions.

Proposition 1.1.1. *Suppose that X and Y are Polish spaces and $\phi\colon X \to Y$ is Borel. Then graph(ϕ) is Borel.*

Proof. See, for example, [Kec95, Proposition 12.4]. ⊠

When X is a compact space and Y is a metric space, we use $C(X, Y)$ to denote the set of continuous functions $\phi \colon X \to Y$, equipped with the *uniform metric* given by $d(\phi, \psi) = \sup_{x \in X} d_Y(\phi(x), \psi(x))$.

Proposition 1.1.2. *Suppose that X is a compact metric space and Y is a Polish metric space. Then $C(X, Y)$ is also a Polish metric space.*

Proof. See, for example, [Kec95, Theorem 4.19]. ⊠

1.2. **Analytic sets.** A topological space is *analytic* if it is a continuous image of a Polish space.

Proposition 1.2.1. *Suppose that X is a topological space and $A \subseteq X$ is non-empty. Then A is analytic if and only if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$.*

Proof. See, for example, [Kec95, Theorem 7.9]. ⊠

A *tree* on $\mathbb{N}^k \times \mathbb{N}$ is a subset of $\bigcup_{n \in \mathbb{N}} (\mathbb{N}^n)^k \times \mathbb{N}^n$ such that whenever $m < n$ are natural numbers and $(s, t) \in (\mathbb{N}^n)^k \times \mathbb{N}^n$ is in T, then so too is $((s(i) \upharpoonright m)_{i<k}, t \upharpoonright m)$. A *branch* through such a tree is a pair $(a, b) \in (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^{\mathbb{N}}$ with the property that $((a(i) \upharpoonright n)_{i<k}, b \upharpoonright n) \in T$, for all $n \in \mathbb{N}$. We use $[T]$ to denote the set of all such branches, and we use $p[T]$ to denote the projection of $[T]$ onto $(\mathbb{N}^{\mathbb{N}})^k$.

Proposition 1.2.2. *Suppose that $k \in \mathbb{N}$ and $A \subseteq (\mathbb{N}^{\mathbb{N}})^k$. Then A is analytic if and only if there is a tree T on $\mathbb{N}^k \times \mathbb{N}$ such that $A = p[T]$.*

Proof. See, for example, [Kec95, Proposition 2.4 and Exercise 14.3]. ⊠

1.3. **Baire category.** A subset of a topological space is *nowhere dense* if its closure does not contain a non-empty open set, *meager* if it is the union of countably many nowhere dense sets, and *comeager* if its complement is meager.

Theorem 1.3.1 (Baire)**.** *Suppose that X is a complete metric space. Then every comeager subset of X is dense.*

Proof. See, for example, [Kec95, Theorem 8.4]. ⊠

A *Baire space* is a topological space in which every comeager set is dense. In particular, it follows that every non-empty open subset of a Baire space is non-meager.

A subset of a topological space has the *Baire property* if its symmetric difference with some open set is meager, and a function between topological spaces is *Baire measurable* if pre-images of open sets have the Baire property.

We say that a set $B \subseteq X$ is *comeager* in a non-empty open set $U \subseteq X$ if $B \cap U$ is comeager in U, where the latter is equipped with the subspace topology, or equivalently, if $U \setminus B$ is meager.

Proposition 1.3.2. *Suppose that X is a Baire space and $B \subseteq X$ has the Baire property. Then exactly one of the following holds:*

(1) *The set B is meager.*
(2) *There is a non-empty open set $U \subseteq X$ in which B is comeager.*

Proof. See, for example, [Kec95, Proposition 8.26]. ⊠

Suppose that $R \subseteq X \times Y$. For each $x \in X$, the x^{th} *vertical section* of R is given by $R_x = \{y \in Y \mid x \mathrel{R} y\}$. For each $y \in Y$, the y^{th} *horizontal section* of R is given by $R^y = \{x \in X \mid x \mathrel{R} y\}$.

Theorem 1.3.3 (Kuratowski-Ulam)**.** *Suppose that X and Y are second countable topological spaces and $R \subseteq X \times Y$ has the Baire property.*

(1) *The set $\{x \in X \mid R_x$ has the Baire property$\}$ is comeager.*
(2) *The set R is comeager if and only if $\{x \in X \mid R_x$ is comeager$\}$ is comeager.*

Proof. See, for example, [Kec95, Theorem 8.41]. ⊠

Concerning the connection between analytic sets and the Baire property, we have the following.

Theorem 1.3.4 (Lusin-Sierpiński)**.** *Every analytic subset of a Polish space has the Baire property.*

Proof. See, for example, [Kec95, Theorem 21.6]. ⊠

Theorem 1.3.5 (Novikov)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is analytic. Then $\{x \in X \mid R_x$ is comeager$\}$ is analytic.*

Proof. See, for example, [Kec95, Theorem 29.22]. ⊠

An equivalence relation on a Baire space is *generically ergodic* if every invariant set with the Baire property is meager or comeager.

Proposition 1.3.6. *Suppose that X is a Baire space and E is the orbit equivalence relation associated with a group Γ of homeomorphisms of X with a dense orbit. Then E is generically ergodic.*

Proof. It is sufficient to show that if $B \subseteq X$ is an E-invariant non-meager set with the Baire property, then B is non-meager in every non-empty open set $U \subseteq X$. Towards this end, fix a non-empty open set $V \subseteq X$ in which B is comeager. As Γ has a dense orbit, there exist $\gamma \in \Gamma$, $x \in U$, and $y \in V$ with $\gamma \cdot x = y$. As γ^{-1} is continuous, there is an open neighborhood $W \subseteq V$ of y such that $\gamma^{-1} \cdot W \subseteq U$. As γ is continuous, the set $\gamma^{-1} \cdot W$ is open. As γ is a homeomorphism, it follows that $\gamma^{-1} \cdot B$ is comeager in $\gamma^{-1} \cdot W$ (see, for example, [Kec95, Exercise 8.45]), thus B is non-meager in U. ⊠

As a first application, we have the following.

Proposition 1.3.7. *The equivalence relation* $\mathbb{E}_0$ *is generically ergodic.*

Proof. As $\mathbb{E}_0$ is generated by the homeomorphisms $T_\tau \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, where $n \in \mathbb{N}$ and τ is a permutation of 2^n, given by $T_\tau(s \frown c) = \tau(s) \frown c$ for $c \in 2^{\mathbb{N}}$ and $s \in 2^n$, the desired result follows from Proposition 1.3.6. ⊠

Generic ergodicity ensures that Borel homomorphisms to equality on $2^{\mathbb{N}}$ are generically constant, providing a means to rule out smoothness.

Proposition 1.3.8. *Suppose that* X *and* Y *are Polish spaces,* E *is a generically ergodic equivalence relation on* X*, and* $\phi \colon X \to Y$ *is a Baire measurable homomorphism from* E *to* $\Delta(Y)$*. Then there is a comeager set* $C \subseteq X$ *on which* ϕ *is constant.*

Proof. Fix a countable basis $\mathcal{V}$ for the topology of Y. Then for each $V \in \mathcal{V}$, the set $\phi^{-1}(V)$ is E-invariant, and is therefore either meager or comeager. Define $C_V = X \setminus \phi^{-1}(V)$ in the former case, and define $C_V = \phi^{-1}(V)$ in the latter. Then the set $C = \bigcap_{V \in \mathcal{V}} C_V$ is comeager, and $\phi(C)$ is a singleton. ⊠

1.4. **Reflection.** A property Φ of subsets of Y is $\mathbf{\Pi^1_1}$*-on-*$\mathbf{\Sigma^1_1}$ if whenever X is a Polish space and $R \subseteq X \times Y$ is analytic, the corresponding set $\{x \in X \mid \Phi(R_x)\}$ is co-analytic.

Theorem 1.4.1. *Suppose that* X *is a Polish space,* Φ *is a* $\mathbf{\Pi^1_1}$*-on-*$\mathbf{\Sigma^1_1}$ *property of subsets of* X*, and* $A \subseteq X$ *is an analytic set on which* Φ *holds. Then there is a Borel set* $B \supseteq A$ *on which* Φ *holds.*

Proof. See, for example, [Kec95, Theorem 35.10]. ⊠

A set B *separates* a set A from a set A' if $A \subseteq B$ and $A' \cap B = \emptyset$. The E*-saturation* of $W \subseteq X$ is given by $[W]_E = \{x \in X \mid \exists w \in W \; w \mathrel{E} x\}$.

Proposition 1.4.2 (Harrington-Kechris-Louveau)**.** *Suppose that* X *is a Polish space,* E *is an analytic equivalence relation on* X*,* $A, A' \subseteq X$ *are disjoint analytic sets, and* A' *is* E*-invariant. Then there is an* E*-invariant Borel set* $B \subseteq X$ *separating* A *from* A'*.*

Proof. As A and A' are analytic, the property (of A) that the E-saturation of A is disjoint from A' is $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$, so by applying Theorem

1.4.1 infinitely many times, we obtain a sequence $(B_n)_{n\in\mathbb{N}}$ of Borel supersets of A such that $[B_n]_E \subseteq B_{n+1}$ and the E-saturation of B_n is disjoint from A', for all $n \in \mathbb{N}$. Define $B = \bigcup_{n\in\mathbb{N}} B_n$. ⊠

1.5. **Uniformization.** The *projection* from $X \times Y$ to X is given by $\mathrm{proj}_X(x, y) = x$. A *uniformization* of a set $R \subseteq X \times Y$ is a function $\phi\colon \mathrm{proj}_X(R) \to Y$ whose graph is contained in R.

Theorem 1.5.1 (Lusin-Novikov)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is a Borel set whose vertical sections are all countable. Then $\mathrm{proj}_X(R)$ is Borel, and R is a countable union of Borel uniformizations.*

Proof. See, for example, [Kec95, Theorem 18.10]. ⊠

We use $\sigma(\mathbf{\Sigma}^1_1)$ to denote the σ-algebra generated by the analytic sets.

Theorem 1.5.2 (Jankov, von Neumann)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is analytic. Then there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformization of R.*

Proof. See, for example, [Kec95, Theorem 18.1]. ⊠

1.6. **Equivalence relations.** In the countable case, smoothness has a useful alternate characterization.

Proposition 1.6.1. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then E is smooth if and only if X is the union of countably many Borel partial transversals of E.*

Proof. This is a straightforward consequence of Theorem 1.5.1. ⊠

It follows that the family of smooth countable Borel equivalence relations is closed downward under Borel subequivalence relations.

