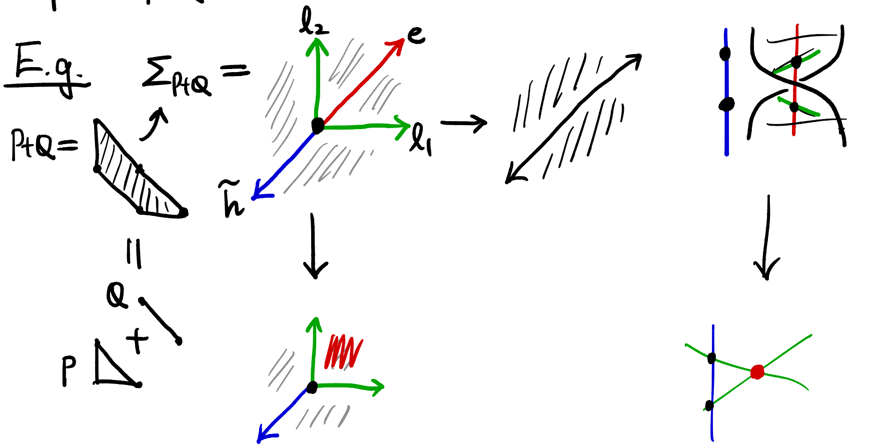
# Lecture 13

**Defn** The Minkowski sum of two polyhedra $P, Q \subseteq M_{\mathbb{R}}$ is
$$P + Q := \{ m \in M_{\mathbb{R}} \mid m = p + q \quad \exists p \in P, q \in Q \}.$$

**Prop** $\Sigma_{P+Q}$ is the coarsest common refinement of $\Sigma_P$ and $\Sigma_Q$.

**E.g.**




$[u:v]$ $[x:y:z]$
$$Bl_{pt}\,\mathbb{P}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$$
$$\| $$
$$\{xv - yu = 0\}$$

(weak Minkowski summand)

**Defn** Let $\Sigma_P$ be a projective fan. A polytope $Q$ is a deformation of $P$ if $\Sigma_Q \preceq \Sigma_P$ (i.e. $\Sigma_Q$ is a coarsening of $\Sigma_P$). $P$ is indeformable if $\lambda P$ for $\lambda > 0$ are the only deformations of $P$. $\mathrm{Def}(P)$ is a cone.

**Exer** Suppose $P$ is a zonotope, i.e. a Minkowski sum of line segments $\{\ell_1, \dots, \ell_k\}$. Then $Q \in \mathrm{Def}(P) \iff$ every edge of $Q$ is $/\!/$ to some $\ell_i$.

Let $P$ be a $M$-lattice polytope. Can define $X_{\Sigma_P}$, and $X_{P \cap M}$ where
$$X_{P\cap M} := \text{closure of image of } \left( T \longrightarrow (\mathbb{C}^*)^{P\cap M} \longrightarrow \mathbb{P}(\mathbb{C}^*)^{P \cap M} \right)$$
$$t \longmapsto (\chi^m(t))_m$$

**E.g.**

vs. $\mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 2e_3)$

**Rem** $P$ has IDP (is normal) if $(kP) \cap M = k(P \cap M)$ $k > 0$.
$P$ is smth if $\Sigma_P$ is ($\Rightarrow$ IDP).

Let $X = X_{\Sigma_P}$ for $P$ smth polytope now. Assume also $\dim P = \dim M_{\mathbb{R}}$.
Let $\mathbb{C}(X)$ be the (constant) sheaf of rat'l fcts on $X$, i.e. $\mathbb{C}(X) = \mathrm{Frac}\, \mathbb{C}[M]$.

Recall: $A^1(X) = \mathrm{Div}(X) / \langle \mathrm{div}(f) \rangle$. Let $\Sigma(1)$ be the rays of $\Sigma = \Sigma_P$.
Let $D_\rho = V(\rho)$. $\quad \rho \rightsquigarrow u_\rho$ the prim. ray vector.

**Thm** ① $0 \longrightarrow M \xrightarrow{\text{div}} \mathrm{Div}_T(X) \longrightarrow A^1(X) \longrightarrow 0$

$m \longmapsto \sum_\rho \langle m, u_\rho \rangle D_\rho$

Exer As claimed, show $\mathrm{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho$

E.g. $e + l_1 = e + l_2 = \tilde{h}$ in $Bl_{pt} \mathbb{P}^2$

② Denote now $D = \sum_\rho a_\rho D_\rho$. Then $\mathcal{O}_X(D)|_{U_\sigma} \simeq \mathbb{C}[S_\sigma] \cdot \chi^{m_\sigma}$ where $\langle m_\sigma, u_\rho \rangle = -a_\rho \ \forall \rho \leq \sigma$.

Since $\chi^m \in \mathbb{C}[S_\sigma] \Leftrightarrow \langle m, u_\rho \rangle \geq 0 \ \forall \rho \leq \sigma$,

have $H^0(X, \mathcal{O}_X(D)) \simeq \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$ where $P_D = \{ m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -a_\rho \ \forall \rho \}$

N.B. $P_{D + \mathrm{div}(\chi^m)} = P_D - m$

③ $D$ is b.p.f. $\Longleftrightarrow$ the piecewise linear fct $\varphi_D : N_\mathbb{R} \longrightarrow \mathbb{R}$ def. by $u_\rho \mapsto -a_\rho$ is "convex"

$\Longleftrightarrow$ $\varphi_D$ is the support fct $u \mapsto \min_{m \in P_D} \langle m, u \rangle$ of $P_D \in \mathrm{Def}(P)$

i.e. (lattice) deformations of $P \longleftrightarrow$ b.p.f. divisors on $X$

$Q \longmapsto D_Q = \sum_\rho - \min_{m \in Q} \langle m, u_\rho \rangle D_\rho$

N.B. ample $\Leftrightarrow$ strictly convex.

$\overline{\mathrm{Nef}(X) = \mathrm{Ample\ cone}} \simeq \mathrm{Def}(Q)/\text{transl.}$

④ (Demazure vanishing) $\chi(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D))$ when $D$ b.p.f.

$= \#(P_D \cap M)$

⑤ For $D$ b.p.f., $\int_X D^{\dim X} = \mathrm{vol}(P_D)$

(the lattice volume, i.e. std splx has vol=1)