Lecture 9

Want: matroid invariants (e.g. #(indep. subsets of size k), or k-th unsigned coeff. of $\chi_M(q)$) as intersection degrees of nef divisor classes on some proj. variety.

Recall: $L = \text{rowspan}\ r\begin{array}{c} E \\ \left[\quad A \quad\right] \end{array} \subseteq k^E$ realizing a (simple) matroid M.

$\rightsquigarrow$ $L_i = L \cap H_i$, $H_i = \{x_i = 0\}$ in k^E for $i \in E$
$L_F = L \cap \bigcap_{i \in F} H_i$ for F a flat.
$\Big\} \longrightarrow$ lattice of flats of M $=$ intersection lattice of $\{L_i\}$.

Goal now: How to compactify L? Or $\mathring{L} = L \setminus (\bigcup_{i\in E} L_i)$? Or $\mathbb{P}\mathring{L} = \mathbb{P}L \cap (k^*)^E/k^*$?

<u>Strategy</u>: Take closure of $\mathbb{P}\mathring{L}$ in a cpt $X \supset (k^*)^E/k^*$

Would like: ① X & $\overline{\mathbb{P}\mathring{L}}$ in X both smth (with simple normal crossing bndry)

② The (co)homology class of $\overline{\mathbb{P}\mathring{L}}$ in $H^\bullet(X)$ shld determine M.

③ X <u>resolves</u> the Cremona map: $\mathbb{P}^n \xrightarrow{\text{crem}} \mathbb{P}^n$ $[x_i]_i \mapsto [\frac{1}{x_i}]_i$.

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^n \end{tikzcd}$$

π_1, π_2 birat'l map (isom. on an open subset) st $\pi_2 \circ \pi_1^{-1} = \varphi$

<u>E.g.</u> $X = \Gamma_\varphi =$ graph of φ $= \overline{\{(x, \varphi(x)) \in \mathbb{P}^n \times \mathbb{P}^n\}}$

$\deg \varphi = \pi_2^* h^n \cap [\Gamma_\varphi]$.

[Huh-Katz '12][Proudfoot-Speyer '06][Terao '02][Speyer '09][Hacking-Keel-Tevelev '06]

<u>Prop</u>☆ The degree of the <u>reciprocal linear space</u> $\mathbb{P}L^{-1} := \overline{\text{crem}(\mathbb{P}\mathring{L})} \subseteq \mathbb{P}^n$ is $|\chi_M(0)|$.

Equivalently, if $Y = (\text{graph of crem}|_{\mathbb{P}L}) \subseteq \mathbb{P}^n \times \mathbb{P}^n$, then

$$\int_{\mathbb{P}^n \times \mathbb{P}^n} \eta_Y \cdot \eta_{\mathbb{P}^n \times H}^{\dim \mathbb{P}L} = |\chi_M(0)|.$$

<u>N.B.</u> If $f: X' \longrightarrow X$ birat., then $\int_X \eta_{Z_1} \cdots \eta_{Z_k} = \int_X f^*\eta_{Z_1} \cdots f^*\eta_{Z_k}$
where $f^*\eta_Z = \eta_{f^{-1}(Z)}$ in good situations. (e.g. $f^{-1}(Z)$ of right codim & gen. red or Z is CM).

<u>Defn</u> Let $M = (E, \mathcal{B})$ of rk >0. The <u>free extension</u> $M+e$ is a matroid on $E \sqcup \{e\}$ defined by $\mathcal{B}(M+e) = \mathcal{B}(M) \cup \{ I \cup e \mid |I| = rk(M) - 1 \text{ and indep. in } M\}$.
The <u>truncation</u> $Tr(M)$ of M is $(M+e)/e$, i.e. $\mathcal{B}(TrM) = \{ I \mid |I| = rk(M)-1, \text{indep}\}$.

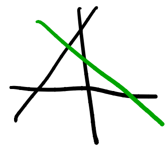
E.g. If vectors $v_0,\dots,v_n$ spanning $L^\vee$ realize M, then $v_0,\dots,v_n,v_e$ realize $M+e$ iff v_e not in any $\mathrm{span}(v_i \mid i\in F)$ for F a proper flat of M.

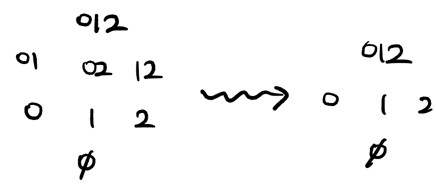
Under $L^\vee \twoheadrightarrow L^\vee/\mathrm{span}(v_e)$, the images $\overline{v_0},\dots,\overline{v_n}$ realize $\mathrm{Tr}(M)$.

Dually: If arr. compl. $\mathbb{P}L^\circ$ realizes M, then $M+e \longleftrightarrow \mathbb{P}L^\circ \setminus H_{\text{general}}$

$\mathrm{Tr}(M) \longleftrightarrow \mathbb{P}L^\circ \cap H_{\text{general}}$

$U_{3,3} \rightsquigarrow U_{3,4} \rightsquigarrow U_{2,3}$





Exer $\mathcal{F}(\mathrm{Tr}(M)) = \mathcal{F}(M)$ with corank 1 layer removed.

$$\Rightarrow \left(\overline{\chi}_M(q) - \overline{\chi}_M(0)\right)/q = \overline{\chi}_{\mathrm{Tr}(M)}(q)$$

Prop $\overline{\chi}_M(0) = \overline{\chi}_{M+e}(1)$ $\left(= \chi_{top}(\mathbb{P}L^\circ \setminus H_{gen.}) \text{ if } \mathbb{P}L^\circ \text{ realizes } M\right)$.

pf) $M+e\setminus e = M$, so $\overline{\chi}_M - \overline{\chi}_{\mathrm{Tr}(M)} = \overline{\chi}_{M+e}$.