Proposition 1.6.2. *Suppose that X is a Polish space, E is a smooth countable Borel equivalence relation on X, and F is a Borel subequivalence relation of E. Then F is also smooth.*

Proof. As partial transversals of E are also partial transversals of F, Proposition 1.6.1 yields the desired result. ⊠

Following the standard abuse of language, we say that an equivalence relation is *finite* if its equivalence classes are all finite.

Proposition 1.6.3. *Suppose that X is a Polish space and E is a finite Borel equivalence relation on X. Then E is smooth.*

Proof. This is also a straightforward consequence of Theorem 1.5.1. ⊠

We say that a subequivalence relation F of E has *finite index* if every E-class is the union of finitely many F-classes.

Proposition 1.6.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and F is a finite index Borel subequivalence relation of E. Then E is smooth if and only if F is smooth.*

Proof. In light of Proposition 1.6.2, we need only show that if F is smooth, then E is smooth. Towards this end, note that the restriction of E to any partial transversal of F is finite, thus Propositions 1.6.1 and 1.6.3 yield the desired result. ⊠

A Borel equivalence relation is *hyperfinite* if it is of the form $\bigcup_{n \in \mathbb{N}} E_n$, where $E_0 \subseteq E_1 \subseteq \cdots$ are finite Borel subequivalence relations.

Proposition 1.6.5 (Jackson-Kechris-Louveau)**.** *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and F is a finite index Borel subequivalence relation of E. Then E is hyperfinite if and only if F is hyperfinite.*

Proof. See [JKL02, Proposition 1.3]. ⊠

The following is one half of a characterization of hyperfiniteness.

Theorem 1.6.6 (Dougherty-Jackson-Kechris)**.** *Suppose that X is a Polish space and E is a hyperfinite Borel equivalence relation on X. Then there is a Borel embedding of E into $\mathbb{E}_0$.*

Proof. See [DJK94, Theorem 1]. ⊠

2. Dichotomies

Here we establish our generalization of the Harrington-Kechris-Louveau Theorem. In §2.1, we present a number of Baire category arguments. In §2.2, we give several applications of reflection. In §2.3, we consider cores for families of finite sets. And in §2.4, we establish our primary results.

2.1. Baire category.

A *graph* on a set X is an irreflexive, symmetric subset G of $X \times X$. The *restriction* of G to a set $Y \subseteq X$ is given by $G \upharpoonright Y = G \cap (Y \times Y)$. We say that Y is *G-independent* if this restriction is empty. A *coloring* of G is a function $c \colon X \to I$ with the property that $c^{-1}(i)$ is G-independent, for all $i \in I$. In the special case that I is countable, we also refer to c as an *$\aleph_0$-coloring.*

Fix sequences $s_n \in 2^n$ with the property that $\{s_n \mid n \in 2\mathbb{N}\}$ is *dense* in $2^{<\mathbb{N}}$, in the sense that $\forall s \in 2^{<\mathbb{N}} \exists n \in 2\mathbb{N}\ s \sqsubseteq s_n$. Recursively define graphs $\mathbb{G}_0(2^n)$ on 2^n by asking that $\mathbb{G}_0(2^{2n+1})$ is the union of the graph $\{(s \frown (i), t \frown (i)) \mid i < 2 \text{ and } s \mathrel{\mathbb{G}_0(2^{2n})} t\}$ with the singleton edge $\{(s_{2n} \frown (i), s_{2n} \frown (1-i)) \mid i < 2\}$, and by asking that $\mathbb{G}_0(2^{2n+2})$ is the graph $\{(s \frown (i), t \frown (i)) \mid i < 2 \text{ and } s \mathrel{\mathbb{G}_0(2^{2n+1})} t\}$. Define $\mathbb{G}_0$ on $2^{\mathbb{N}}$ by $\mathbb{G}_0 = \bigcup_{n \in \mathbb{N}} \{(s \frown c, t \frown c) \mid c \in 2^{\mathbb{N}} \text{ and } s \mathrel{\mathbb{G}_0(2^n)} t\}$.

Proposition 2.1.1 (Kechris-Solecki-Todorcevic)**.** *Suppose that $B \subseteq 2^{\mathbb{N}}$ is a $\mathbb{G}_0$-independent set with the Baire property. Then B is meager.*

Proof. Suppose, towards a contradiction, that B is not meager. Then there exists $s \in 2^{<\mathbb{N}}$ such that B is comeager in $\mathcal{N}_s$. Fix $n \in 2\mathbb{N}$ such that $s \sqsubseteq s_n$. Then B is comeager in $\mathcal{N}_{s_n}$. As the involution $I \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ which flips coordinate n is a homeomorphism, it sends meager sets to meager sets, so $B \cap I(B)$ is comeager in $\mathcal{N}_{s_n}$. Fix $c \in B \cap I(B) \cap \mathcal{N}_{s_n}$, and observe that $c \mathrel{\mathbb{G}_0} I(c)$, the desired contradiction. ⊠

Corollary 2.1.2 (Kechris-Solecki-Todorcevic)**.** *Suppose that $B \subseteq 2^{\mathbb{N}}$ is a non-meager set with the Baire property. Then no Baire measurable function $c \colon 2^{\mathbb{N}} \to \mathbb{N}$ is a coloring of $\mathbb{G}_0$ on B.*

We use $t(0)$ and $t(1)$ to denote the coordinates of $t \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. Fix $t_n \in 2^n \times 2^n$ such that $\{t_n \mid n \in 2\mathbb{N}+1\}$ is *dense* in $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, i.e., $\forall t \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \exists n \in 2\mathbb{N}+1 \ (t(0) \sqsubseteq t_n(0) \text{ and } t(1) \sqsubseteq t_n(1))$. Recursively define graphs $\mathbb{H}_0(2^n)$ on 2^n, this time by asking that $\mathbb{H}_0(2^{2n+1})$ is the graph $\{(s \frown (i), t \frown (i)) \mid i < 2 \text{ and } s \mathrel{\mathbb{H}_0(2^{2n})} t\}$, and that $\mathbb{H}_0(2^{2n+2})$ is the union of the graph $\{(s \frown (i), t \frown (i)) \mid i < 2 \text{ and } s \mathrel{\mathbb{H}_0(2^{2n+1})} t\}$ with the singleton edge $\{(t_{2n+1}(i) \frown (i), t_{2n+1}(1-i) \frown (1-i)) \mid i < 2\}$. Define $\mathbb{H}_0$ on $2^{\mathbb{N}}$ by $\mathbb{H}_0 = \bigcup_{n \in \mathbb{N}} \{(s \frown c, t \frown c) \mid c \in 2^{\mathbb{N}} \text{ and } s \mathrel{\mathbb{H}_0(2^n)} t\}$, and let $\mathbb{E}_0'(2^n)$ and $\mathbb{E}_0'$ denote the corresponding equivalence relations.

Again, all $\mathbb{H}_0$-independent sets with the Baire property are meager.

Proposition 2.1.3. *The equivalence relation $\mathbb{E}_0'$ is generically ergodic.*

Proof. This follows directly from Propositions 1.3.6 and 1.3.8. ⊠

The following observation ties all of these objects together.

Proposition 2.1.4. *Suppose that E and F are equivalence relations on $2^{\mathbb{N}}$ with the Baire property, $E \cap \mathbb{G}_0 = \emptyset$, and $\mathbb{E}_0' \setminus \Delta(2^{\mathbb{N}}) \subseteq E \setminus F$. Then there is a continuous embedding $\pi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{E}_0$ into the restriction of E to a partial transversal of F.*

Proof. As E is an equivalence relation with the Baire property, Theorem 1.3.3 ensures that all of its classes have the Baire property. As E is disjoint from $\mathbb{G}_0$, Proposition 2.1.1 therefore implies that every equivalence class of E is meager, so one more application of Theorem

1.3.3 ensures that E is meager. As F is disjoint from $\mathbb{H}_0$, an analogous argument shows that F is also meager.

Fix a decreasing sequence of dense open sets $U_n \subseteq (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \Delta(2^{\mathbb{N}})$ whose intersection is disjoint from $E \cup F$. We will now recursively construct natural numbers $k_n \in \mathbb{N}$ and functions $\pi_n \colon 2^n \to 2^{k_n}$. We begin by setting $k_0 = 0$. Given k_n and π_n, set $k_{n+1} = k_n + \ell_n + 1$ and $\pi_{n+1}(s \frown (i)) = \pi_n(s) \frown u_n(i) \frown (i)$, where both $\ell_n \in \mathbb{N}$ and $u_n \in 2^{\ell_n} \times 2^{\ell_n}$ are chosen via a recursion of finite length, so as to ensure that $\mathcal{N}_{\pi_n(s) \frown u_n(i)} \times \mathcal{N}_{\pi_n(t) \frown u_n(1-i)} \subseteq U_n$ for all $i < 2$ and $s, t \in 2^n$, and $\pi_n((i)^n) \frown u_n(i) = t_{k_n + \ell_n}(i)$ for all $i < 2$. Define $\pi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\pi(c) = \bigcup_{n \in \mathbb{N}} \pi_n(c \restriction n)$. The first condition ensures that π is an injective cohomomorphism from $\mathbb{E}_0$ to $E \cup F$. As $\mathbb{E}_0$ is generated by $\bigcup_{n \in \mathbb{N}} \{((0)^n \frown c, (1)^n \frown c) \mid c \in 2^{\mathbb{N}}\}$, and the latter condition implies that π is a homomorphism from this graph to $\mathbb{H}_0$, it follows that it is a homomorphism from $\mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}})$ to $\mathbb{E}_0' \setminus \Delta(2^{\mathbb{N}})$, and therefore from $\mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}})$ to $E \setminus F$. ⊠

2.2. **Reflection.** Suppose that k is a positive integer and $R \subseteq X^k$. We say that a sequence $(X_i)_{i<k}$ of subsets of X is *R-independent* if $\prod_{i<k} X_i$ is disjoint from R.

Proposition 2.2.1. *Suppose that k is a positive integer, X is a Polish space, $R \subseteq X^k$ is analytic, and $(A_i)_{i<k}$ is an R-independent sequence of analytic subsets of X. Then there is an R-independent sequence $(B_i)_{i<k}$ of Borel subsets of X such that $A_i \subseteq B_i$ for all $i < k$.*

Proof. As all of the A_i and R are analytic, it follows that for each $i < k$, the property (of A_i) that $\prod_{i<k} A_i$ is R-independent is $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$, thus the desired result can be obtained as a consequence of k successive applications of Theorem 1.4.1. ⊠

We say that a set $A \subseteq X$ is *R-independent* if the constant sequence of length k with value A is R-independent.

