Lecture 24

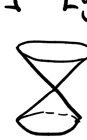
Variations on the theme: stellahedral case.

Let the ground set be $[n] = \{1, \cdots, n\}$ now; M of rank r on $[n]$.

For a realization $L \subseteq \mathbb{C}^n$ of M, consider $\overline{L} \subseteq (\mathbb{P}^1)^n$.
$\overline{L}$ is called the <u>(matroid) Schubert variety</u>.

<u>E.g.</u> $L = V(x_1 + x_2 + x_3) \subset \mathbb{C}^3$, $M = U_{2,3}$.

$\overline{L} = V(x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3) \subset (\mathbb{P}^1)^3 = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \times \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \right\}$

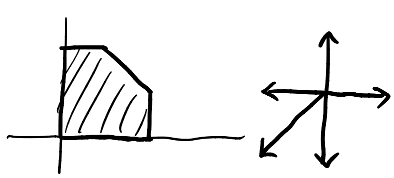
$\overline{L}$ around $(0,0,0) = L$, $\overline{L}$ around (∞, ∞, ∞):  (singular)

<u>Prop</u> $\eta_{\overline{L}} \in H^\bullet((\mathbb{P}^1_{\mathbb{C}})^n) = \mathbb{Z}[h_1, \cdots, h_n] / \langle h_1^2, \cdots, h_n^2 \rangle$ is $\sum_{B \in \mathcal{B}(M)} \prod_{i \in [n] \setminus B} h_i$.
(follows from [Ardila–Boocher '16]).

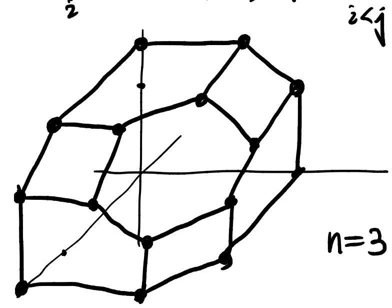
Equiv.: $[\overline{L}]$ as a MW on $(\Sigma_1)^n$ is const. wt 1 on $\bigcup_{B \in \mathcal{B}} \mathbb{R}^B$.

<u>Defn</u> The stellahedron $St_n = \sum_i \mathrm{Conv}(0, e_i) + \sum_{i<j} \mathrm{Conv}(0, e_i, e_j)$

<u>E.g.</u>



$n=2$



$n=3$

Note that: $\mathrm{star}(\Sigma_{St_n}, -e_{[n]}) \simeq \Sigma_{A_{n-1}}$, and $\mathbb{R}^n_{\geq 0}$ is a cone of Σ_{St_n}.

<u>Thm</u> ① X_{St_n} is a $(\mathbb{C}^n, +)$-eqv. compactification of $\mathbb{C}^n$ (as well as being the toric var. compactifying $(\mathbb{C}^*)^n$) such that $\overline{L}$ in X_{St_n} is a smth $(L, +)$-eqv. compactification of L.
[EHL]

② $[\overline{L}]$ as MW on Σ_{St_n} is the <u>augmented Bergman class</u> Δ_M.
$(\mathrm{trop}((L+b) \cap (\mathbb{C}^*)^n) = \Delta_M \longleftrightarrow$ usual Bergman class of free coext. of $M)$.

Let $Z_{r,n} = \left\{ \sum_i a_i M_i \;\middle|\; a_i \in \mathbb{Z},\ M_i \text{ matroid of rk } r \text{ on } E \right\}$

Thm [EHL] Following three equiv. notions coincide:

(Val) $\sum_i a_i M_i \sim_{\text{val}} 0$ if $\sum_i a_i 1_{P(M_i)} = 0$, where $1_{P(M_i)} : \mathbb{R}^n \to \mathbb{Z}$, $x \mapsto \begin{cases} 1 & x \in P(M_i) \\ 0 & \text{else} \end{cases}$

(Hom) $\sum_i a_i M_i \sim_{\text{hom}} 0$ if $\sum_i a_i \Delta_{M_i} = 0$

(Num) Let $\langle M, M' \rangle := \begin{cases} 1 & \text{if } M \,\&\, M' \text{ has a disjoint bases} \\ 0 & \text{else} \end{cases}$

$\sum_i a_i M_i \sim_{\text{num}} 0$ if $\langle \sum_i a_i M_i, - \rangle = 0$ on $Z_{n-r,n}$

Cor $\bigoplus_{\bullet=0}^{n} Z_{\bullet,n} / \sim_{\text{val/hom/num}} \; \simeq \; A^\bullet(X_{stn})$ **Rem** # positroids

Thm $T_M(1+q, q^{-1})$ log-conc. (cf. [Postnikov-Shapiro-Shapiro])

$\boxed{\text{KEY}}$

$$\begin{tikzcd} \mathbb{C}^{2n} \arrow[d, dashed] \arrow[dr, dashed, "{[A^\perp \mid A^\perp]}"] & & \\ \boxplus \mathcal{O}(1) \arrow[r] \arrow[d, dash] & Q_L \arrow[r] & 0 \\ X_{stn} & & \end{tikzcd}$$

&

$$\begin{tikzcd} K(X_{stn}) \arrow[r, "\phi"] \arrow[d, "\chi"'] & A^\bullet(X_{stn}) \arrow[d, "\int(-)\cdot c(\boxplus \mathcal{O}(1))"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$