Tautological classes of matroids

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Methods inspired from the algebraic geometry of realizable matroids has recently led to fruitful developments in matroid theory. We introduce a new framework that recovers, unifies, and extends these developments. For notations and conventions regarding matroids, we point to [11] as a standard reference.

Let M be a matroid of rank r on a nonempty ground set $E = \{0, 1, \ldots, n\}$. Let $T = (\mathbb{C}^*)^E$ be the algebraic torus with the standard action on $\mathbb{C}^E$. A realization (over $\mathbb{C}$) of M is an r-dimensional linear subspace $L \subseteq \mathbb{C}^E$ such that the set of bases of M equals the subcollection $\{B \in \binom{E}{r} \mid L \cap \bigcap_{i \in B} H_i = \{0\}\}$ of size r subsets of E. Here H_i denotes the i-th coordinate hyperplane of $\mathbb{C}^E$. Let X_E be the permutohedral variety of dimension n, which is obtained from $\mathbb{P}^n$ by sequentially blowing up from lower to higher dimensions (the strict transforms of) all coordinate subspaces of $\mathbb{P}^n$. It is a toric variety with the open torus $T/\mathbb{C}^*$, the quotient of T by the diagonal copy of $\mathbb{C}^*$.

Given a realization $L \subseteq \mathbb{C}^E$ of a matroid M, we define two T-equivariant vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$ on the permutohedral variety X_E as follows.

Definition 1. The **tautological subbundle** $\mathcal{S}_L$ (resp. the **tautological quotient bundle** $\mathcal{Q}_L$) is the unique torus-equivariant vector bundle whose fiber over a point $\bar{t}$ in the open torus $T/\mathbb{C}^*$ of X_E is $t^{-1}L$ (resp. $\mathbb{C}^E/t^{-1}L$).

The T-equivariant K-classes of $\mathcal{S}_L$ and $\mathcal{Q}_L$ depend only on the matroid M that L realizes, and one can thus define T-equivariant K-classes $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ on X_E for an arbitrary, not necessarily realizable matroid M. The Chern classes of these tautological classes recover previously studied geometric models of matroids:

- The first Chern class $c_1(\mathcal{Q}_M)$ equals the nef divisor class on X_E corresponding to the base polytope [8] of the matroid M.
- The top Chern class $c_{|E|-r}(\mathcal{Q}_M)$ equals the Bergman class Δ_M of the matroid M as studied in [12, 9], which coincides with the homology class in X_E of the wonderful compactification [5] of a realization L of M when M has a realization.
- The products of Chern classes $c_i(\mathcal{S}_M)c_{|E|-r}(\mathcal{Q}_M)$ for $0 \leq i \leq r$ equal the Chern-Schwartz-MacPherson (CSM) classes of a matroid M introduced in [10], and coincides with the CSM classes of the associated hyperplane arrangement complement when the matroid has a realization.

The permutohedral variety X_E resolves the rational Cremona map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Let α and β be divisor classes on X_E obtained as the pullbacks of the hyperplane class from each $\mathbb{P}^n$. We express the Tutte polynomial of a matroid, which is the universal deletion-contraction invariant of matroids, in terms of intersection multiplicities of α, β, and Chern classes of $\mathcal{S}_M$ and $\mathcal{Q}_M$.

Theorem 1. Let $\int_{X_E} : A^\bullet(X_E) \to \mathbb{Z}$ be the degree map on X_E, and $T_M(u,v)$ the Tutte polynomial of a rank r matroid M on ground set E. Then, one has

$$\sum_{i+j+k+\ell=n} \left(\int_{X_E} \alpha^i \beta^j c_k(\mathcal{S}_M^\vee) c_\ell(\mathcal{Q}_M) \right) x^i y^j z^k w^\ell = \\ (x+y)^{-1}(y+z)^r (x+w)^{|E|-r} T_M\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big).$$

We also establish a log-concavity property for the Tutte polynomial. For a homogeneous polynomial $f \in \mathbb{R}[x_1,\dots,x_N]$ of degree d with nonnegative coefficients, we say that its coefficients form a *log-concave unbroken array* if, for any $1 \le i < j \le N$ and a monomial $x^{\boldsymbol{m}}$ of degree $d' \le d$, the coefficients of $\{x_i^k x_j^{d-d'-k} x^{\boldsymbol{m}}\}_{0 \le k \le d-d'}$ in f form a log-concave sequence with no internal zeros.