Proposition 2.2.2. *Suppose that k is a positive integer, X is a Polish space, $R \subseteq X^k$ is analytic, and $A \subseteq X$ is an R-independent analytic set. Then there is an R-independent Borel set $B \supseteq A$.*

Proof. Proposition 2.2.1 yields an R-independent sequence $(B_i)_{i<k}$ of Borel subsets of X with $A \subseteq B_i$ for all $i < k$. Set $B = \bigcap_{i<k} B_i$. ⊠

As a corollary, we obtain the following.

Proposition 2.2.3. *Suppose that k is a positive integer, X is a Polish space, E is an analytic equivalence relation on X, F is a Borel subequivalence relation of E, and $A \subseteq X$ is an analytic set on which*

E has index at most k over F. Then there is an F-invariant Borel set $B \supseteq A$ on which E has index at most k over F.

Proof. As E is analytic and F is co-analytic, it follows that the set $R = \{(x_0, \ldots, x_k) \mid \forall i, j \leq k \ (i \neq j \implies x_i \ (E \setminus F) \ x_j)\}$ is analytic. As F is analytic, F-saturations of analytic sets are analytic. Mimicking the proof of Proposition 1.4.2, by applying Proposition 2.2.2 infinitely many times, we obtain an increasing sequence $(B_n)_{n \in \mathbb{N}}$ of Borel supersets of A, on which E has index at most k over F, such that $[B_n]_F \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Define $B = \bigcup_{n \in \mathbb{N}} B_n$. ⊠

2.3. **Cores.** Let $[X]^n$ denote the family of subsets of X of cardinality n, equipped with the standard Borel structure it inherits from X^n. Let $[X]^n_E$ denote the subspace consisting of those sets which are contained in a single E-class, and let $[X]^n_{E,F}$ denote the further subspace consisting of such sets which are partial transversals of F. Set $[X]^{<\aleph_0} = \bigcup_{n \in \mathbb{N}} [X]^n$ and $[X]^{<\aleph_0}_{E,F} = \bigcup_{n \in \mathbb{N}} [X]^n_{E,F}$, equipping both with the corresponding standard Borel structures.

We say that two subsets of X are *F-disjoint* if their F-saturations are disjoint. We say that a family $\mathscr{A} \subseteq [X]^{<\aleph_0}_{E,F}$ is an *F-antichain* if any two distinct elements of $\mathscr{A}$ are F-disjoint, and we say that such a set is an *E-local F-antichain* if $\bigcup \mathscr{A}$ is contained in a single E-class. We say that a set $C \subseteq X$ is a *core* for $\mathscr{A}$ if the intersection of C with every non-empty set in $\mathscr{A}$ is itself non-empty. For each non-empty set $a' \in [X]^{<\aleph_0}_{E,F}$, define $[a', \mathscr{A}]_F = \{[a]_F \mid a \in \mathscr{A} \text{ and } a' \subseteq [a]_F\}$.

Proposition 2.3.1. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, F is a Borel subequivalence relation of E, and $\mathscr{A} \subseteq [X]^{<\aleph_0}_{E,F}$ is an analytic family of sets of bounded finite cardinality such that the E-local F-antichains of sets in $\mathscr{A}$ also have bounded finite cardinality. Then there is an F-invariant Borel core $C \subseteq X$ for $\mathscr{A}$ on which E has bounded finite index over F.*

Proof. Recursively define $f(m, n) = m(mn)^m + \sum_{0<k<m} f(k, n)$, for positive integers m and n. We will establish the stronger fact that if every set in $\mathscr{A}$ has cardinality at most m, and every E-local F-antichain of sets in $\mathscr{A}$ has cardinality at most n, then there is an F-invariant Borel core for $\mathscr{A}$ on which E has index at most $f(m, n)$ over F.

We proceed by induction on m. The base case $m = 1$ is just a rephrasing of Proposition 2.2.3, so suppose that $m \geq 2$ and we have already established the proposition strictly below m. We will construct analytic families $\mathscr{A}_k \subseteq \mathscr{A}$ and $\mathscr{A}'_k \subseteq [X]^k_{E,F}$, as well as F-invariant Borel sets $B_k \subseteq X$, which satisfy the following conditions:

(1) $\forall k < m$ $\mathscr{A}_k = \{a \in \mathscr{A}_{k+1} \mid a \cap B_{k+1} = \emptyset\}$.
(2) $\forall 1 \leq k \leq m$ $\mathscr{A}'_k = \{a' \in [X]^k_{E,F} \mid |[a', \mathscr{A}_k]_F| > (mn)^{m-k}\}$.
(3) $\forall 1 \leq k \leq m$ B_k is a core for $\mathscr{A}'_k$.
(4) $\forall 1 \leq k \leq m$ E has index at most $f(k, n)$ over F on B_k.

We proceed by reverse recursion, beginning with $\mathscr{A}_m = \mathscr{A}$, $\mathscr{A}'_m = \emptyset$, and $B_m = \emptyset$. Suppose now that $0 < k < m$ and we have already found $\mathscr{A}_{k+1}$, $\mathscr{A}'_{k+1}$, and B_{k+1}. Conditions (1) and (2) then yield $\mathscr{A}_k$ and $\mathscr{A}'_k$.

Lemma 2.3.2. *Suppose that $a' \in \mathscr{A}'_k$ and $x \in [a']_E \setminus [a']_F$. Then the family $[a' \cup \{x\}, \mathscr{A}_k]_F$ has cardinality at most $(mn)^{m-(k+1)}$.*

Proof. We can clearly assume that $[a' \cup \{x\}, \mathscr{A}_k]_F \neq \emptyset$. Given any set $a \in [a' \cup \{x\}, \mathscr{A}_k]_F$, it follows from condition (1) that $a \cap B_{k+1} = \emptyset$, thus $(a' \cup \{x\}) \cap B_{k+1} = \emptyset$. Condition (3) therefore ensures that that $a' \cup \{x\} \notin \mathscr{A}'_{k+1}$, so $|[a' \cup \{x\}, \mathscr{A}_{k+1}]_F| \leq (mn)^{m-(k+1)}$ by condition (2). As condition (1) also implies that $\mathscr{A}_k \subseteq \mathscr{A}_{k+1}$, the lemma follows. ⊠

Lemma 2.3.3. *Suppose that $a' \in \mathscr{A}'_k$ and $b \subseteq [a']_E \setminus [a']_F$ has cardinality at most mn. Then there exists $a \in \mathscr{A}_k$ with $a' \subseteq [a]_F$ and $[a]_F \cap [b]_F = \emptyset$.*

Proof. Lemma 2.3.2 ensures that $|[a' \cup \{x\}, \mathscr{A}_k]_F| \leq (mn)^{m-(k+1)}$ for all $x \in b$, so $[a', \mathscr{A}_k]_F$ contains at most $(mn)^{m-k}$ sets which intersect $[b]_F$, thus condition (2) ensures that $[a', \mathscr{A}_k]_F$ contains a set which does not intersect $[b]_F$. ⊠

Lemma 2.3.4. *Every E-local F-antichain of sets in $\mathscr{A}'_k$ has cardinality at most n.*

Proof. Suppose, towards a contradiction, that $(a'_i)_{i \leq n}$ is an injective enumeration of an E-local F-antichain of sets in $\mathscr{A}'_k$. By repeatedly applying Lemma 2.3.3, we can recursively find a sequence $(a_i)_{i \leq n}$ of sets in $\mathscr{A}_k$ with $a'_i \subseteq [a_i]_F$ for all $i \leq n$, and $[a_i]_F \cap [a_j]_F = \emptyset$ for all $i < j \leq n$. In particular, it follows that $(a_i)_{i \leq n}$ is an injective enumeration of an E-local F-antichain of sets in $\mathscr{A}_k$, the desired contradiction. ⊠

As the induction hypothesis ensures that there is an F-invariant Borel core $B_k \subseteq X$ for $\mathscr{A}'_k$ on which E has index at most $f(k, n)$ over F, this completes the recursive construction. Define $A_0 = \bigcup \mathscr{A}_0$.

Lemma 2.3.5. *The relation E has index at most $m(mn)^m$ over F on A_0.*

Proof. Given $x \in X$, fix a maximal F-antichain $\mathscr{A}_x \subseteq \mathscr{A}_0 \cap [x]^{<\aleph_0}_E$. Then $|\mathscr{A}_x| \leq n$, so the set $A_x = \bigcup \mathscr{A}_x$ has cardinality at most mn, and its F-saturation is a core for $\mathscr{A}_0 \cap [x]^{<\aleph_0}_E$. If $y \in A_x$, then there exists $a \in \mathscr{A}_x$ such that $y \in a$. As $a \in \mathscr{A}_0$, condition (1) ensures that $a \cap B_1 = \emptyset$, thus $y \notin B_1$. Condition (3) then implies that $\{y\} \notin \mathscr{A}_1'$, so $|[\{y\}, \mathscr{A}_1]_F| \leq (mn)^{m-1}$ by condition (2). As $\mathscr{A}_0 \subseteq \mathscr{A}_1$ by condition (1), it follows that $|[\{y\}, \mathscr{A}_0]_F| \leq (mn)^{m-1}$ as well, and since there are only (mn)-many possibilities for y, the desired result follows. ⊠

Proposition 2.2.3 now ensures that that there is an F-invariant Borel set $B_0 \supseteq A_0$ on which E has index at most $m(mn)^m$ over F. It only remains to note that for each $a \in \mathscr{A}$, there is a least $k \leq m$ for which $a \in \mathscr{A}_k$, in which case $a \cap B_k$ is non-empty, thus the set $B = \bigcup_{k<m} B_k$ is a core for $\mathscr{A}$. ⊠

Remark 2.3.6. Similar results appear in [CCCM11].

Given any binary relation R on X, we will also use R to denote the extension to $[X]^{<\aleph_0}_{E,F}$ given by $a \mathrel{R} b \iff a \times b \subseteq R$.

