Lecture 11

Toric varieties : Alg. geom. :: Polyhedra : Combinatorics. [Fulton '93][Cox-Little-Schenck '11]

Let M, N be lattices (i.e. fin. gen. free abel. grp) with perfect pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$.
$M_{\mathbb{R}} = M \otimes \mathbb{R}$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$.

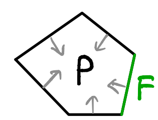
Defn A <u>polyhedron</u> $P \subseteq M_{\mathbb{R}}$ is a finite intersection of half-spaces, i.e. $\exists u_i \in N$, $a_i \in \mathbb{R}$ $(i \in I,\ |I| < \infty)$

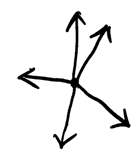
$$P = \{ m \in M_{\mathbb{R}} \mid \langle m, u_i \rangle \geq a_i \ \ \forall i \} = \bigcap_i H^+_{u_i, a_i}$$

<u>polytope</u> if bounded. <u>lattice polytope</u> if vertices of P are all in M.

Defn A <u>face</u> of a polyhedron $P \subseteq M_{\mathbb{R}}$ a polyhedron F that arise as $P \cap H_{u,a}$ where $P \subseteq H^+_{u,a}$. Vertices are 0-dim'l faces, edges 1-dim'l, facets max'l proper faces. Write $F \preceq P$ in this case.

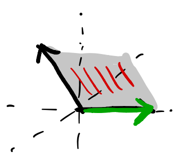
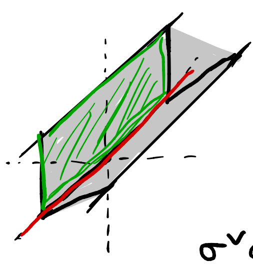
Defn $\dim(P) = \dim(\text{affine span of } P)$

E.g. 

$\Sigma_P =$ 

Defn A <u>(rational) polyhedral cone</u> $\sigma \subseteq N_{\mathbb{R}}$ is $\overset{\text{Cone}(u_1, \dots, u_k)}{\rotatebox{90}{=}} \mathbb{R}_{\geq 0}\{u_1, \dots, u_k\}$ $\exists u_1, \dots, u_k \in N_{\mathbb{R}}$ (N).
Its <u>dual cone</u> is $\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \ \forall u \in \sigma \}$. Denote $\sigma^\perp = \{ m \mid \langle m, u \rangle = 0 \ \forall u \in \sigma \}$.
<u>Lineality</u> $\text{lin}(\sigma)$ of σ is the largest $\mathbb{R}$-vect. sp. contained in σ. (= min'l face)

Thm $\exists$ codim. reversing bijection: $\{\text{faces of } \sigma\} \longleftrightarrow \{\text{faces of } \sigma^\vee\}$
$\tau \preceq \sigma \longleftrightarrow \sigma^\vee \cap \tau^\perp$

E.g.  $\longleftrightarrow$ 

$\sigma \subset \mathbb{R}^3$ $\qquad$ $\sigma^\vee \subset \mathbb{R}^3$

(<u>strongly convex</u> $\iff \text{lin}(\sigma) = \{0\}$)

<u>rays</u> = 1-dim'l faces of a strongly convex σ.

Exer If $\dim(\sigma) = \dim N_{\mathbb{R}}$ and strongly convex, do σ & $\sigma^\vee$ have same #rays?

Defn A (rat'l) <u>fan</u> $\Sigma \subset N_{\mathbb{R}}$ is a collection of (rat'l) cones $\{\sigma \subset N_{\mathbb{R}}\}$ st
(1) $\sigma \in \Sigma$ and $\tau \preceq \sigma \Rightarrow \tau \in \Sigma$, and (2) $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2 \preceq \sigma_1$ & $\sigma_1 \cap \sigma_2 \preceq \sigma_2$

Defn For $P \subseteq M_{\mathbb{R}}$ polyhedron, its <u>normal fan</u> $\Sigma_P = \{ (\mathbb{R}_{\geq 0}(P - p))^\vee \mid p \in P \}$.
N.B. Codim reversing bijection btw $\{\text{cones of } \Sigma_P\} \longleftrightarrow \{\text{faces of } P\}$

We will always assume that cones in $N_{\mathbb{R}}$ are rat'l now.

