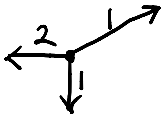
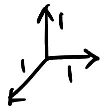
Lecture 16

Last time: $A^\bullet(\Sigma) = \mathbb{Z}[x_\rho \mid \rho \in \Sigma(1)]/\sim$, $A^k(\Sigma)$ gen. by $\{x_\sigma \mid \sigma \in \Sigma(k)\}$.
Minkowski weights of dim. k, denoted $\Delta \in MW_k(\Sigma)$.

E.g. ① $MW_0(\Sigma) \simeq \mathbb{Z}$ and $MW_n(\Sigma) = \mathbb{Z}$ (latter only needs Σ complete)
⇝ fundamental class $\Delta_\Sigma \approx [X_\Sigma]$

②





Prop One has an isomorphism

$$t_\Sigma : MW_k(\Sigma) \xrightarrow{\sim} \mathrm{Hom}(A^k(X_\Sigma), \mathbb{Z})$$
$$\Delta \longmapsto (x_\sigma \mapsto \Delta(\sigma))$$

Rem Above holds for Σ simplicial, not necessarily complete or smth (with suitable defn for MW).

Prop The cap product

$$A^k(\Sigma) \times MW_d(\Sigma) \longrightarrow MW_{d+k}(\Sigma)$$
$$(\xi, \Delta) \mapsto \xi \cap \Delta : (\tau \mapsto (t_\Sigma \Delta)(\xi \cdot x_\tau))$$

given an isomorphism $\delta_\Sigma : A^k(\Sigma) \xrightarrow{\sim} MW_{n-k}(\Sigma)$ by $\xi \mapsto \xi \cap \Delta_\Sigma$.

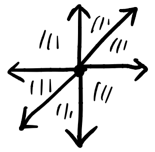
$\int_{X_\Sigma} : A^n(X_\Sigma) \longrightarrow \mathbb{Z}$ coincides with $\delta_\Sigma : A^n(X_\Sigma) \longrightarrow MW_0(\Sigma) \simeq \mathbb{Z}$.
$\xi \longmapsto \xi \cap \Delta_\Sigma$

For $Y \subset T_N$ a very affine subvariety, and $\overline{Y}$ its closure in $X_\Sigma \supset T_N$,
what is $[\overline{Y}] = \eta_{\overline{Y}} \cap \Delta_\Sigma \in MW_\bullet(\Sigma)$? See [Maclagan-Sturmfels] for reference.

Defn Let $f = \sum_{\alpha \in \mathcal{A} \subset \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$. Define $f_{trop} : \mathbb{R}^n \to \mathbb{R}$ by
$f_{trop} = \min\{\alpha \cdot x \mid \alpha \in \mathcal{A}\}$, and $V_{trop}(f_{trop}) = \{w \in \mathbb{R}^n \mid \text{min. in } f \text{ achieved at least twice at } w\}$
≈ "kinks" of the PL fct trop(f)

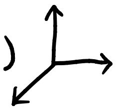
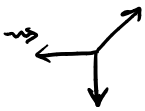
For $I \subset \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$, define $V_{trop}(I) = \bigcap_{f \in I} V_{trop}(f_{trop})$
If $Y \subset (\mathbb{C}^*)^n$ given by the ideal I, write $\mathrm{trop}(Y) := V_{trop}(I)$.

Thm trop(Y) is a weighted balanced polyhedral cplx (say with Gröbner fan str).
(pure of dim = dim Y, if Y irred.) ↳ i.e. any fan str on trop(Y) makes it into a MW.

E.g. $\Sigma =$ . $\mathrm{Def}(\Sigma)/_{\text{transl.}}$ has extremal rays $\Delta, \nabla, |, -, \backslash$.

$\cap$

$\mathbb{Z}^6/\mathbb{Z}^2$

$ax+by+c \rightsquigarrow \min(x,y,0)$ , $axy+by+cx \rightsquigarrow \min(x+y, y, x) \rightsquigarrow$ 

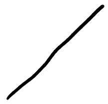
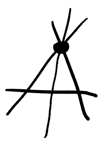
Exer For $D \in A^1(X_\Sigma)$ b.p.f. and $\sigma \in \Sigma(n-1)$, let P_D^σ be the face of P_D corresp. to σ.
Then, $D \cap \Delta_\Sigma = V_{\text{trop}}(\varphi_D) : \sigma \mapsto \#(P_D^\sigma \cap M) - 1$

Rem In fact, $D^k \cap \Delta_\Sigma : \tau \mapsto$ k-dim'l lattice vol. of P_D^τ.
↳ k! Euclidean vol.

$k = n-1$ case: $\sum_{F \text{ facets}} \mathrm{vol}(F)\, \vec{n}_F = 0.$

Thm [Tevelev][Katz] If Σ contains a refinement of $\mathrm{trop}(Y)$ as a subfan, then $\mathrm{trop}(Y) = [\overline{Y}]$ as elts in $MW_\bullet(\Sigma)$.
If $\Sigma' \succeq \Sigma$, then $\mathrm{trop}(Y) = [\overline{Y}^{X_{\Sigma'}}] = \pi^*[\overline{Y}^{X_\Sigma}] = [\pi^{-1}\overline{Y}^{X_\Sigma}]$.

⚠ $Y = \{x+y=0\} \subset \mathbb{P}(\mathbb{C}^*)^3 \subset \mathbb{P}^2 \longleftarrow Bl_{[0:0:1]}\mathbb{P}^2$ from  $\preceq$ 

$\mathrm{trop}(Y) =$ . $[\overline{Y}^{\mathbb{P}^2}] = [H]$, but $[\overline{Y}^{Bl}] \neq \pi^*[H]$. 

Rem Ring of conditions: $\mathcal{R}((\mathbb{C}^*)^n) := \varinjlim_{\Sigma} A^\bullet(X_\Sigma) =$ ring of MWs (tropical cycles) in $(\mathbb{Z}^n)_{\mathbb{R}}$

Let $L \subseteq \mathbb{C}^E$ represent a loopless matroid M.

$$\begin{tikzcd} W_L \arrow[r, hook, "\iota"] & X_{A_n} \arrow[r, squiggly] & A^\bullet(X_{A_n}) \arrow[r, "\iota^*"] \arrow[d, equal] & A^\bullet(W_L) \arrow[d, equal] \\ & & \mathbb{Z}[x_S \mid \emptyset \subsetneq S \subsetneq E]/\sim \arrow[r] & \mathbb{Z}[x_F \mid \emptyset \subsetneq F \subsetneq E \text{ flat}]/\sim \end{tikzcd}$$

Have the above, $x_S \mapsto \begin{cases} x_S & S \text{ flat} \\ 0 & \text{else} \end{cases}$

$$\underset{\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_{r-1} \subsetneq E}{x_{S_1} \cdots x_{S_{r-1}}} \cap [W_L] = \begin{cases} 1 & \text{if } S_1, \dots, S_{r-1} \text{ chain of flats of } M. \\ 0 & \text{else} \end{cases}$$

Prop $\mathrm{trop}(\mathring{L}) = \Delta_M$ the Bergman class defined by putting weight 1 for cones in the Bergman fan Σ_M whose cones are chains of flats.

Exer For arbitrary (not necessarily realizable) loopless matroid M, Δ_M as above is a MW on Σ_{A_n}.