Lecture 7

Let $A^\bullet = \bigoplus_{i=0}^{n} A^i$ be a graded $\mathbb{R}$-algebra with Poincare duality,

i.e. $\exists \int : A^n \xrightarrow{\sim} \mathbb{R}$ st $A^i \times A^{n-i} \longrightarrow A^n \xrightarrow{\sim} \mathbb{R}$ is non-degen. $\forall\, 0 \le i \le n$.

E.g. $A^\bullet = H^{2\bullet}(X)_{\mathbb{R}}$, $\quad A^\bullet = H^{\bullet,\bullet}(X) = \bigoplus_{i=0}^{n} H^{i,i}(X)$, $\quad A^\bullet = A^\bullet(X)_{\mathbb{R}}$ (if X "nice").

Defn For $(A^\bullet, \int)$ with $n \ge 2$, and a subset $K \subset A^1$, say that $(A^\bullet, \int, K)$ satisfies mixed Hodge–Riemann relations in deg ≤ 1 ($HR^{\le 1}$) if:

(HR^0) $\displaystyle\underset{\ell_1 \cdots \ell_n}{\int \ell^n} > 0 \quad \underset{\ell_1, \dots, \ell_n}{\forall \ell \in K}$

(HR^1) The symmetric pairing $A^1 \times A^1 \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \displaystyle\int x y \underset{\ell_1 \cdots \ell_{n-2}}{\ell^{n-2}}$
has signature $(+,-,-,\dots,-)$ $\underset{\ell_1, \dots, \ell_{n-2}}{\forall \ell \in K}$.

Lem (Cauchy interlacing thm). Let A be a $n\times n$ real symmetric matrix, and B an $(n-c)\times(n-c)$ principal submatrix. Then $\lambda_i(A) \le \lambda_i(B) \le \lambda_{i+c}(A)$ $\forall\, i = 1, \dots, n-c$.

$$\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \cdots \quad \lambda_{n-1} \quad \lambda_n$$

Thm $(A^\bullet = H^{\bullet,\bullet}(X), \int_X, K(X)_{\mathbb{Q}})$ satisfies mixed $HR^{\le 1}$ for X smth proj. $\mathbb{C}$-var of dim $= n$.

pf) May assume very ample. Bertini $\Rightarrow$ mixed HR^0, and $\exists Y$ smth proj. surface with $\eta_Y = \eta_X\, \ell_1 \cdots \ell_{n-2}$. The pairing $A^1(X) \times A^1(X) \longrightarrow \mathbb{R}$ is the restriction of the Poincare pairing $A^1(Y) \times A^1(Y) \longrightarrow \mathbb{R}$ to the image of $A^1(X)$ under $A^\bullet(X) \xrightarrow{i^*} A^\bullet(Y)$. Now apply weak Lefschetz along with Cauchy interlacing to the Hodge index thm for surfaces

Thm (Weak Lefschetz) $X \subseteq \mathbb{P}^N_{\mathbb{C}}$. Then $H^i(X) \xrightarrow{i^*} H^i(X \cap H)$ is isom. for $i \le n-2$
injec. for $i \le n-1$.

Prop $(A^\bullet, \int, \mathcal{K})$ has mixed $HR^{\le 1}$ $\Rightarrow$ for any $\alpha, \beta \in \overline{\mathcal{K}}$, the sequence $(\alpha^{n-i} \cdot \beta^i)_i$ is log-concave with no internal zeroes.

Defn For $\partial_1, \dots, \partial_m \in A^1$ in a Poincare duality algebra $(A^\bullet, \int)$, the <u>volume polynomial</u> w/r/t $\partial_1, \dots, \partial_m$ is $VP(w_1, \dots, w_m) := \int (w_1 D_1 + \dots + w_m D_m)^n$

Prop (Macaulay inverse system) There is a bijection

$$\left\{ \begin{array}{c} \text{homogeneous polyom.} \\ f \in \mathbb{R}[w_1, \dots, w_m] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Poincare duality } \mathbb{R}\text{-algebras that} \\ \text{are quotients of } \mathbb{R}[\partial_1, \dots, \partial_m] \end{array} \right\}$$

Lorentzian $\qquad$ with mixed $HR^{\le 1}$ for $K_f = \mathbb{R}_{\ge 0}\{\partial_1, \dots, \partial_m\}$

$$f \text{ of deg } d \longmapsto A_f^\bullet = \frac{\mathbb{R}[\partial_1, \dots, \partial_m]}{\langle g \in \mathbb{R}[\partial_i\text{'s}] \mid g \cdot f = 0 \rangle} \quad (\partial_i = \frac{\partial}{\partial w_i})$$

$$\int g = g \cdot \frac{f}{d!} \quad (\deg g = d)$$

$$VP \text{ w/r/t } \partial_1, \dots, \partial_m \longleftarrow\!\!\shortmid$$

Prop If $(A^\bullet, \int, K)$ has HR^1 and mixed HR^0, then VP w/r/t to $\partial_1, \dots, \partial_m \in \overline{K}$ as a real fct $\mathbb{R}^m \to \mathbb{R}$ is log-concave on $\mathbb{R}^m_{>0}$ (i.e. the Hessian of $\log VP$ is negative semidef. on $\mathbb{R}^m_{>0}$), or identically zero.

Exer f Lorentzian $\Rightarrow$ $(A_f^\bullet, \int, K_f)$ has $HR^{\le 1}$ $\Rightarrow$ f log-conc. on pos. orth.
But each $\Rightarrow$ is strict (converse fails).

E.g. $A^\bullet = \dfrac{\mathbb{R}[\partial_x, \partial_y]}{\langle \partial_y^2, \partial_x^3 - \partial_x^2 \partial_y \rangle} \longleftrightarrow VP = x^3 + 3x^2 y$