Lecture 6

Let X be smth complete $\mathbb{C}$-variety (irred.) of $\dim = n$ ($2n$-dim'l real mfd)

(1) $H_{2d}(X) = \mathbb{Z}\cdot[X]$, $[X] =$ <u>fundamental class</u> of X.

(2) Poincaré duality: $H^k(X) \xrightarrow{\sim} H_{2n-k}(X)$, $\eta \mapsto \eta \cap [X]$.

(3) $Y \overset{\iota}{\hookrightarrow} X$ subvar. of $\operatorname{codim}_X Y = c$, then

$$\begin{array}{ccccc} H_{2n-2c}(Y) & \xrightarrow{\iota_*} & H_{2n-2c}(X) & \xrightarrow{\sim} & H^{2c}(X) \\ [Y] & \longmapsto & \iota_*[Y] & \longmapsto & \eta_Y \end{array} \quad (\eta_Y \cap [X] = \iota_*[Y])$$

(lands in $H^{c,c}(X)$ actually)

<u>Notation</u> Let $A^\bullet(X) =$ subring of $H^{2\bullet}(X)$ gen. by $\{\eta_Y \mid Y \subseteq X \text{ subvar.}\}$.

<u>Rem</u> In this course, we call $A^\bullet(X)$ the "Chow ring of X."
But this actually a quotient of the genuine Chow ring of X.
(Somewhat forgivable since for most varieties we will see, they coincide).
[Totaro '14: "Linear varieties"]

<u>N.B.</u> (4) $\eta_{Y_1}\cdot\eta_{Y_2} = \eta_{Y_1\cap Y_2}$ if $Y_1 \cap Y_2$ (generically) transversally.

(5) The <u>degree map</u> $\int_X : A^n(X) \xrightarrow{\sim} H_0(X) \simeq \mathbb{Z}\cdot[pt]$

<u>E.g.</u> $X = \mathbb{P}^n_{\mathbb{C}}$. Coho. ring $= A^\bullet(X) = \mathbb{Z}[h]/h^{n+1}$, $h = \eta_H$ $H \subset \mathbb{P}^n_{\mathbb{C}}$ hyperplane

$Y \subset \mathbb{P}^n_{\mathbb{C}}$ subvar. of $\deg Y =$ deg. of f if $Y = V(f)$ a hypersurface
$=$ leading coeff./$(\dim Y)!$ of Hilb. polynom. of Y
$= \#(Y \cap L)$ for L general linear subvar. of complementary dim.
↳ counted w/ mult.

$\eta_Y = (\deg Y)\, h^{\operatorname{codim} Y} \Leftarrow$ $\quad \eta_L = h^{\dim Y}$

☆(6) $$\deg Y = \int_X \eta_Y \cdot h^{\dim Y} > 0.$$
$$\parallel$$
$$(\eta_Y \cdot h^{\dim Y})$$

"Codim 1 classes"

(7) (Weil) divisor grp $\{D = \sum_i a_i Y_i \mid a_i \in \mathbb{Z},\ Y_i \subset X \text{ codim } 1 \text{ subvar.}\}$

$$0 \longrightarrow \underset{\{\operatorname{div}(f) \mid f \in K(X)\}}{\underset{\|}{\operatorname{Prin}(X)}} \longrightarrow \overset{\|}{\operatorname{Div}(X)} \longrightarrow \overset{=\{\text{line bdls}\}/\text{isom}\ (\mathcal{L}\otimes\mathcal{L}')}{\operatorname{Pic}(X)} \longrightarrow 0 ,\qquad \begin{array}{ccc} \operatorname{Pic}(X) & \xrightarrow{c_1} & H^2(X) \\ \mathcal{O}_X(D) & \longmapsto & \eta_D \end{array}$$

$$D \longmapsto \mathcal{O}_X(D)|_U = \mathcal{O}_X \cdot f_U^{-1} \quad \text{where } D|_U = V(f_U).$$

$\underline{\operatorname{supp}}(D) := \bigcup_{a_i \neq 0} Y_i$. $\qquad D \sim D'$ if $D - D' \in \operatorname{Prin}(X)$ (linearly equiv.)

$$\Downarrow$$
$$\eta_D = \eta_{D'}$$

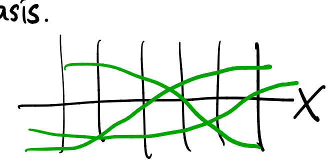
(8) $|D| := \mathbb{P}H^0(X, \mathcal{O}_X(D)) = \{D' \sim D \mid D' \text{ effective } (a_i \geq 0\ \forall Y_i)\}$

$$s \longmapsto V(s) := \{s = 0\}$$

D base-point-free (bpf) if $\forall x \in X\ \exists D' \in |D|$ st $x \notin D'$!

$\rightsquigarrow$ choose $s_0, \ldots, s_n$ spanning $H^0(D)$, often a basis.

