Lecture 4

Recall: $\mathcal{B}$, $\mathcal{I}$, $\mathcal{C}$, rk_M, $\mathcal{F}$

⚠ Let M be realized by identifying E with a set of vectors. Then
$\mathrm{span}(F_1) \cap \mathrm{span}(F_2) \supsetneq \mathrm{span}(F_1 \cap F_2)$ (equality when ?)

<u>Defn</u> The <u>closure</u> of $S \subseteq E$ in M, denoted $cl_M(S)$ or $\overline{S}$, is the min'l flat in M containing S.

<u>N.B.</u> coloop : in every basis of M ($\leftrightarrow$ bridge in a graph).
cocircuit: the complement of a hyperplane of M
coindependent: complement of spanning

<u>Prop</u> $rk_{M^\perp}(S) = rk_M(E \setminus S) + |S| - r$

Equivalently, denoting $\mathrm{corank}_M(S) := \mathrm{rank}(M) - rk_M(S)$, $\mathrm{nullity}_M(S) := |S| - rk_M(S)$,
have $\mathrm{null}_{M^\perp}(E \setminus S) = \mathrm{corank}_M(S)$

<u>Defn</u> Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ be matroids on disjoint E_1 and E_2.
Then the <u>direct sum</u> $M_1 \oplus M_2$ is a matroid on $E_1 \cup E_2$ with bases $\{ B_1 \cup B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2 \}$.

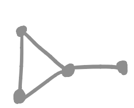
<u>E.g.</u> loops & coloops are direct summands.

<u>Prop</u> (1) $M_1 = (E_1, \mathcal{F}_1)$, $M_2 = (E_2, \mathcal{F}_2)$. Then $\mathcal{F}(M_1 \oplus M_2) \simeq \mathcal{F}_1 \times \mathcal{F}_2$.

(2) M is connected (i.e. not a nontrivial direct sum) $\Leftrightarrow$ $\forall i,j \in E\ \exists C \in \mathcal{C}$ st $i,j \in C$.
In fact, ($i \sim j$ if ij in a circuit) is an equivalence relation [Oxley '11, Ch. 4]

⚠ For a graph G, components(G) as a graph can be a strict coarsening of components$(M(G))$.

<u>E.g.</u>



<u>Rem</u> G is 2-conn. $\Longleftrightarrow$ $M(G)$ is conn.

N.B. The chromatic polynomial $\chi_G(q)$ satisfies (defining) property:

$$\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q).$$

Defn For matroid M on E, and $e\in E$, define matroids $M\setminus e$ and M/e on ground set $E\setminus e$ by:

$$rk_{M\setminus e}(\cdot) := rk_M(\cdot) \quad \text{and} \quad rk_{M/e}(\cdot) = rk_M(\cdot \cup e) - rk_M(e).$$

Equivalently, $\mathcal{B}(M\setminus e) = \{B \mid B \not\ni e\}$ if e not a coloop $\quad (rk(M\setminus e) = rk(M))$

$\mathcal{B}(M/e) = \{B\setminus e \mid B \ni e\}$ if e not a loop $\quad (rk(M/e) = rk(M) - 1)$

($M/e = M\setminus e$ when e a loop or a coloop).

Defn The characteristic polynomial $\chi_M(q)$ of a matroid M defined by:

(1) $\chi_M = \chi_{M\setminus e} - \chi_{M/e}$ if e neither loop nor coloop,

(2) $\chi_M = 0$ if M has a loop,

(3) $\chi_M = (q-1)\chi_{M/e}$ if e a coloop, and

(4) $\chi_{U_{1,1}} = q-1$ (or equiv. $\chi_{U_{0,0}} = 1$).

Exer $\chi_G(q) = q^{\#\mathrm{comp}(G)} \chi_M(q)$.

Thm Suppose $L \subseteq \mathbb{C}^E$ realizes M, and recall $\mathring{L} = L \cap (\mathbb{C}^*)^E$. Then,

$$(-q)^r \chi_M\left(-\frac{1}{q}\right) = P_{\mathring{L}}(q) := \sum_{i\geq 0} \operatorname{rank} H^i(\mathring{L}, \mathbb{Z})\, q^i.$$

(May replace $\mathbb{C}$ by any algebraically closed field k, and consider $H^i_{et}(\mathring{L}, \mathbb{Z}_\ell)$, $\ell \neq \mathrm{char} k$)

pf drawn from [Kim '94]) If M has a loop, $\mathring{L} = \phi$, so both zero. Assume loopless now.

① For $e \in E$, let $\mathring{L}_{\setminus e} := L \setminus (\bigcup_{i\neq e} L_i)$ and $\mathring{L}_{/e} := L_e \setminus (\bigcup_{i\neq e} L_i)$.

Note that $\mathring{L}_{\setminus e} / \bigcap_{i\neq e} L_i$ realizes $M\setminus e$, and $\mathring{L}_{/e}$ realizes M/e.

② Purity/Gysin seq. on $\mathring{L}_{/e} \xhookrightarrow[\text{closed}]{} \mathring{L}_{\setminus e} \xhookleftarrow[\text{open}]{} \mathring{L}$:

$$\cdots \longrightarrow H^i_{\mathring{L}_{/e}}(\mathring{L}_{\setminus e}) \xrightarrow{0} H^i(\mathring{L}_{\setminus e}) \longrightarrow H^i(\mathring{L}) \longrightarrow H^{i+1}_{\mathring{L}_{/e}}(\mathring{L}_{\setminus e}) \xrightarrow{0} \cdots$$

$H^i_{\mathring{L}_{/e}}(\mathring{L}_{\setminus e}) \cong H^{i-2}(\mathring{L}_{/e})(-1)$ (Thom isom.)

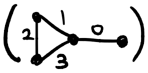
by noting the weights $W_\ell H^i$

Proj. Topological / Igusa / Motivic zeta functions of hyperplane arrangements.

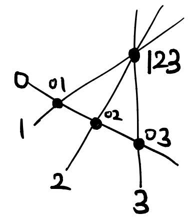
Defn For P a finite poset, the Möbius fct $\mu_P : P \times P \longrightarrow \mathbb{Z}$ is defined by

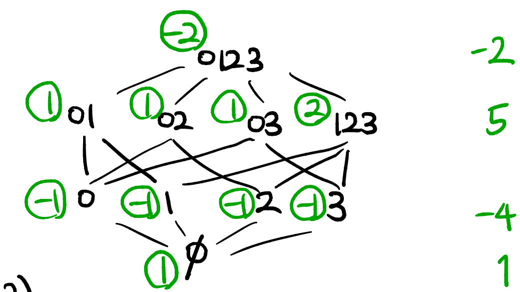
(1) $\displaystyle\sum_{x \le z \le y} \mu(x,z) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \le y \end{cases}$

(2) $\mu(x,y) = 0$ if $x \not\le y$

E.g. Lattice of flats $\mathcal{F}$ of M(

) :





$\mu(\emptyset, F)$

$\chi_G(q) = q(q-1)^2(q-1) = q(q^3 - 4q^2 + 5q - 2)$