Proposition 2.3.7. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, F is a Borel subequivalence relation of E, and $(\mathscr{A}, \mathscr{A}')$ is an $(E \setminus F)$-independent pair of analytic subfamilies of $[X]^{<\aleph_0}_{E,F}$ of sets of bounded finite cardinality. Then there is an E-invariant Borel set $B \subseteq X \setminus \bigcup \mathscr{A}'$ for which there is an F-invariant Borel core $C \subseteq X$ for $\mathscr{A} \cap [X \setminus B]^{<\aleph_0}$ on which E has bounded finite index over F.*

Proof. Set $A' = [\bigcup \mathscr{A}']_E$. If every set in $\mathscr{A}'$ has cardinality at most n, then every E-local F-antichain of sets in $\mathscr{A} \cap [A']^{<\aleph_0}$ has cardinality at most n, so Proposition 2.3.1 ensures that there is an F-invariant Borel core $C \subseteq X$ for $\mathscr{A} \cap [A']^{<\aleph_0}$ on which E has bounded finite index over F. As the analytic set $A = \bigcup (\mathscr{A} \cap [X \setminus C]^{<\aleph_0})$ is disjoint from the E-invariant analytic set A', Proposition 1.4.2 ensures that there is an E-invariant Borel set $B \subseteq X$ separating A from A', and our choice of A ensures that C is a core for $\mathscr{A} \cap [X \setminus B]^{<\aleph_0}$. ⊠

2.4. **Main results.** Suppose that $R_1, R_2 \subseteq X \times X$ and $S_1, S_2 \subseteq Y \times Y$. We say that a function $\pi \colon X \to Y$ is a *homomorphism* from (R_1, R_2) to (S_1, S_2) if it is simultaneously a homomorphism from R_1 to S_1 and a homomorphism from R_2 to S_2.

Theorem 2.4.1. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, F is a Borel equivalence relation on X, and G is an analytic graph on X. Then exactly one of the following holds:*

1. *There is a Borel set $B \subseteq X$, off of which E has σ-bounded finite index over $E \cap F$, for which there is a smooth Borel equivalence relation $E' \supseteq E$ such that $E' \cap G$ has a Borel $\aleph_0$-coloring on B.*

(2) *There is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from the pair $(\mathbb{E}_0' \setminus \Delta(2^{\mathbb{N}}), \mathbb{G}_0)$ to the pair $(E \setminus F, G)$.*

Proof. To see that conditions (1) and (2) are mutually exclusive, suppose that both hold, and let B_0, E_0, E_0', F_0, and G_0 denote the pullbacks of B, E, E', F, and G through ϕ. Note that $\mathbb{E}_0' \subseteq E_0 \subseteq E_0'$, so the fact that E_0' is smooth, along with Propositions 1.3.8 and 2.1.3, ensures that there is a comeager E_0'-class $C_0 \subseteq 2^{\mathbb{N}}$, thus $G_0 \restriction C_0 \subseteq E_0' \cap G_0$. As there is a Borel $\aleph_0$-coloring of $(E' \cap G) \restriction B$, there is also a Borel $\aleph_0$-coloring of $(E_0' \cap G_0) \restriction B_0$, and therefore of $G_0 \restriction (B_0 \cap C_0)$. As $\mathbb{G}_0 \subseteq G_0$, Corollary 2.1.2 ensures that B_0 is meager. However, the fact that $\mathbb{E}_0' \cap F_0 = \Delta(2^{\mathbb{N}})$ ensures that $\mathbb{E}_0'$ has σ-bounded finite index over $\Delta(2^{\mathbb{N}})$ on $2^{\mathbb{N}} \setminus B_0$, so there is a non-meager Borel set $D_0 \subseteq 2^{\mathbb{N}}$ on which $\mathbb{E}_0'$ is finite, and therefore smooth, by Proposition 1.6.3. But this contradicts Propositions 1.3.8 and Proposition 2.1.3.

It remains to show that at least one of conditions (1) and (2) holds. For notational purposes, it will be convenient to assume that $X = \mathbb{N}^{\mathbb{N}}$. To see that this special case is sufficient to establish the theorem, note that we can assume there is a continuous surjection $\psi \colon \mathbb{N}^{\mathbb{N}} \to X$, and let E_0, F_0, and G_0 denote the pullbacks of E, F, and G through ψ. If B_0 and E_0' witness the analog of condition (1) for the pullbacks, then their pushforwards witness the weakening of condition (1) for the original structures in which all of the sets involved are merely analytic, rather than Borel. However, this problem can be rectified by appealing to reflection (Propositions 1.4.2, 2.2.2, and 2.2.3). On the other hand, if the analog of condition (2) holds of the pullbacks as witnessed by $\phi \colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, then the function $\pi = \psi \circ \phi$ witnesses that condition (2) holds of the original structures.

As the sets $E \setminus F$ and G are analytic, and we can clearly assume that they are non-empty, Proposition 1.2.2 yields trees $T_{E \setminus F}$ and T_G on $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ for which $E \setminus F = p[T_{E \setminus F}]$ and $G = p[T_G]$.

We recursively define increasing sequences $(B^\alpha)_{\alpha<\omega_1}$ and $(C^\alpha)_{\alpha<\omega_1}$ of Borel subsets of X, as well as a decreasing sequence $(E^\alpha)_{\alpha<\omega_1}$ of smooth Borel superequivalence relations of E, such that $E^\alpha \cap G$ has a Borel $\aleph_0$-coloring on B^α, and E has σ-finite bounded index over $E \cap F$ on C^α, for all $\alpha < \omega_1$. We begin by setting $B^0 = C^0 = \emptyset$ and $E^0 = \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. At limit ordinals $\lambda < \omega_1$, we set $B^\lambda = \bigcup_{\alpha<\lambda} B^\alpha$, $C^\lambda = \bigcup_{\alpha<\lambda} C^\alpha$, and $E^\lambda = \bigcap_{\alpha<\lambda} E^\alpha$. In order to describe the construction at successor ordinals, we must first introduce some terminology and establish several preliminary results.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi_k^a)_{k<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \mathbb{N}^{n^a}$, $\psi_k^a \colon 2^{n^a-(k+1)} \to \mathbb{N}^{n^a}$ for all even $k < n^a$, and

$\psi^a_k\colon [t_k(0)]_{\mathbb{E}_0'(2^k)} \times [t_k(1)]_{\mathbb{E}_0'(2^k)} \times 2^{n^a-(k+1)} \to \mathbb{N}^{n^a}$ for all odd $k < n^a$. An approximation a is *extended* by an approximation b if $n^a \leq n^b$, $s \sqsubseteq t \implies \phi^a(s) \sqsubseteq \phi^b(t)$ for all $s \in \mathrm{dom}(\phi^a)$ and $t \in \mathrm{dom}(\phi^b)$, and $s \sqsubseteq t \implies \psi^a_k(s) \sqsubseteq \psi^b_k(t)$ for all $k < n^a$, $s \in \mathrm{dom}(\psi^a_k)$, and $t \in \mathrm{dom}(\psi^b_k)$. When $n^b = n^a + 1$, we say that ϕ^b is a one-step extension of ϕ^a.

A *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_k)_{k<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma\colon 2^{n^\gamma} \to \mathbb{N}^\mathbb{N}$, $\psi^\gamma_k\colon 2^{n^\gamma-(k+1)} \to \mathbb{N}^\mathbb{N}$ for all even $k < n^\gamma$, and $\psi^\gamma_k\colon [t_k(0)]_{\mathbb{E}_0'(2^k)} \times [t_k(1)]_{\mathbb{E}_0'(2^k)} \times 2^{n^\gamma-(k+1)} \to \mathbb{N}^\mathbb{N}$ for all odd $k < n^\gamma$, with

$$((\phi^\gamma(s_k \frown (0) \frown s), \phi^\gamma(s_k \frown (1) \frown s)), \psi^\gamma_k(s)) \in [T_G]$$

for all even $k < n^\gamma$ and $s \in 2^{n^\gamma-(k+1)}$, and

$$((\phi^\gamma(r_0 \frown (0) \frown s), \phi^\gamma(r_1 \frown (1) \frown s)), \psi^\gamma_k(r_0, r_1, s)) \in [T_{E\setminus F}]$$

for all odd $k < n^\gamma$, $r_0 \in [t_k(0)]_{\mathbb{E}_0'(2^k)}$, $r_1 \in [t_k(1)]_{\mathbb{E}_0'(2^k)}$, and $s \in 2^{n^\gamma-(k+1)}$.

We say that a configuration γ is *compatible* with an approximation a if $n^a = n^\gamma$, $\phi^a(s) \sqsubseteq \phi^\gamma(s)$ for all $s \in \mathrm{dom}(\phi^a)$, and $\psi^a_k(s) \sqsubseteq \psi^\gamma_k(s)$ for all $k < n^a$ and $s \in \mathrm{dom}(\psi^a_k)$. We say that γ is *compatible* with B^α if $\phi^\gamma(s) \notin B^\alpha$ for all $s \in \mathrm{dom}(\phi^\gamma)$, and similarly for C^α. And we say that γ is compatible with E^α if $\phi^\gamma(s)\ E^\alpha\ \phi^\gamma(t)$ for all $s, t \in \mathrm{dom}(\phi^\gamma)$.

We say that a is *α-terminal* if for no one-step extension b of a is there a configuration compatible with b, B^α, C^α, and E^α. In particular, if a is not α-terminal, then it has a compatible configuration.

An approximation a is *even* if n^a is even. For such approximations, we use $A_\alpha(a)$ to denote the analytic set of points of the form $\phi^\gamma(s_{n^a})$, where γ ranges over configurations compatible with a, B^α, C^α, and E^α.

Lemma 2.4.2. *Suppose that a is an even approximation with the property that $A_\alpha(a)$ is not $(E^\alpha \cap G)$-independent. Then a is not α-terminal.*

Proof. Fix configurations γ_0 and γ_1, compatible with a, B^α, C^α, and E^α, such that $\phi^{\gamma_0}(s_{n^a})\ (E^\alpha \cap G)\ \phi^{\gamma_1}(s_{n^a})$. Then there exists $b \in \mathbb{N}^\mathbb{N}$ with the property that $((\phi^{\gamma_0}(s_{n^a}), \phi^{\gamma_1}(s_{n^a})), b) \in [T_G]$. Let γ denote the configuration given by

- $n^\gamma = n^a + 1$.
- $\forall i < 2 \forall s \in 2^{n^a}\ \phi^\gamma(s \frown (i)) = \phi^{\gamma_i}(s)$.
- $\psi^\gamma_{n^a}(\emptyset) = b$.
- $\forall i < 2 \forall k < n^a$ even$\forall s \in 2^{n^a-(k+1)}\ \psi^\gamma_k(s \frown (i)) = \psi^{\gamma_i}_k(s)$.
- $\forall i < 2 \forall k < n^a$ odd$\forall r_0 \in [t_k(0)]_{\mathbb{E}_0'(2^k)} \forall r_1 \in [t_k(1)]_{\mathbb{E}_0'(2^k)} \forall s \in 2^{n^a-(k+1)}$
  $\psi^\gamma_k(r_0, r_1, s \frown (i)) = \psi^{\gamma_i}_k(r_0, r_1, s)$.