Theorem 2. The coefficients of the polynomial

$$t_M(x,y,z,w) = (x+y)^{-1}(y+z)^r (x+w)^{|E|-r} T_M\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big)$$

form a log-concave unbroken array.

The two theorems together unify, recover, and extend several previous geometric interpretations for the Tutte polynomial and the log-concavity properties for the characteristic polynomial of a matroid, as given in [1, 3, 7, 9, 10, 2]. We prove Theorem 1 by using the method of localization in torus-equivariant geometry, and Theorem 2 by using methods from tropical Hodge theory. Previous geometric frameworks for studying matroids were disjoint in the sense that one could not easily use both of these two fundamental methods within one framework.

In order to use Theorem 1 to recover previous K-theoretic interpretations of the Tutte polynomial of a matroid, we develop an exceptional Hirzebruch-Riemann-Roch type formula for permutohedral varieties.

Theorem 3. There exists a ring isomorphism $\zeta_{X_E} : K_0(X_E) \xrightarrow{\sim} A^\bullet(X_E)$ which satisfies

$$\chi\big([\mathcal{E}]\big) = \int_{X_E} (1 + \alpha + \cdots + \alpha^n) \cdot \zeta_E([\mathcal{E}])$$

for any $[\mathcal{E}] \in K_0(X_E)$. Denote by $\bigwedge^i$ for the i-th exterior power and $c(\mathcal{E},u) := \sum_{i\ge 0} c_i(\mathcal{E}) u^i$ the Chern polynomial of $[\mathcal{E}]$. If $[\mathcal{E}]$ "has simple Chern roots" (which $\mathcal{S}_M^\vee$ and $\mathcal{Q}_M^\vee$ do) and rank $\mathrm{rk}(\mathcal{E})$, then we have

$$\sum_{i\ge 0} \zeta_{X_E}\big([\textstyle\bigwedge^i \mathcal{E}]\big) u^i = (u+1)^{\mathrm{rk}(\mathcal{E})} c(\mathcal{E}, \tfrac{u}{u+1}), \quad \text{and}$$
$$\sum_{i\ge 0} \zeta_{X_E}\big([\textstyle\bigwedge^i \mathcal{E}^\vee]\big) u^i = (u+1)^{\mathrm{rk}(\mathcal{E})} c(\mathcal{E},1)^{-1} c(\mathcal{E}, \tfrac{1}{u+1}).$$

The map ζ_E is not the Chern character map, and the Chow class $(1+\alpha+\cdots+\alpha^n)$ is not the Todd class of X_E. The proof of Theorem 3 is a purely algebraic. We use the localization methods for T-equivariant K-theory and T-equivariant Chow rings of toric varieties along with the Atiyah-Bott localization formula.

Question 1. Is there a geometric interpretation or a proof of this Hirzebruch-Riemann-Roch type formula (Theorem 3)?

Remark 1. One can show that the map ζ_E is the unique isomorphism that sends $[\mathcal{O}_{W_L}]$, the K-class of the structure sheaf of the wonderful compactification W_L associated to a realization L of a matroid, to the Chow class $[W_L]$ in X_E of W_L as a subvariety of X_E. However, this description of ζ_E makes it unclear why such isomorphism should even exist.

Theorem 3 also opens new questions about wonderful compactifications. For instance, it implies that for a realization $L \subseteq \mathbb{K}^E$ of a matroid M over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic, one has that the Euler characteristic of the line bundle $\det \mathcal{Q}_M$ pulled back to W_L satisfies

$$\chi(\det \mathcal{Q}_M; W_L) = |\mu(M)|,$$

where $\mu(M)$ is the constant coefficient of the characteristic polynomial of M. Separately, one can show by computation that also $h^0(\det \mathcal{Q}_M; W_L) = |\mu(M)|$.

Question 2. Is $H^i(\det \mathcal{Q}_M; W_L) = 0$ for all $i > 0$?

When the realization L is over a field of characteristic zero, one can show via Kawamata-Viehweg vanishing theorem that $H^i(\det \mathcal{Q}_M; W_L) = 0$ for all $i > 0$.

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