Lecture 21

Thm [Rosu, Knutson '03][Vezzosi–Vistoli '03][Nielsen '74] Let Σ be a smth proj. fan ($\mathrm{lin} \neq 0$ ok)

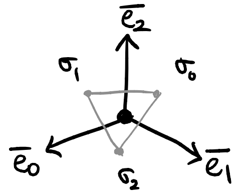
Then $K_T(X_\Sigma) \hookrightarrow \prod_{\sigma \in \Sigma_{max}} K_T(p_\sigma)$ whose image is $(f_\sigma)_{\sigma \in \Sigma_{max}}$ st

$$[\mathcal{F}]^T \longmapsto [\mathcal{F}]^T_{p_\sigma} \qquad\qquad f_{\sigma_1} - f_{\sigma_2} \equiv 0 \mod (1 - \chi^{m_{\sigma_1,\sigma_2}})$$

$\forall\, \sigma_1, \sigma_2$ st $\sigma_1 \cap \sigma_2$ codim 1.

$K_T(X_\Sigma) \twoheadrightarrow K(X_\Sigma)$ (ker = ideal gen. by const. $(f)_\sigma$ st $f(1,1,\cdots,1) = 0$).

E.g.

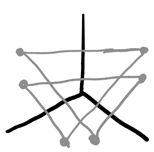


$\mathbb{P}^2$ as a $T = (\mathbb{C}^*)^3$-variety

$\mathcal{O}(-1) \longrightarrow \underline{\mathbb{C}}^3$ where $T \curvearrowright \underline{\mathbb{C}}^3$ standard

$$[\mathcal{O}(-1)]^T_{p_{\sigma_0}} = \chi^{e_0}, \qquad [\mathcal{O}(-1)]^T_{p_{\sigma_2}} = \chi^{e_2}$$

$$\chi^{e_0} - \chi^{e_2} \equiv 0 \mod (1 - \chi^{e_0 - e_2})$$



See [Klyachko][Payne] for more on toric vector bundles.

Defn For a permutation $\sigma \in \mathfrak{S}_E$, let $<_\sigma$ be the ordering $\sigma(0) <_\sigma \sigma(1) <_\sigma \cdots <_\sigma \sigma(n)$.

The <u>lex-first-basis</u> of M for σ is $B_\sigma(M) \in \mathcal{B}(M)$ obtained by the greedy algorithm that minimizes the weights as given by the ordering.

N.B. $-e_{B_\sigma(M)}$ the vertex of $-P(M)$ minimizing $\langle x, n\bar{e}_{\sigma(0)} + (n-1)\bar{e}_{\sigma(1)} + \cdots + \bar{e}_{\sigma(n-1)} \rangle$

Prop If $L \subseteq \mathbb{C}^E$ realizes M, then for $\sigma \in \mathfrak{S}_E$ ($\leftrightarrow$ cone of chain $\emptyset \subsetneq \sigma(1) \subsetneq \sigma(1,2) \subsetneq \cdots \subsetneq E$)

$$[S_L]^T_\sigma = \sum_{i \in B_\sigma(M)} T_i^{-1}$$

$$[Q_L]^T_\sigma = \sum_{i \in B_\sigma(M)^c} T_i^{-1}$$

Prop Can define $[S_M]^{(T)}$ & $[Q_M]^{(T)}$ for arbitrary matroids M.
Also $c^{(T)}(S_M)$ and $c^{(T)}(Q_M)$

Recall: For a chain $\mathcal{C}: \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$, have

$$X_E \supseteq V(\sigma_{\mathcal{C}}) \simeq X_{S_1} \times X_{S_2 \setminus S_1} \times \cdots \times X_{E \setminus S_k}$$

Prop ① $[\mathcal{S}_M]\big|_{V(\sigma_{\mathcal{L}})} = [\mathcal{S}_{M|S_1}] + [\mathcal{S}_{M|S_2\setminus S_1}] + \cdots + [\mathcal{S}_{M/S_k}]$

and likewise for $\mathcal{Q}_L$.

② $M \mapsto$ any polynomial expression in $\mathbb{S}_\lambda(\mathcal{S}_M)$ and $\mathbb{S}_\lambda(\mathcal{Q}_M)$ or in the Chern classes of $\mathcal{S}_M/\mathcal{Q}_M$ is valuative. (E.g. $M \mapsto \Delta_M$ is valuative).

pf) ① Exer.

② $\mathrm{Mat}_{r,E} \longrightarrow \mathbb{Z}\{\langle S\rangle \mid S \in \binom{E}{r}\}$, $M \mapsto \langle B_\sigma(M)\rangle$ is valuative for any fixed $\sigma \in \mathfrak{S}_E$. [Ardila-Fink-Rincon '10] [E.-Sanchez-Supina '21]

Recall $\alpha, \beta \in A^1(X_E)$ (Also, $\alpha = c_1(\mathcal{Q}_{U_{n,E}})$, $\beta = c_1(\mathcal{S}^\vee_{U_{1,E}})$)

Thm [Berget-E-Spink-Tseng '21]

$$\sum_{i,j,k,l}\left(\int_{X_E} \alpha^i \beta^j c_k(\mathcal{S}_M^\vee) c_l(\mathcal{Q}_M)\right) x^i y^j z^k w^l$$
$$= (x+y)^{-1}(y+z)^r(x+w)^{|E|-r}\, T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right)$$

pf) Let $\xi_M = \frac{1}{1-\alpha x}\frac{1}{1-\beta y} c(\mathcal{S}_M^\vee, z)\, c(\mathcal{Q}_M, w)$, and $\pi: X_E \longrightarrow X_{E\setminus i}$.

Then
$$\pi_*\xi_M = \begin{cases} (x+y)\,\xi_{M\setminus i} & i \text{ a loop} \\ (x+y)\,\xi_{M/i} & i \text{ a coloop} \\ (x+w)\,\xi_{M\setminus i} + (y+z)\,\xi_{M/i} & \text{neither} \end{cases}$$

Cor ① [Huh-Katz '12] formula. $\left(z=0, \left(\frac{\partial}{\partial w}\right)^{|E|-r}\right)$

② All main thms of [Lucia de Medrano-Rincon-Shaw '21]

③ $y=w=0 \rightsquigarrow x^{n-r}y^r\, T_M\left(\frac{x}{z}, 1\right) \rightsquigarrow$ h-vec. of $IN(M)$.