Note that γ is compatible with B^α, C^α, and E^α. As the unique approximation b with which γ is compatible is a one-step extension of a, it follows that a is not α-terminal. $\boxtimes$

Lemma 2.4.3. *Suppose that a is an α-terminal even approximation. Then there is an $(E^\alpha \cap G)$-independent Borel set $B_\alpha(a) \supseteq A_\alpha(a)$.*

Proof. As Lemma 2.4.2 ensures that $A_\alpha(a)$ is an analytic $(E^\alpha \cap G)$-independent set, Proposition 2.2.2 yields the desired set. ⊠

An approximation a is *odd* if n^a is odd. For such a, we use $\mathscr{A}_\alpha(a,i)$ to denote the analytic family of sets of the form $\phi^\gamma([t_{n^a}(i)]_{\mathbb{E}_0'(2^{n^a})})$, where γ ranges over configurations compatible with a, B^α, C^α, and E^α.

Lemma 2.4.4. *Suppose that a is an odd approximation with the property that $(\mathscr{A}_\alpha(a,0), \mathscr{A}_\alpha(a,1))$ is not $(E \setminus F)$-independent. Then a is not α-terminal.*

Proof. Fix configurations γ_0 and γ_1, compatible with a, B^α, C^α, and E^α, for which $\phi^{\gamma_0}([t_{n^a}(0)]_{\mathbb{E}_0'(2^{n^a})}) \times \phi^{\gamma_1}([t_{n^a}(1)]_{\mathbb{E}_0'(2^{n^a})}) \subseteq E \setminus F$. Then there exists $f \colon [t_{n^a}(0)]_{\mathbb{E}_0'(2^{n^a})} \times [t_{n^a}(1)]_{\mathbb{E}_0'(2^{n^a})} \to \mathbb{N}^\mathbb{N}$ with the property that $((\phi^{\gamma_0}(r_0), \phi^{\gamma_1}(r_1)), f(r_0,r_1)) \in [T_{E\setminus F}]$ for all $r_0 \in [t_{n^a}(0)]_{\mathbb{E}_0'(2^{n^a})}$ and $r_1 \in [t_{n^a}(1)]_{\mathbb{E}_0'(2^{n^a})}$. Let γ denote the configuration given by

- $n^\gamma = n^a + 1$.
- $\forall i < 2 \forall s \in 2^{n^a}$ $\phi^\gamma(s \frown (i)) = \phi^{\gamma_i}(s)$.
- $\forall r_0 \in [t_{n^a}(0)]_{\mathbb{E}_0'(2^{n^a})} \forall r_1 \in [t_{n^a}(1)]_{\mathbb{E}_0'(2^{n^a})}$ $\psi^\gamma_{n^a}(r_0, r_1, \emptyset) = f(r_0, r_1)$.
- $\forall i < 2 \forall k < n^a$ even$\forall s \in 2^{n^a-(k+1)}$ $\psi^\gamma_k(s \frown (i)) = \psi^{\gamma_i}_k(s)$.
- $\forall i < 2 \forall k < n^a$ odd$\forall r_0 \in [t_k(0)]_{\mathbb{E}_0'(2^k)} \forall r_1 \in [t_k(1)]_{\mathbb{E}_0'(2^k)} \forall s \in 2^{n^a-(k+1)}$ $\psi^\gamma_k(r_0, r_1, s \frown (i)) = \psi^{\gamma_i}_k(r_0, r_1, s)$.

Note that γ is compatible with B^α, C^α, and E^α. As the unique approximation b with which γ is compatible is a one-step extension of a, it follows that a is not α-terminal. ⊠

Lemma 2.4.5. *Suppose that a is an α-terminal odd approximation. Then there exists an E-invariant Borel set $D_\alpha(a) \subseteq \mathbb{N}^\mathbb{N} \setminus \bigcup \mathscr{A}_\alpha(a,1)$ for which there is an $(E \cap F)$-invariant Borel core $C_\alpha(a) \subseteq \mathbb{N}^\mathbb{N}$ for $\mathscr{A}_\alpha(a,0) \cap [\mathbb{N}^\mathbb{N} \setminus D_\alpha(a)]^{<\aleph_0}$ on which E has bounded finite index over $E \cap F$.*

Proof. As the pair $(\mathscr{A}_\alpha(a,0), \mathscr{A}_\alpha(a,1))$ of analytic sets is $(E \setminus F)$-independent by Lemma 2.4.4, Proposition 2.3.7 yields the desired sets. ⊠

Let T^α_{even} and T^α_{odd} be the sets of α-terminal even and odd approximations. Set $B^{\alpha+1} = B^\alpha \cup \bigcup_{a \in T^\alpha_{\text{even}}} B_\alpha(a)$ and $C^{\alpha+1} = C^\alpha \cup \bigcup_{a \in T^\alpha_{\text{odd}}} C_\alpha(a)$, and let $E^{\alpha+1}$ denote the smooth Borel equivalence relation given by

$$x \; E^{\alpha+1} \; y \iff (x \; E^\alpha \; y \text{ and } \forall a \in T^\alpha_{\text{odd}} \; (x \in D_\alpha(a) \iff y \in D_\alpha(a))).$$

This completes the recursive construction.

Lemma 2.4.6. *Suppose that a is an approximation whose one-step extensions are all α-terminal. Then a is $(\alpha + 1)$-terminal.*

Proof. Suppose, towards a contradiction, that there is a one-step extension b of a for which there is a configuration γ compatible with b, $B^{\alpha+1}$, $C^{\alpha+1}$, and $E^{\alpha+1}$. If n^b is even, then the fact that b is α-terminal ensures that $\phi^\gamma(s_{n^b}) \in B_\alpha(b)$, contradicting the fact that γ is compatible with $B^{\alpha+1}$. If n^b is odd, then the fact that b is α-terminal ensures that $\phi^\gamma(2^{n^b})$ intersects $C_\alpha(b) \cup D_\alpha(b)$, contradicting the fact that γ is compatible with $C^{\alpha+1}$ and $E^{\alpha+1}$. ⊠

As every α-terminal approximation is β-terminal whenever $\alpha < \beta$, and there are only countably many approximations, there exists $\alpha < \omega_1$ such that every $(\alpha + 1)$-terminal approximation is α-terminal.

If the unique approximation a with the property that $n^a = 0$ is α-terminal, then $X = B^{\alpha+1} \cup C^{\alpha+1}$. Our original requirements that $E^{\alpha+1}$ is smooth, $E^{\alpha+1} \cap G$ has a Borel $\aleph_0$-coloring on $B^{\alpha+1}$, and E has σ-finite bounded index over $E \cap F$ on $C^{\alpha+1}$ therefore ensure that the set $B = B^{\alpha+1}$ and the relation $E' = E^{\alpha+1}$ satisfy condition (1).

Otherwise, repeated application of Lemma 2.4.6 gives rise to a sequence $(a_n)_{n\in\mathbb{N}}$ of non-α-terminal approximations such that a_{n+1} is a one-step extension of a_n. Define a continuous function $\phi \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by setting $\phi(c) = \bigcup_{n\in\mathbb{N}} \phi^{a_n}(c \restriction n)$. For each even $k \in \mathbb{N}$, define a continuous function $\psi_k \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\psi_k(c) = \bigcup_{n>k} \psi_k^{a_n}(c \restriction (n - k - 1))$. And for each odd $k \in \mathbb{N}$, define $\psi_k \colon [t_k(0)]_{\mathbb{E}_0'(2^k)} \times [t_k(1)]_{\mathbb{E}_0'(2^k)} \times 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\psi_k(r_0, r_1, c) = \bigcup_{n>k} \psi_k^{a_n}(r_0, r_1, c \restriction (n - k - 1))$.

To see that ϕ is a homomorphism from $\mathbb{G}_0$ to G, we show that if $c \in 2^\mathbb{N}$ and $k \in \mathbb{N}$ is even, then $\phi(s_k \frown (0) \frown c) \mathrel{G} \phi(s_k \frown (1) \frown c)$. In fact, we show that $((\phi(s_k \frown (0) \frown c), \phi(s_k \frown (1) \frown c)), \psi_k(c)) \in [T_G]$, or equivalently, that $((\phi^{a_n}(s_k \frown (0) \frown s), \phi^{a_n}(s_k \frown (1) \frown s)), \psi_k^{a_n}(s)) \in T_G$ for all $n > k$, where $s = c \restriction (n - k - 1)$. Towards this end, it is sufficient to observe that if γ_n is a configuration compatible with a_n, then $((\phi^{\gamma_n}(s_k \frown (0) \frown s), \phi^{\gamma_n}(s_k \frown (1) \frown s)), \psi_k^{\gamma_n}(s)) \in [T_G]$.

To see that ϕ is a homomorphism from $\mathbb{E}_0' \setminus \Delta(2^\mathbb{N})$ to $E \setminus F$, we show that if $c \in 2^\mathbb{N}$, $k \in \mathbb{N}$ is odd, $r_0 \in [t_k(0)]_{\mathbb{E}_0'(2^k)}$, and $r_1 \in [t_k(1)]_{\mathbb{E}_0'(2^k)}$, then $\phi(r_0 \frown (0) \frown c) \mathrel{(E \setminus F)} \phi(r_1 \frown (1) \frown c)$. In fact, we show that $((\phi(r_0 \frown (0) \frown c), \phi(r_1 \frown (1) \frown c)), \psi_k(r_0, r_1, c)) \in [T_G]$. Now, as before, this latter statement is itself equivalent to the statement that $((\phi^{a_n}(r_0 \frown (0) \frown s), \phi^{a_n}(r_1 \frown (1) \frown s)), \psi_k^{a_n}(r_0, r_1, s)) \in T_G$ for all $n > k$, where $s = c \restriction (n - k - 1)$. Once more as before, for this it is sufficient to note that if γ_n is a configuration compatible with a_n, then $((\phi^{\gamma_n}(r_0 \frown (0) \frown s), \phi^{\gamma_n}(r_1 \frown (1) \frown s)), \psi_k^{\gamma_n}(r_0, r_1, s)) \in [T_{E\setminus F}]$. ⊠

Generalizing the Harrington-Kechris-Louveau Theorem, we now have the following.