Thm [Dimca-Papadima '03] Let $f\in\mathbb{C}[x_0,\dots,x_n]$ homog., and $U_f = \mathbb{P}^n_{\mathbb{C}} \setminus \{f=0\}$. Then for a general hyperplane $H\subset\mathbb{P}^n$, $\deg(\mathrm{grad}(f)) = (-1)^n \chi_{top}(U_f\setminus H)$.
Here, $\mathrm{grad}(f): \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $x \longmapsto [\frac{\partial f}{\partial x_0}(x),\dots,\frac{\partial f}{\partial x_n}(x)]$.

pf) Let $F_1 = \{f=1\}\subset\mathbb{C}^{n+1}$ (smth), and $d=\deg(f)$. Under $\mathbb{C}^{n+1}\setminus\{0\}\longrightarrow\mathbb{P}^n_{\mathbb{C}}$, $F_1\longrightarrow U_f$ and $F_1\cap\hat{H}\longrightarrow U_f\cap H$ are d-sheeted covering maps.
Reduce to showing at $\deg(\mathrm{grad}(f)) = (-1)^n \frac{1}{d}\chi_{top}(F_1\setminus\hat{H})$.
Apply polar curves results from "cplx Morse thry" / "Picard-Lefschetz":

$$\#(\text{tangent hyperplanes to } F_1 \mathbin{/\!/} \text{ to a fixed general } H)$$
$$= \#(n\text{-cells to attach to } F_1\cap\hat{H} \text{ to get homotopy equiv. to } F_1)$$
$$= (-1)^n\left(\chi_{top}(F_1) - \chi_{top}(F_1\cap\hat{H})\right) = (-1)^n\chi_{top}(F_1\setminus\hat{H}).$$

pf of Prop*) For $\mathbb{C}^r\simeq L\subseteq\mathbb{C}^E$ realizing a loopless matroid M, have

$$\begin{tikzcd} \mathbb{P}^{r-1} \arrow[r, dashed, "\mathrm{grad}(\ell)"] \arrow[d, hook, "\iota"'] & \mathbb{P}^{r-1} \\ \mathbb{P}^n \arrow[r, dashed, "\mathrm{grad}(x_0\cdots x_n)"] & \mathbb{P}^n \arrow[u, dashed, "\iota^\vee"'] \end{tikzcd}$$

where ℓ is $x_0x_1\cdots x_n$ restricted to $\mathbb{P}L$, so $\{\ell=0\} = \bigcup_{i\in E} L_i$, and $\mathrm{crem} = \mathrm{grad}(x_0\cdots x_n)$, ι is the linear inclusion, and $\iota^\vee$ the dual projection.

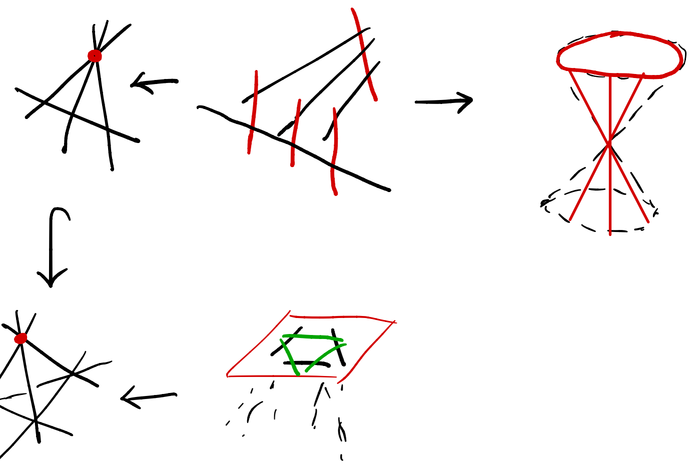
Applying Thm now to $\overline{\chi}_{M+e}(1) = \chi_M(0)$ yields Prop*.

Defn The <u>wonderful compactification</u> (w/r/t maximal building set) of $\mathbb{P}L^\circ$ is a proj. smth variety W_L of dim. $r-1$ obtained from $\mathbb{P}L$ by:

blowing up all $\mathbb{P}L_F$ for all rank $r-1$ flats, then
blowing up all $\widetilde{\mathbb{P}L_F}$ for all rank $r-2$ flat, and so forth.

E.g. $M = U_{1,1} \oplus U_{2,3}$

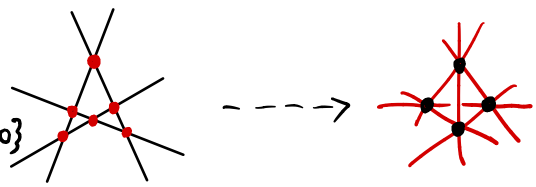
$\mathbb{P}L = \{x_1 + x_2 + x_3 = 0\}$
$\subset \mathbb{P}^3$



$\{x_1x_2 + x_1x_3 + x_2x_3 = 0\}$
projective cone over conic curve

$M = U_{3,4}$

$\mathbb{P}L = \{x_0 + \cdots + x_3 = 0\}$
$\subset \mathbb{P}^3$



$\{x_0x_1x_2 + \cdots + x_1x_2x_3 = 0\}$
Cayley nodal cubic
(has 9 lines)