Lecture 20

Defn/Thm [Mac '74] Let X be smth proj. $\mathbb{C}$-var. Let $\mathcal{C}(X) := \{1_V \mid V \subseteq X \text{ (constructible) subvar.}\}$.
$\exists$ a map (really a natural transf. $\mathcal{C} \to H_\bullet$) called CSM class

$$\mathrm{csm} : \mathcal{C}(X) \longrightarrow H_\bullet(X) \quad \text{st} \quad 1_X \mapsto c(\mathcal{T}_X) \cap [X].$$

N.B. $\mathrm{csm}_0(1_Y) = \chi_{top}(Y)$ as elts in $H_0(X) \simeq \mathbb{Z}$.

Thm [Alu '99] $U \subset X$ open st $X \setminus U = \bigcup_{i=1}^{\ell} D_i$, a union of smth snc divisors. Then

$$c(\mathcal{T}_X(-\log(X\setminus U))) \cap [X] = \mathrm{csm}(1_U)$$

pf) Have $0 \to \mathcal{T}_X(-\log \bigcup_i D_i) \to \mathcal{T}_X \to \bigoplus_i \mathcal{O}_{D_i}(D_i) \to 0$. Induct on ℓ & $\dim X$.

$$c(\mathcal{T}_X(-\log \textstyle\bigcup_i D_i)) \cap [X] = c(\mathcal{T}_X) \Big/ \prod_i (1+D_i) \cap [X].$$

Whereas:

$$1_{X \setminus (\bigcup_{i=1}^{\ell-1} D_i)} - 1_{D_\ell \setminus (\bigcup_{i=1}^{\ell-1} D_i)} = c(\mathcal{T}_X) \Big/ \prod_{i=1}^{\ell-1}(1+D_i) \cap [X] - \underbrace{c(\mathcal{T}_{D_\ell}) \Big/ \prod_{i=1}^{\ell-1}(1+D_i)}_{= \, c(\mathcal{T}_X)/(1+D_\ell)} \cap [D]$$

$$= c(\mathcal{T}_X)\left(\frac{1+D_\ell}{\prod_{i=1}^{\ell}(1+D_i)} - \frac{D_\ell}{\prod_{i=1}^{\ell}(1+D_i)} \right) \cap [X]$$

④

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \mathcal{T}_{W_L}(-\log \partial W_L) & \to & \mathcal{T}_{X_E}(-\log \partial X_E)|_{W_L} & \to & N_{W_L/X_E} & \to & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \to & \mathcal{S}_L|_{W_L} & \to & \underline{\mathbb{C}}^E & \to & \mathcal{Q}_L|_{W_L} & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \underline{\mathbb{C}} & = & \underline{\mathbb{C}} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Cor

$$\mathrm{csm}(1_{\mathbb{P}L^\circ}) = c(\mathcal{T}_{W_L}(-\log \partial W_L)) \cap [W_L]$$
$$= c(\mathcal{S}_L)\, c_{|E|-r}(\mathcal{Q}_L) \cap [X_E] \in H_\bullet(X_E)$$

$$\alpha^i c_{r-1-i}(\mathcal{S}_L^\vee)\, c_{|E|-r}(\mathcal{Q}_L) \cap [X_E] = i^{th}\text{-coeff. of } T_M(x,0)/x.$$

Rem [Brylawski '84] conj. : coeff's of $T_M(x,0)$ log-conc. Solved: [Huh '15]

Rem log Poincaré-Hopf $\Rightarrow \int_{X_E} c_{r-1}(\mathcal{S}_L)\, c_{|E|-r}(\mathcal{Q}_L) = \int_{W_L} c_{r-1}(\mathcal{T}_{W_L}(-\log \partial W_L)) = \overline{\chi}_M(1)$.

$$c_{r-1}(\mathcal{S}_L)\, c_{|E|-r}(\mathcal{Q}_L) \;\big\|\; \int_{X_E} \varphi_L^*(\text{hook Young diagram})$$

[Speyer '09]
[Hacking-Keel-Tevelev '06]
[Varchenko '95, Silvotti '96]

Recall: $K(X) = \mathbb{Z}\{\text{vec. bdls on } X\}/_{SES}$ E.g. $K_T(pt) = \mathbb{Z}[\mathrm{Char}(T)]$.

$$\uparrow$$

$K_T(X) = \mathbb{Z}\{T\text{-eqv. vec. bdls on } X\}/_{SES}$

Thm [Rosu, Knutson '03][Vezzosi-Vistoli '03][Nielsen '74] Let Σ be a smth proj. fan ($\mathrm{lin} \neq 0$ ok). Then $K_T(X_\Sigma) \hookrightarrow \prod_{\sigma \in \Sigma_{max}} K_T(p_\sigma)$ whose image is $(f_\sigma)_{\sigma \in \Sigma_{max}}$ st

$$[\mathcal{F}]^T \longmapsto [\mathcal{F}]^T_{p_\sigma}$$

$f_{\sigma_1} - f_{\sigma_2} \equiv 0 \mod (1 - \chi^{m_{\sigma_1\sigma_2}})$
$\forall\, \sigma_1, \sigma_2$ st $\sigma_1 \cap \sigma_2$ codim 1.

$K_T(X_\Sigma) \twoheadrightarrow K(X_\Sigma)$ (ker = ideal gen. by const. $(f)_\sigma$ st $f(1,1,\ldots,1) = 0$).

Rem See also [Fink-Speyer '12] & [Dinu-E.-Seynnaeve '21]