Theorem 2.4.7. *Suppose that X is a Polish space and E and F are Borel equivalence relations on X. Then exactly one of the following holds:*

1. *There is an E-smooth Borel set $B \subseteq X$ off of which E has σ-bounded-finite-index over $E \cap F$.*
2. *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into the restriction of E to a partial transversal of F.*

Proof. To see that conditions (1) and (2) are mutually exclusive, suppose that both hold, let B_0 denote the pullback of B through π, and observe that both B_0 and its complement are $\mathbb{E}_0$-smooth, contradicting Propositions 1.3.7 and 1.3.8.

To see that at least one of conditions (1) and (2) holds, let G denote the complement of E. If condition (1) of Theorem 2.4.1 holds, then there is a Borel set $B \subseteq X$, off of which E has σ-bounded finite index over $E \cap F$, for which there exists a smooth Borel equivalence relation $E' \supseteq E$ such that $E' \cap G$ has a Borel $\aleph_0$-coloring on B. Proposition 2.2.3 then implies that B is contained in the union of countably many E-invariant $(E' \cap G)$-independent Borel sets $B_n \subseteq X$. In particular, it follows that if $x, y \in B$, then

$$x \mathrel{E} y \iff (x \mathrel{E'} y \text{ and } \forall n \in \mathbb{N} \; (x \in B_n \iff y \in B_n)),$$

so $E \restriction B$ is smooth, thus condition (1) holds.

Otherwise, condition (2) of Theorem 2.4.1 holds, yielding a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{E}_0' \setminus \Delta(2^{\mathbb{N}}), \mathbb{G}_0)$ to $(E \setminus F, G)$. Letting E' and F' denote the pullbacks of E and F through ϕ, Proposition 2.1.4 then implies that there is a continuous embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}})$ into $E' \setminus F'$, thus $\phi \circ \psi$ is the desired continuous embedding of $\mathbb{E}_0$ into the restriction of E to a partial transversal of F. ⊠

In the countable case, we obtain the following strengthening.

Theorem 2.4.8. *Suppose that X is a Polish space, E and F are Borel equivalence relations on X, and E is countable. Then exactly one of the following holds:*

1. *The relation E has σ-bounded-finite-index over $E \cap F$.*
2. *There is a continuous embedding of $\mathbb{E}_0$ into the restriction of E to a partial transversal of F.*

Proof. Although substantially easier to establish than Theorem 2.4.7, this is a direct consequence of the latter and Proposition 1.6.1. ⊠

The theorem also generalizes significantly beyond Polish spaces.

Theorem 2.4.9. *Suppose that X is an analytic Hausdorff space and E and F are Borel equivalence relations on X. Then exactly one of the following holds:*

1. *There is an E-smooth Borel set $B \subseteq X$ off of which E has σ-bounded-finite-index over $E \cap F$.*
2. *There is a continuous embedding of $\mathbb{E}_0$ into the restriction of E to a partial transversal of F.*

Proof. Our proof of Theorem 2.4.7 goes through in this generality. ⊠

We say that a set $Y \subseteq X$ is *$\sigma(\boldsymbol{\Sigma}^1_1)$-$E$-smooth* if there is a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable reduction of $E \restriction Y$ to equality on $2^{\mathbb{N}}$. We say that E has *$\sigma(\boldsymbol{\Sigma}^1_1)$-$\sigma$-bounded finite index* over F if the underlying space is the union of countably many $\sigma(\boldsymbol{\Sigma}^1_1)$ sets on which E has bounded finite index over F.

Theorem 2.4.10. *Suppose that X is an analytic Hausdorff space and E and F are Borel equivalence relations on X. Then the following are equivalent:*

1. *There is an E-smooth Borel set $B \subseteq X$ off of which E has σ-bounded-finite-index over $E \cap F$.*
2. *There is a $\sigma(\boldsymbol{\Sigma}^1_1)$-$E$-smooth $\sigma(\boldsymbol{\Sigma}^1_1)$ set $B \subseteq X$ off of which E has $\sigma(\boldsymbol{\Sigma}^1_1)$-$\sigma$-bounded finite index over $E \cap F$.*

Proof. It is sufficient to establish that the second condition implies the first, and for this, it is enough to show that the former is incompatible with the second condition of Theorem 2.4.9. But the argument given there is sufficiently general so as to establish this as well. ⊠

We say that a subequivalence relation F of E has *countable index* if every E-class is the union of countably many F-classes.

Theorem 2.4.11. *Suppose that X is a Polish space, E is a smooth Borel equivalence relation on X, and F is a countable index Borel subequivalence relation of E. Then F is also smooth.*

Proof. Note that if F is non-smooth, then Theorem 2.4.7 (or just the Harrington-Kechris-Louveau Theorem) yields a Borel set on which F is both countable and non-smooth, and therefore on which E is also non-smooth, by Proposition 1.6.2. ⊠

Along similar lines, we have the following.

Theorem 2.4.12. *Suppose that X is a Polish space, E is a Borel equivalence relation on X, and F is a finite index Borel subequivalence relation of E. Then E is smooth if and only if F is smooth.*

Proof. Note that if either E or F is non-smooth, then Theorem 2.4.7 (or just the Harrington-Kechris-Louveau Theorem) yields a Borel set on which the corresponding equivalence relation is both countable and non-smooth, and therefore on which both are non-smooth, by Proposition 1.6.4. ⊠

3. Rigidity

Here we establish our rigidity results connecting each Borel equivalence relation E with the corresponding σ-ideal $\mathcal{I}_E$ generated by the family of E-smooth Borel sets. In §3.1, we give several further corollaries of reflection. And in §3.2, we present our main results.

3.1. **Reflection.**

We begin by checking that smoothness is closed under saturation.

Proposition 3.1.1. *Suppose that X is a Polish space, E is a Borel equivalence relation, and $A \subseteq X$ is an E-smooth analytic set. Then so too is $[A]_E$.*

Proof. Suppose that $\pi \colon A \to 2^{\mathbb{N}}$ is a Borel reduction of $E \restriction A$ to equality, and observe that the function $\pi' \colon [A]_E \to 2^{\mathbb{N}}$ given by

$$\pi'(x) = c \iff \exists w \in A \ (w \mathrel{E} x \text{ and } \pi(w) = c)$$

is a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable reduction of $E \restriction [A]_E$ to equality, thus Theorem 2.4.10 (or just the analogous consequence of the Harrington-Kechris-Louveau Theorem) ensures that $[A]_E$ is E-smooth. ⊠

We next check that E-smoothness is closed under countable unions. In particular, it follows that $\mathcal{I}_E$ is the downwards closure of the family of all E-smooth Borel sets.

Proposition 3.1.2. *Suppose that X is a Polish space, E is a Borel equivalence relation on X, and $(A_n)_{n \in \mathbb{N}}$ is a sequence of E-smooth analytic subsets of X. Then $\bigcup_{n \in \mathbb{N}} A_n$ is also E-smooth.*

Proof. By Proposition 3.1.1, we can assume that each of the sets A_n is E-invariant. Fix Borel reductions $\pi_n \colon A_n \to 2^{\mathbb{N}}$ of $E \restriction A_n$ to equality, and note that the map $\pi \colon \bigcup_{n \in \mathbb{N}} A_n \to 2^{\mathbb{N}} \times \mathbb{N}$, given by $\pi(x) = (\pi_n(x), n)$ for $x \in A_n \setminus \bigcup_{m<n} A_m$, is a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable reduction of $E \restriction \bigcup_{n \in \mathbb{N}} A_n$ to equality, thus Theorem 2.4.10 (or just the analogous consequence of the Harrington-Kechris-Louveau Theorem) ensures that $\bigcup_{n \in \mathbb{N}} A_n$ is E-smooth. ⊠

The following result implies that E-smooth analytic sets are in $\mathcal{I}_E$.

Proposition 3.1.3. *Suppose that X is a Polish space, E is a Borel equivalence relation on X, and $A \subseteq X$ is an E-smooth analytic set. Then there is an E-smooth Borel set $B \supseteq A$.*

Proof. Theorem 2.4.9 (or just the analogous generalization of the Harrington-Kechris-Louveau Theorem) ensures that A is E-smooth if and only if there is no continuous function $\pi \colon 2^{\mathbb{N}} \to X$ satisfying the following conditions:

(1) The pre-image of A under π is comeager.
(2) The set $\{c \in 2^{\mathbb{N}} \mid \forall d \in [c]_{\mathbb{E}_0}\ \pi(c)\ E\ \pi(d)\}$ is comeager.
(3) The set $\{(c,d) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid \neg\pi(c)\ E\ \pi(d)\}$ is comeager.

As Proposition 1.1.2 and Theorems 1.3.5 and 1.5.1 ensure that this is a $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$ property of A, the desired result is a consequence of Theorem 1.4.1. ⊠

We say that a set $R \subseteq X \times Y$ *induces* a partial function from X/E to $[Y/F]^{\leq k}$ if for all $x \in X$, the restriction of F to $\bigcup_{w \in [x]_E} R_w$ has at most k classes. When R is analytic, we say that such a partial function is *smooth-to-one* if the analytic set $\bigcup_{z \in [y]_F} R^z$ is E-smooth, for all $y \in Y$.

Proposition 3.1.4. *Suppose that k is a positive integer, X and Y are Polish spaces, E is a Borel equivalence relation on X, F is a Borel equivalence relation on Y, and $R \subseteq X \times Y$ is an analytic set which induces a smooth-to-one partial function from X/E to $[Y/F]^{\leq k}$. Then there is an $(E \times F)$-invariant Borel set $S \supseteq R$ which also induces a smooth-to-one partial function from X/E to $[Y/F]^{\leq k}$.*

Proof. As E is analytic and F is co-analytic, the property (of R) of inducing a partial function from X/E to $[Y/F]^{\leq k}$ is $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$. As E is Borel and F and R are analytic, Theorem 2.4.9 (or just the analogous generalization of the Harrington-Kechris-Louveau Theorem) ensures that this partial function is smooth-to-one if and only if there do not exist $y \in Y$ and a continuous function $\pi \colon 2^{\mathbb{N}} \to X$ satisfying the following conditions:

(1) The pre-image of $\bigcup_{z \in [y]_F} R^z$ under π is comeager.
(2) The set $\{c \in 2^{\mathbb{N}} \mid \forall d \in [c]_{\mathbb{E}_0}\ \pi(c)\ E\ \pi(d)\}$ is comeager.
(3) The set $\{(c,d) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid \neg\pi(c)\ E\ \pi(d)\}$ is comeager.

