Primer on blow-ups

Let X smth var. $/k=\bar{k}$, and $Y \subset X$ smth subvar. defined by ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$.

<u>Defn</u> $\mathrm{Bl}_Y X := \mathrm{Proj}_X \bigoplus_{i\geq 0} \mathcal{I}^i$ (blow-up of X along Y)

$$\begin{tikzcd} \mathrm{Bl}_Y X \arrow[d, "\pi"'] & \pi^{-1}(Y) = E \arrow[l, hook'] \arrow[d] \\ X & Y \arrow[l, hook'] \end{tikzcd}$$

$\pi^{-1}(Y) = E := \mathbb{P}_Y(N_{Y/X}) := \mathrm{Proj}_Y \bigoplus_{i\geq 0} (\mathcal{I}/\mathcal{I}^2)^i$ (exceptional divisor)

$\mathbb{P}_Y(N_{Y/X}) \;\rotatebox{90}{$\subseteq$}\; \{(y,n) \mid y \in Y,\ n \in \mathbb{P}(\text{subspace of } T_yX \text{ normal to } T_yY)\}$

<u>Facts</u> (1) π isom. on $X \setminus Y$.

(2) For $Z \hookrightarrow X$ closed, have

$$\begin{tikzcd} \mathrm{Bl}_{Y\cap Z} Z \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Y X \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

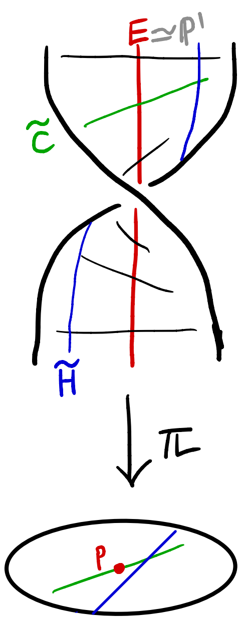
$\mathrm{Bl}_{Y\cap Z} Z$ = "strict transform of Z"

(3) $\mathcal{O}(1)$ on $\tilde{X} = \mathrm{Bl}_Y X$ is the inverse image ideal sheaf $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ (i.e. $0 \to \mathcal{O}_{\tilde{X}}(1) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_E \to 0$)

$\Rightarrow H^0(E, \mathcal{O}_E(E)) = 0 \Rightarrow h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(E)) = 1$, i.e. E doesn't move.

(4) If Y the base loci of a linear series $V \subseteq H^0(X, \mathcal{O}(D))$, then $\mathrm{Bl}_Y X$ resolves $X \dashrightarrow \mathbb{P}V^*$, and $V = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*D - E))$.

<u>E.g.</u> (Blow-up of a point in a plane). $\mathrm{Bl}_p \mathbb{P}^2 \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^1_{[u:v]}$ as $\{xv - yu = 0\}$, $p = [0:0:1]$



$\pi^* h = \tilde{H} = \tilde{C} + E$ in $A^1(X)$, which is gen. by $\tilde{H}$ & E.

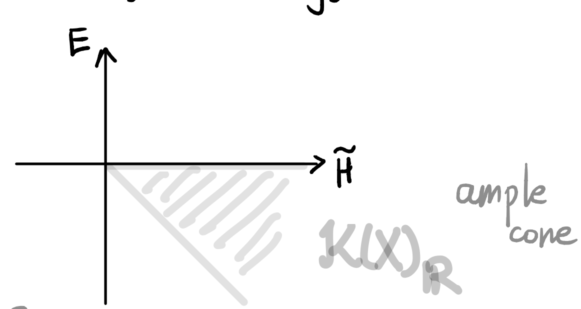
$\tilde{H}^2 = 1$, $E^2 = -1$, $\tilde{H} \cdot E = 0$

E "doesn't move" $h^0(E) = 1$ $(0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{O}_E(E) \to 0)$

$|\tilde{H}| \simeq \{\text{lines in } \mathbb{P}^2\}$, $|\tilde{H} - E| \simeq \{\text{lines in } \mathbb{P}^2 \text{ thru } p\}$

$H^0(\tilde{H}) \simeq \mathbb{C}\{x,y,z\}$ $\qquad H^0(\tilde{H} - E) \simeq \mathbb{C}\{x,y\}$

$$\begin{tikzcd} & \mathrm{Bl}_p \mathbb{P}^2 \arrow[dl, "\varphi_{\tilde{H}}"'] \arrow[dr, "\varphi_{\tilde{H}-E}"] \arrow[rr, hook, "\varphi_{2\tilde{H}-E}"] & & \mathbb{P}^4 \arrow[d, hook] \\ \mathbb{P}^2 & \times & \mathbb{P}^1 \arrow[r, hook] & \mathbb{P}^5 \end{tikzcd}$$



<u>Side note:</u> $2\tilde{H} - E \cdot (-3\tilde{H} + E + 2\tilde{H} - E) = -2$

Adjunction formula $\Rightarrow$ $\mathrm{Bl}_p \mathbb{P}^2 \hookrightarrow \mathbb{P}^4$ gen. hyperpl. section is $\simeq \mathbb{P}^1$

In fact, $(2\tilde{H} - E)^2 = 3 \Rightarrow$ twisted cubic curve.