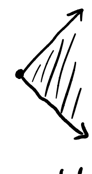
Lecture 12

Let M, N be Char, Cochar lattice of an alg. torus T.

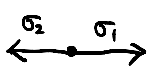
Defn A strongly convex cone $\sigma \subset N_{\mathbb{R}}$ (if $\sigma \subset N_{\mathbb{R}}$ general, take $\sigma/_{\mathrm{lin}(\sigma)} \subset N/_{\mathrm{lin}(\sigma)}$) is <u>simplicial</u> if the primitive vectors of rays of σ are linearly indep., and <u>smooth</u> (unimodular) if they can be extended to a $\mathbb{Z}$-basis of N.
(Equiv., $U_\sigma \simeq (k^*)^i \times k^j$ if smth, $U_\sigma \simeq (k^*)^i \times (k^j/_{\text{fin. ab. grp}})$ if simplicial).

E.g. $\sigma = \mathrm{Cone}(e_1, e_1+e_2)$  vs. $\sigma = \mathrm{Cone}(e_1-e_2, e_1+e_2)$ 

N.B. If $\tau \preceq \sigma$, then $S_\tau = S_\sigma + \mathbb{Z}\{m \in \sigma^\vee \cap \tau^\perp\}$, i.e. $U_\tau \underset{\text{distinguished open}}{\subseteq} U_\sigma$
(i.e. $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$ for $m \in \mathrm{relint}(\sigma^\vee \cap \tau^\perp)$).

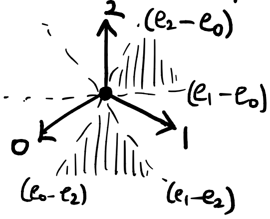
More generally, if $\tau \subseteq \sigma$, then $\exists$ (toric) morphism $U_\tau \to U_\sigma$

Defn Let $\Sigma \subset N_{\mathbb{R}}$ be a rat'l fan. The X_Σ is the variety obtained by gluing together U_σ for each $\sigma \in \Sigma$.

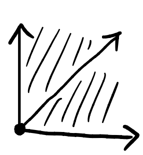
E.g. 

$U_{\sigma_1} = k[x]$, $U_{\sigma_2} = k[x^{-1}]$ $(x = \chi^{e_1})$
$\Rightarrow X_\Sigma \simeq \mathbb{P}^1$

E.g. $\Sigma \subset \mathbb{Z}^{n+1}$ by $(n+1)$ cones $\sigma_i = \mathrm{Cone}(e_{j \neq i}\text{'s}, \pm \mathbb{1}) \rightsquigarrow \mathbb{P}^n$ with $(k^*)^{n+1}$-action



Σ has lineality $\mathbb{R}\mathbb{1}$, and $(\mathbb{Z}^{n+1}/\mathbb{Z}\mathbb{1})^\vee = \mathbb{1}^\perp$
Writing $\chi^{e_i} = x_i$, have $U_i = \mathrm{Spec}\, k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$

E.g.  $= \Sigma \rightsquigarrow X_\Sigma = Bl_{(0,0)} \mathbb{A}^2$

Exer Realize $Bl_{pt} \mathbb{A}^n$ as a toric variety of a fan.

Thm (Orbit-cone corresp.) Let $\Sigma \subset N_{\mathbb{R}}$, and $T = T_N$. There is a bijection:

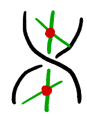
$$\left\{ \overset{(k+\dim(\mathrm{lin}(\sigma)))\text{-dim'l cones}}{\text{codim. } \ell \text{ cones of } \Sigma} \right\} \longleftrightarrow \left\{ \overset{k\text{-codim'l } T\text{-orbits}}{\ell\text{-dim'l } T\text{-orbits of } X_\Sigma} \right\}$$

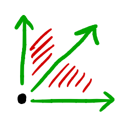
$$\sigma \longmapsto O(\sigma) := U_\sigma \setminus \Big(\bigcup_{\tau \not\leq \sigma} U_\tau \Big) \simeq T_{N/\mathrm{span}(\sigma)\cap N}$$

and $V(\sigma) = \overline{O(\sigma)} = \bigcup_{\tau \succeq \sigma} O(\sigma)$, $U_\sigma = \bigcup_{\tau \preceq \sigma} O(\sigma)$.

E.g. $\mathbb{P}^2$

E.g. Verify on $Bl_{pt} \mathbb{A}^2$:





Exer $U_\sigma \cap V(\tau) \hookrightarrow U_\sigma$ given by $\mathbb{C}[S_\sigma] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M]$

For a fan $\Sigma \subset N_{\mathbb{R}}$, its <u>support</u> $|\Sigma|$ is $\bigcup_{\sigma \in \Sigma} \sigma$. Say Σ is <u>complete</u> if $|\Sigma| = N_{\mathbb{R}}$, and Σ is <u>projective</u> if $\Sigma = \Sigma_P$ $\exists$ lattice polytope P.