$\rightsquigarrow X \xrightarrow{\varphi_D} \mathbb{P}^n \quad \varphi^{-1}(H_i) = V(s_i)$



D very ample if $\varphi_D : X \hookrightarrow \mathbb{P}^n$ closed embedding

ample if mD very ample $\exists m \gg 0$

Exer D bpf & D' ample $\Rightarrow$ $D + D'$ ample.

(9) $f : X \longrightarrow X'$ morphism, $\operatorname{Div}(X') = \sum_i a_i Y_i'$. Then

$f^* \mathcal{O}_{X'}(D') = \mathcal{O}_X\left(\sum_i a_i f^{-1}(Y_i')\right)$ if $f^{-1}(Y_i')$ has codim 1 & is generically reduced $\forall a_i \neq 0$.

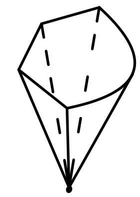
E.g. $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \simeq \mathbb{C}[x_0, \ldots, x_n]_d$. $X \xrightarrow{\varphi_D} \mathbb{P}^n \rightsquigarrow \varphi^* h = \eta_D$ (often written $h|_X$)

$|dH| = \{\deg d \text{ hypersurfaces}\}$

Thm (Bertini) $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ smth proj. var. Then for general $H \subset \mathbb{P}^n$, $X \cap H$ is smth (and irred. if $\dim X > 1$).

Thm (weak Lefschetz) $H^k(X) \xrightarrow{\iota^*} H^k(X \cap H)$ is isom. for $k \leq n-2$, injec. for $k = n-1$.

Let $\mathcal{K}(X) = \{\eta_D \mid D \text{ ample on } X\}$.



Let $\mathcal{K}(X)_{\mathbb{R}}$ = convex hull of $\mathcal{K}(X)$ in $A^1(X)_{\mathbb{R}} = A^1(X) \otimes \mathbb{R}$.

Exer above $\Rightarrow$ If η a $\mathbb{Q}$-element of $\mathcal{K}(X)$, $m\eta$ ample $\exists m \in \mathbb{Z}_{>0}$.

<u>Defn</u> $\mathcal{K}(X)_{\mathbb{R}}$ is called the <u>ample cone</u> of X. (It is open in $A^1(X)_{\mathbb{R}}$)

Let $\mathrm{Nef}(X) :=$ closure of $\mathcal{K}(X)_{\mathbb{R}}$.

$\quad\quad \eta \in \mathrm{Nef}(X)$ a ($\mathbb{R}$-)nef divisor class. <u>E.g.</u> b.p.f. $\Rightarrow$ nef

(6′) $\eta \in \mathcal{K}(X)_{\mathbb{R}} \Rightarrow \eta^{\dim Y} \cap [Y] > 0 \quad \forall\, Y \subseteq X$ subvar.

<u>Thm</u> (Nakai–Moishezon–Kleiman) $\Leftarrow$ holds in above.

<u>Thm</u> (Khovanskii–Teissier) If $\alpha, \beta \in A^1(X)_{\mathbb{R}}$ nef on smth proj. var. X of $\dim = n$, then

$$\left(\int_X \alpha^n,\ \int_X \alpha^{n-1}\beta,\ \cdots,\ \int_X \beta^n \right) \text{ form a log-concave sequence.}$$

pf) May assume α, β ample, in fact very ample. For any $1 \leq i \leq n-1$, need $(\alpha^{n-i+1} \cdot \beta^{i-1},\ \alpha^{n-i} \cdot \beta^{i},\ \alpha^{n-i-1} \cdot \beta^{i+1})$ log-conc.

By Bertini, $\exists Y \subseteq X$ smth var. st $\eta_Y = \alpha^{n-i-1}\beta^{i-1}$, so reduce to $\dim X = 2$.

<u>Thm</u> (Hodge index thm) If X smth proj. surface, H ample, and $D.H = 0$. Then $D^2 \leq 0$.

pf) $H' = D + kH$ ample for $k \gg 0$. If $D^2 > 0$, then $D.H' > 0$, which implies mD effective for $m \gg 0$ by RR. But then $mD.H > 0$. ※

$\Rightarrow \begin{bmatrix} \alpha^2 & \alpha \cdot \beta \\ \alpha \cdot \beta & \beta^2 \end{bmatrix}$ cannot have signature $(+,+)$ (nor $(-,-)$ since $\alpha^2 > 0$).

Scholium: log-conc. seq. <u>with no internal zeroes</u> ($\forall i<j<k$, $a_i a_k \neq 0 \Rightarrow a_j \neq 0$).

Goal now: Extract the combinatorial & linear algebraic essence of this ineq.

$\rightsquigarrow$ Lorentzian polynomials [Brändén–Huh '21]

[Anari–(Liu)–Oveis Gharan–Vinzant '18]