Let k be an alg. closed field.

<u>Defn</u> An <u>algebraic torus</u> $T \simeq (k^*)^n$ $\exists n \in \mathbb{Z}_{\geq 0}$.

<u>Facts</u> ① $M = \mathrm{Char}(T) = \mathrm{Hom}_{\text{alg-grp.}}(T, k^*) \simeq \mathbb{Z}^n$ <u>character lattice</u>

$$T = \operatorname{Spec} k[M], \quad k[M] := \{ \textstyle\sum_i a_i \chi^{m_i} \mid a_i \in k, m_i \in M \} \quad \chi^m \chi^{m'} = \chi^{m+m'}$$
$$\simeq k[t_1^{\pm}, \dots, t_n^{\pm}]$$

$N = \mathrm{Cochar}(T) = \mathrm{Hom}(k^*, T)$ <u>cocharacter lattice</u>. Write $T = T_N$.

$M \times N \longrightarrow \mathbb{Z}$ by $t^{\langle m, u\rangle} = \chi^m(\phi_u(t))$.

② If V a f.d. T-rep., then $V = \bigoplus_m V_m$ where
$V_m = \{ v \in V \mid t \cdot v = \chi^m(t) v \}$.

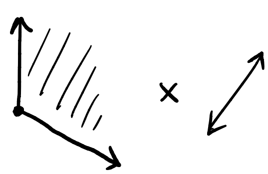
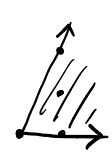
③ T acts on $k[M]$ by $t \cdot \chi^m = \chi^{-m}(t) \cdot \chi^m$ (This is really a <u>convention</u>, not fact)

<u>Prop</u> If $\sigma^\vee \subset M_{\mathbb{R}}$ rat'l, then $\sigma^\vee \cap M$ is fin. gen. semigrp, i.e.
(Gordon's Lem) $\sigma^\vee \cap M = \mathbb{Z}_{\geq 0}\{m_1, \dots, m_k\}$ $\exists m_1, \dots, m_k \in M$.

<u>Defn</u> For a rat'l cone $\sigma \subset N_{\mathbb{R}}$, denote $S_\sigma = \sigma^\vee \cap M$, and $U_\sigma := \operatorname{Spec} k[S_\sigma]$.
U_σ is the <u>affine toric variety</u> assoc. to σ.

<u>N.B.</u> U_σ has a (dense) T-action (since T acts on $k[S_\sigma]$) with stabilizer T' where $\mathrm{Cochar}(T') = \mathrm{lin}(\sigma) \cap N$. So, if σ strongly convex, $U_\sigma \supset T$.

<u>E.g.</u> $\sigma = \mathrm{Cone}(2e_1 - e_2, e_2, \pm e_3) \subset \mathbb{R}^3 \longleftrightarrow \sigma^\vee = \mathrm{Cone}(e_1, e_1 + 2e_2) \subset \mathbb{R}^3$

Writing $x_1 = \chi^{e_1}, x_2 = \chi^{e_2}, x_3 = \chi^{e_3}$, have $k[S_\sigma] = k[x_1, x_1x_2, x_1x_2^2]$

$U_\sigma \simeq \{(a,b,c) \in k^3 \mid ac - b^2 = 0\}$ $\simeq k[a, b, c] / \langle ac - b^2 \rangle$

T $(t_1, t_2, t_3) \cdot (a, b, c) = (t_1 a, t_1 t_2 b, t_1^2 t_2 c)$, $\mathrm{Stab} = \{(1, 1, t_3)\}$

<u>Rem</u> Every affine normal toric variety arises as U_σ for some σ and N.