As Proposition 1.1.2 and Theorems 1.3.5 and 1.5.1 ensure that this is a $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$ property of R, it follows that so too is the property (of R) of inducing a smooth-to-one partial function from X/E to $[Y/F]^{\leq k}$. Mimicking the proof of Proposition 1.4.2, by applying Theorem 1.4.1 infinitely many times, we obtain a sequence $(R_n)_{n \in \mathbb{N}}$ of Borel supersets of R, inducing smooth-to-one partial functions from X/E to $[Y/F]^{\leq k}$, such that $[R_n]_{E \times F} \subseteq R_{n+1}$, for all $n \in \mathbb{N}$. Define $R = \bigcup_{n \in \mathbb{N}} R_n$. ⊠

3.2. **Main results.** We begin this section by showing that appropriate morphisms between Borel equivalence relations are automatically morphisms between the corresponding smooth ideals.

Proposition 3.2.1. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and $\phi \colon X \to Y$ is a smooth-to-one Borel homomorphism from E to F. Then ϕ is a cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$.*

Proof. Suppose, towards a contradiction, that there is a set $W \subseteq X$ for which $W \notin \mathcal{I}_E$ but $\phi(W) \in \mathcal{I}_F$. Fix an F-smooth Borel set $B \supseteq \phi(W)$, as well as a Borel reduction $\pi \colon B \to 2^{\mathbb{N}}$ of $F \upharpoonright B$ to equality, and observe that the set $A = \phi^{-1}(B)$ is E-non-smooth, since $W \subseteq A$. By Theorem 2.4.7 (or just the Harrington-Kechris-Louveau Theorem), there is a continuous embedding $\psi \colon 2^{\mathbb{N}} \to A$ of $\mathbb{E}_0$ into $E \upharpoonright A$. As Proposition 1.3.7 ensures that $\mathbb{E}_0$ is generically ergodic, Proposition 1.3.8 yields a comeager Borel set $C \subseteq 2^{\mathbb{N}}$ on which $\pi \circ \phi \circ \psi$ is constant. Let c denote this constant value. As ϕ is smooth-to-one, it follows that $(\pi \circ \phi)^{-1}(c)$ is E-smooth. But then $\psi \upharpoonright C$ is a reduction of $\mathbb{E}_0 \upharpoonright C$ to a smooth Borel equivalence relation, contradicting Propositions 1.3.7 and 1.3.8. ⊠

Proposition 3.2.2. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and $\phi \colon X \to Y$ is a Borel reduction of E to F. Then ϕ is a reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$.*

Proof. In light of Proposition 3.2.1, it is enough to show that if $W \subseteq X$ and $W \in \mathcal{I}_E$, then $\phi(W) \in \mathcal{I}_F$. Towards this end, fix an E-smooth Borel set $A \supseteq W$, and appeal to Theorem 1.5.2 to obtain a $\sigma(\mathbf{\Sigma^1_1})$-measurable function $\psi \colon \phi(A) \to A$ such that $y = (\phi \circ \psi)(y)$ for all $y \in \phi(A)$. As any such function is a reduction of $F \upharpoonright \phi(A)$ to $E \upharpoonright A$, Theorem 2.4.10 (or just the analogous consequence of the Harrington-Kechris-Louveau Theorem) ensures that $\phi(A)$ is F-smooth, in which case Proposition 3.1.3 yields an F-smooth Borel set $B \supseteq \phi(A)$. As $\phi(W) \subseteq B$, it follows that $\phi(W) \in \mathcal{I}_F$. ⊠

We next provide weakenings of the converses of these results.

Theorem 3.2.3. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and $\phi \colon X \to Y$ is Borel. Then ϕ is a cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is an E-smooth Borel set off of which ϕ is a smooth-to-one σ-quasi-homomorphism from E to F.*

Proof. Suppose first that ϕ is a cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$, and let F' denote the pullback of F through ϕ. Proposition 3.2.2 then implies that $\mathcal{I}_{F'} \subseteq \mathcal{I}_E$, so there is no continuous embedding of $\mathbb{E}_0$ into the restriction of E to a partial transversal of F', thus Theorem 2.4.7 yields an E-smooth Borel set $B \subseteq X$ off of which E has σ-bounded finite index over $E \cap F'$. As the restriction of ϕ to any set on which E has bounded finite index over $E \cap F'$ is a quasi-homomorphism from the corresponding restriction of E to F, it follows that ϕ is a smooth-to-one σ-quasi-homomorphism from E to F off of B.

To establish the converse, it is sufficient to consider the special case in which ϕ is a smooth-to-one quasi-homomorphism from E to F. Towards this end, again let F' denote the pullback of F through ϕ, and note that ϕ is a smooth-to-one homomorphism from $E \cap F'$ to F, by Theorem 2.4.12. As the latter also ensures that $\mathcal{I}_{E \cap F'} \subseteq \mathcal{I}_E$, and Proposition 3.2.1 implies that ϕ is a cohomomorphism from $\mathcal{I}_{E \cap F'}$ to $\mathcal{I}_F$, it follows that ϕ is also a cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$. ⊠

Theorem 3.2.4. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and $\phi \colon X \to Y$ is Borel. Then ϕ is a reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is an E-smooth Borel set, whose image under ϕ is F-smooth, off of which ϕ is a σ-quasi-reduction of E to F.*

Proof. Suppose first that ϕ is a reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$, and once more let F' denote the pullback of F through ϕ. Proposition 3.2.2 then implies that $\mathcal{I}_E = \mathcal{I}_{F'}$, so there is no continuous embedding of $\mathbb{E}_0$ into the restriction of either E or F' to a partial transversal of the other, thus two applications of Theorem 2.4.7 yield an E-smooth Borel set $B \subseteq X$ (whose image under ϕ is necessarily F-smooth) off of which E and F' have σ-bounded finite index over $E \cap F'$. As the restriction of ϕ to any set on which E and F' have bounded finite index over $E \cap F'$ is a quasi-reduction of E to F, it follows that ϕ is a σ-quasi-reduction of E to F off of B.

Proposition 3.1.2 ensures that in order to establish the converse, it is sufficient to consider the special case in which ϕ is a quasi-reduction of E to F. In light of Theorem 3.2.3, it is enough to show that ϕ is a homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$. Towards this end, again let F' denote the pullback of F through ϕ. As two applications of Theorem 2.4.12 ensure that $\mathcal{I}_E \subseteq \mathcal{I}_{E \cap F'} \subseteq \mathcal{I}_{F'}$, and Proposition 3.2.2 implies that ϕ is a homomorphism from $\mathcal{I}_{F'}$ to $\mathcal{I}_F$, it follows that ϕ is also a homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$. ⊠

In appropriate special cases, we can now obtain characterizations of the existence of morphisms between smooth ideals.

Theorem 3.2.5. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and E or F is countable. Then there is a Borel cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is a smooth-to-one Borel quasi-homomorphism from E to F.*

Proof. In light of Theorem 3.2.3 (and the fact that a countable union of smooth-to-one Borel quasi-homomorphisms, whose domains have pairwise disjoint E-saturations, is itself a smooth-to-one Borel σ-quasi-homomorphism), it is sufficient to show that if $B \subseteq X$ is a Borel set on which there is a smooth-to-one Borel quasi-homomorphism $\phi \colon B \to Y$ from E to F, then there is an E-invariant Borel set $C \supseteq B$ on which ϕ extends to a smooth-to-one Borel quasi-homomorphism from E to F.

To handle the case that E is countable, set $C = [B]_E$, appeal to Theorem 1.5.1 to obtain a Borel retraction $\psi \colon C \to B$ whose graph is contained in E, and observe that $\phi \circ \psi$ is the desired extension.

To handle the case that F is countable, fix a positive integer k such that graph(ϕ) induces a smooth-to-one partial function from X/E to $[Y/F]^{\leq k}$, and apply Proposition 3.1.4 to obtain an $(E \times F)$-invariant Borel set $R \supseteq \mathrm{graph}(\phi)$ which induces a smooth-to-one partial function from X/E to $[Y/F]^{\leq k}$. By Theorem 1.5.1, the set $C = \mathrm{proj}_X(R)$ is Borel, and there is a Borel function $\psi \colon C \to Y$ such that $\mathrm{graph}(\psi) \subseteq R$. By redefining ψ to agree with ϕ on B, we obtain the desired extension. ⊠

Theorem 3.2.6. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, E or F is countable, and F has uncountably many classes. Then there is a Borel reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is a Borel quasi-reduction of E to $F \times \Delta(\mathbb{N})$.*

Proof. By Theorem 2.4.7 (or just Silver's Theorem), every smooth Borel equivalence relation is Borel reducible to F. So by Theorem 3.2.4 (and the fact that a countable union of Borel quasi-reductions, whose domains have pairwise disjoint E-saturations and whose ranges have pairwise disjoint $(F \times \Delta(\mathbb{N}))$-saturations, is itself a Borel σ-quasi-reduction), it is enough to show that if $B \subseteq X$ is a Borel set for which there is a Borel quasi-reduction $\phi \colon B \to Y$ of $E \upharpoonright B$ to F, then there is an E-invariant Borel set $C \supseteq B$ on which ϕ extends to a Borel quasi-reduction of E to F.

To handle the case that E is countable, set $C = [B]_E$, appeal to Theorem 1.5.1 to obtain a Borel retraction $\psi \colon C \to B$ whose graph is contained in E, and observe that $\phi \circ \psi$ is the desired extension.

To handle the case that F is countable, fix a positive integer k with the property that $\mathrm{graph}(\phi)$ and $\mathrm{graph}(\phi^{-1})$ induce partial functions from X/E to $[Y/F]^{\leq k}$ and from Y/F to $[X/E]^{\leq k}$, and apply Proposition 3.1.4 to obtain $(E \times F)$-invariant and $(F \times E)$-invariant Borel sets $R \supseteq \mathrm{graph}(\phi)$ and $S \supseteq \mathrm{graph}(\phi^{-1})$ which induce partial functions from X/E to $[Y/F]^{\leq k}$ and from Y/F to $[X/E]^{\leq k}$. By Theorem 1.5.1, the set $C = \mathrm{proj}_X(R \cap S^{-1})$ is Borel, and there is a Borel function $\psi \colon C \to Y$ such that $\mathrm{graph}(\psi) \subseteq R \cap S^{-1}$. By redefining ψ to agree with ϕ on B, we obtain the desired extension. ⊠

4. Homogeneity

Here we characterize homogeneity of smooth σ-ideals. In §4.1, we give several further corollaries of reflection. And in §4.2, we establish our primary result.

4.1. **Reflection.**

We begin this section with a result on finite equivalence relations.

Proposition 4.1.1. *Suppose that X and Y are Polish spaces, E is a co-analytic equivalence relation on X, F is a finite analytic equivalence relation on Y, and $\pi \colon X \to Y$ is a Borel cohomomorphism from E to F. Then π is a cohomomorphism from E to a finite Borel superequivalence relation F' of F.*

Proof. Let Φ denote the property of sets $R \subseteq Y \times Y$ that the smallest equivalence relation containing R is finite, and let Ψ denote the property of sets $R \subseteq Y \times Y$ that π is a cohomomorphism from E to the smallest equivalence relation containing R. As both Φ and Ψ are $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$, so too is their conjunction, thus Theorem 1.4.1 yields a Borel set $S \supseteq F$ satisfying both properties. As S is Borel and has finite sections, Theorem 1.5.1 ensures that the smallest equivalence relation containing S is as desired. ⊠

As a corollary, we obtain the following.

Corollary 4.1.2. *Suppose that X and Y are Polish spaces, E is a co-analytic equivalence relation on X, $(F_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite analytic equivalence relations on Y, and $\pi \colon X \to Y$ is a Borel cohomomorphism from E to the equivalence relation $F = \bigcup_{n\in\mathbb{N}} F_n$. Then there is an increasing sequence of finite Borel superequivalence relations F'_n of F_n for which π is a cohomomorphism from E to the equivalence relation $F' = \bigcup_{n\in\mathbb{N}} F'_n$.*

Proof. By Proposition 4.1.1, there are finite Borel superequivalence relations F_n^* of F_n for which π is a cohomomorphism from E to F_n^*, for all $n \in \mathbb{N}$. Define $F_n' = \bigcap_{m \geq n} F_m^*$. ⊠

We say that an analytic equivalence relation is *hyperfinite* if it is the union of an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite analytic equivalence relations. Corollary 4.1.2 ensures that in the Borel case, this notion is compatible with that given earlier.

We say that an analytic equivalence relation is *essentially hyperfinite* if it is Borel reducible to a hyperfinite analytic equivalence relation. Corollary 4.1.2 again ensures that when the equivalence relation in question is Borel, this is equivalent to the existence of a Borel reduction to a hyperfinite Borel equivalence relation.

Proposition 4.1.3. *Suppose that X is a Polish space, E is a Borel equivalence relation on X, and there is a finite index Borel subequivalence relation E' of E which is essentially hyperfinite. Then E is essentially hyperfinite.*

Proof. Fix a Polish space Y, a hyperfinite Borel equivalence relation F' on Y, and a Borel reduction $\pi \colon X \to Y$ of E' to F'. Then every class of the equivalence relation on Y generated by the set $R = (\pi \times \pi)(E)$ is contained in a union of finitely many equivalence classes of F'. As F' is co-analytic, this is a $\mathbf{\Pi^1_1}$-on-$\mathbf{\Sigma^1_1}$ property of R, so Theorem 1.4.1 yields a Borel set $S \supseteq R$ with the same property. As Theorem 1.5.1 ensures that the equivalence relation F generated by S is Borel, Proposition 1.6.5 implies that F is hyperfinite, thus so too is $\Delta(Y) \cup R$, and it follows that E is essentially hyperfinite. ⊠

4.2. **Main results.**

Before getting to our final result, we will need the following analog of Proposition 3.2.1 for essentially hyperfinite Borel equivalence relations.

Proposition 4.2.1. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, F is hyperfinite, and $\phi \colon X \to Y$ is a smooth-to-one Borel homomorphism from E to F. Then E is essentially hyperfinite.*

Proof. Fix an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite Borel equivalence relations whose union is F, and define $E_n = E \cap (\phi \times \phi)^{-1}(F_n)$ for each $n \in \mathbb{N}$. Then E_0 has countable index below E, thus $\mathcal{I}_E \subseteq \mathcal{I}_{E_0}$ by Proposition 2.4.11. In particular, it follows that ϕ is a smooth-to-one homomorphism from E_0 to F_0, so Proposition 3.2.1 ensures that E_0 is smooth. Fix a Borel reduction $\pi \colon X \to 2^{\mathbb{N}}$ from E_0 to $\Delta(2^{\mathbb{N}})$. Then each of the equivalence relations $E_n' = \Delta(2^{\mathbb{N}}) \cup (\pi \times \pi)(E_n)$ is finite, so the equivalence relation $E' = \bigcup_{n \in \mathbb{N}} E'_n$ is hyperfinite, thus E is essentially hyperfinite. ⊠

At long last, we are now ready to establish our homogeneity result.

Theorem 4.2.2. *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then the following are equivalent:*

(1) *The equivalence relation E is essentially hyperfinite.*
(2) *The ideal $\mathcal{I}_E$ is reduction homogeneous.*
(3) *The ideal $\mathcal{I}_E$ is cohomomorphism homogeneous.*

Proof. To see (1) $\implies$ (2), note that if $B \subseteq X$ is an E-non-smooth Borel set, then Theorem 2.4.7 (or just the Harrington-Kechris-Louveau Theorem) yields a continuous embedding $\phi \colon 2^{\mathbb{N}} \to B$ of $\mathbb{E}_0$ into E. As Theorem 1.6.6 ensures the existence of a Borel reduction $\psi \colon X \to 2^{\mathbb{N}}$ of E to $\mathbb{E}_0$, it follows that the function $\pi = \phi \circ \psi$ is a Borel reduction of E to $E \restriction B$, so Proposition 3.2.2 implies that it is a reduction of $\mathcal{I}_E$ to $\mathcal{I}_E \restriction B$.

As (2) $\implies$ (3) is trivial, it only remains to show $\neg(1) \implies \neg(3)$. Towards this end, note that if E is not essentially hyperfinite, then Theorem 2.4.7 (or just the Harrington-Kechris-Louveau Theorem) yields an E-non-smooth Borel set $B \subseteq X$ on which E is hyperfinite. Suppose, towards a contradiction, that there is a Borel cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_E \restriction B$. Theorem 3.2.5 then yields a smooth-to-one Borel quasi-homomorphism $\phi \colon X \to B$ from E to $E \restriction B$. Let F denote the intersection of E with the pullback of $E \restriction B$ through ϕ. Theorem 2.4.12 then ensures that ϕ is a smooth-to-one Borel homomorphism from F to $E \restriction B$, so Proposition 4.2.1 implies that F is essentially hyperfinite, thus Proposition 4.1.3 yields that E is essentially hyperfinite, the desired contradiction. ⊠

While beyond the scope of this paper, we close by noting that Theorems 3.2.5 and 3.2.6 can also be used to obtain additional complexity results for smooth ideals. For instance, the arguments of [AK00, Gao02, LR05, HK05] can be adapted to show that every analytic quasi-order on a Polish space can be Borel embedded into the quasi-orders of smooth-to-one Borel quasi-homomorphism and Borel quasi-reducibility of countable Borel equivalence relations, and therefore into the quasi-orders of Borel cohomomorphism and Borel reducibility of smooth ideals associated with such relations, and local versions of these results can be obtained using the techniques of [CM14].

References

[AK00] S. Adams and A.S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), no. 4, 909–943 (electronic). MR 1775739 (2001g:03086)

[CCCM11] A.E. Caicedo, J.D. Clemens, C.T. Conley, and B.D. Miller, *Definability of small puncture sets*, Fund. Math. **215** (2011), no. 1, 39–51. MR 2851700

[CM14] C.T. Conley and B.D. Miller, *Definable cardinals just beyond* $\mathbb{R}/\mathbb{Q}$, preprint, 2014.

[DJK94] R. Dougherty, S. Jackson, and A.S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), no. 1, 193–225. MR 1149121 (94c:03066)

[Eff65] E.G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), 38–55. MR 0174987 (30 #5175)

[Gao02] S. Gao, *Some applications of the Adams-Kechris technique*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 863–874 (electronic). MR 1866043 (2002i:03053)

[Gli61] J. Glimm, *Type I C^*-algebras*, Ann. of Math. (2) **73** (1961), 572–612. MR 0124756 (23 #A2066)

[Har76] L.A. Harrington, *A powerless proof of Silver's theorem*, Unpublished notes, 1976.

[HK05] G. Hjorth and A.S. Kechris, *Rigidity theorems for actions of product groups and countable Borel equivalence relations*, Mem. Amer. Math. Soc. **177** (2005), no. 833, viii+109. MR 2155451 (2006f:03078)

[HKL90] L.A. Harrington, A.S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928. MR 1057041 (91h:28023)

[JKL02] S. Jackson, A.S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, J. Math. Log. **2** (2002), no. 1, 1–80. MR 1900547 (2003f:03066)

[Kec95] A.S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)

[LR05] A. Louveau and C. Rosendal, *Complete analytic equivalence relations*, Trans. Amer. Math. Soc. **357** (2005), no. 12, 4839–4866 (electronic). MR 2165390 (2006d:03081)

[Sil80] J.H. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Ann. Math. Logic **18** (1980), no. 1, 1–28. MR MR568914 (81d:03051)

[Sou17] M.Y. Souslin, *Sur une définition des ensembles mesurables B sans nombres transfinis*, C. R. Math. Acad. Sci. Paris **164** (1917), 88–91.

[SW82] Saharon Shelah and Benjamin Weiss, *Measurable recurrence and quasi-invariant measures*, Israel J. Math. **43** (1982), no. 2, 154–160. MR 689974 (84d:28025)

[Wei84] B. Weiss, *Measurable dynamics*, Conference in modern analysis and probability (New Haven, Conn., 1982), Contemp. Math., vol. 26, Amer. Math. Soc., Providence, RI, 1984, pp. 395–421. MR 737417 (85j:28027)

[Zap08] J. Zapletal, *Forcing idealized*, Cambridge Tracts in Mathematics, vol. 174, Cambridge University Press, Cambridge, 2008. MR 2391923 (2009b:03002)

John D. Clemens, Southern Illinois University, Mathematics Department, Neckers, 1245 Lincoln Drive, Mailstop 4408, Carbondale, IL 62901, USA

E-mail address: clemens@siu.edu

URL: http://www.math.siu.edu/faculty-staff/faculty/clemens.php

Clinton T. Conley, Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213, USA

E-mail address: clintonc@andrew.cmu.edu

URL: http://www.math.cmu.edu/math/faculty/Conley

Benjamin D. Miller, Kurt Gödel Research Center for Mathematical Logic, Universität Wien, Währinger Strasse 25, 1090 Wien, Austria, and Institut für mathematische Logik und Grundlagenforschung, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany

E-mail address: glimmeffros@gmail.com

URL: http://wwwmath.uni-muenster.de/u/ben.miller