Divisors on matroids and their volumes

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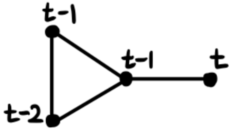
1. Graphs

G: a finite simple graph

chromatic polynomial of G:

$\chi_G(t) :=$ # of ways to color vertices of G with at most t many colors with no adjacent vertices same color

Example



$$\begin{aligned}\chi_G(t) &= t(t-1)(t-1)(t-2)\\ &= t(t^3-4t^2+5t-2)\end{aligned}$$

Conjecture [Rota '71]

The unsigned coefficients of χ_G are unimodal ($\nearrow\searrow$) .

2. Matroids

A **matroid** $M = (E, \mathcal{I})$:

- a finite set E, the **ground set**
- a collection $\mathcal{I}$ of subsets of E, the **indepedent subsets**

Examples

- realizable matroids: $E = \{v_0, \dots, v_n\}$ vectors, independent $=$ linearly independent
- graphical matroids: $E =$ edges of a graph G, independent $=$ no cycles

characteristic polynomial of M: $\chi_M(t)$

Conjecture [Rota '71, Heron '72, Welsh '74]

The unsigned coefficients of $\chi_M(t)$ are unimodal.

3. History

Resolution: graphs [Huh '12], realizable matroids [Huh-Katz '12]

KEY: coefficients of χ_M = intersection numbers of (nef) divisors

$\rightarrow$ mixed volumes of convex bodies (Newton-Okounkov bodies)

Hodge theory of matroids [Adiprasito-Huh-Katz '18]:
Chow ring / "cohomology ring" of a matroid

$\rightarrow$ where are the convex bodies & their (mixed) volumes?

4. Goal today

Today: the volume polynomial of the Chow ring of a matroid

- (Comb) new invariants of matroids, "Hopf-y structures," volumes of generalized permutohedra
- (Alg Geom) degrees of certain varieties, E.g. $\overline{\mathcal{M}}_{0,n}$ and $\overline{L}_n$
- (Trop Geom) first step in tropical Newton-Okounkov bodies

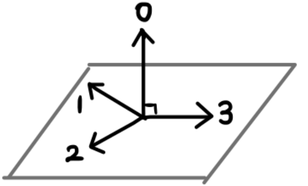
5. More matroids

M a matroid of rank r: $E = \{v_0, \ldots, v_n\}$ spanning $V \simeq \mathbb{C}^r$.

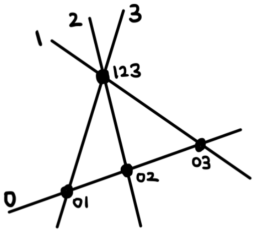
- For $S \subseteq E$, set $\mathrm{rk}_M(S) := \dim_{\mathbb{C}} \mathrm{span}(S)$.
- $F \subseteq E$ is a **flat** of M if $\mathrm{rk}(F \cup \{x\}) > \mathrm{rk}(F)\ \forall x \notin F$.
- hyperplane arrangement $\mathcal{A}_M = \{H_i\}$ in $\mathbb{P}V^*$, where $H_i := \{f \in \mathbb{P}V^* \mid v_i(f) = 0\}$.

Example

M as 4 vectors in 3-space



$\mathcal{A}_M$ drawn on $\mathbb{P}^2$



6. Chow rings of matroids

- M a matroid of rank $r = d + 1$ with ground set E,
- $\overline{\mathcal{L}_M} :=$ the set of nonempty proper flats of M.

Definition [Feichtner-Yuzvinsky '04, de Concini-Procesi '95]

Chow ring $A^\bullet(M)$: a graded $\mathbb{R}$-algebra $A^\bullet(M) = \bigoplus_{i=0}^{d} A^i(M)$

$$A^\bullet(M) := \frac{\mathbb{R}[x_F : F \in \overline{\mathscr{L}_M}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \ni i} x_F - \sum_{G \ni j} x_G \mid i, j \in E\rangle}$$

Elements of $A^1(M)$ called **divisors** on M.

$A^\bullet(M) =$ cohomology ring of the wonderful compactification X_M:

- built via blow-ups from $\mathbb{P}V^*$; compactifies $\mathbb{P}V^* \setminus \bigcup \mathcal{A}_M$
- E.g. $\overline{\mathcal{M}}_{0,n}$, $\overline{L}_n$ (moduli of stable rational curves with marked points)

7. Poincaré duality & the volume polynomial

Theorem [6.19, Adiprasito-Huh-Katz '18]

The ring $A^\bullet(M)$ satisfies Poincaré duality:

1. the degree map $\deg_M : A^d(M) \xrightarrow{\sim} \mathbb{R}$ (where $\deg_M(x_{F_1} x_{F_2} \cdots x_{F_d}) = 1$ for every maximal chain $F_1 \subsetneq \cdots \subsetneq F_d$ of nonempty proper flats)
2. non-degenerate pairings $A^i(M) \times A^{d-i}(M) \to A^d(M) \simeq \mathbb{R}$.

Macaulay inverse system:
Poincaré duality algebras $\leftrightarrow$ volume polynomials

Definition

The **volume polynomial** $VP_M(\underline{t}) \in \mathbb{R}[t_F : F \in \overline{\mathscr{L}_M}]$ **of** M

$$VP_M(\underline{t}) = \deg_M \Big(\sum_{F \in \overline{\mathscr{L}_M}} x_F t_F \Big)^d$$

(where $\deg_M : A^d(M) \to \mathbb{R}$ is extended to $A^d[t_F\text{'s}] \to \mathbb{R}[t_F\text{'s}]$).

8. Formula for VP_M

- M be a matroid of rank $r = d + 1$ on a ground set E,
- $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E$ a chain of flats of M with ranks $r_i := \operatorname{rk} F_i$,
- $d_1, \ldots, d_k \in \mathbb{Z}_{>0}$ such that $\sum_i d_i = d$, and $\widetilde{d}_i := \sum_{j=1}^{i} d_j$

Theorem [E '18]

The coefficient of $t_{F_1}^{d_1} \cdots t_{F_k}^{d_k}$ in $VP_M(\underline{t})$ is

$$(-1)^{d-k} \binom{d}{d_1, \ldots, d_k} \prod_{i=1}^{k} \binom{d_i - 1}{\widetilde{d}_i - r_i} \mu^{\widetilde{d}_i - r_i}(M|F_{i+1}/F_i),$$

$\{\mu^i(M')\}$ = unsigned coefficients of the reduced characteristic polynomial $\overline{\chi}_{M'}(t) = \mu^0(M')t^{\operatorname{rk} M'-1} - \mu^1(M')t^{\operatorname{rk} M'-2} + \cdots + (-1)^{\operatorname{rk} M'-1}\mu^{\operatorname{rk} M'-1}(M')$ of a matroid M'.

9. First applications

1. $M = M(K_{n-1})$: $VP_M \to$ embedding degrees of $\overline{\mathcal{M}}_{0,n}$
   - not a Mori dream space [Castravet-Tevelev '15]
2. $M = U_{n,n}$: $VP_M \to$ volumes of generalized permutohedra
   - Relation to [Postnikov '09]? cf. [Backman-E.-Simpson '19]
3. The operation $M \mapsto VP_M \in \mathbb{R}[t_S \mid S \subseteq E]$ is valuative.
   - "The construction of the Chow ring of a matroid respects its type A structure." Hodge theory of matroids of arbitrary Lie type?

10. Shifted rank volume I

Nef divisors "=" submodular functions

Definition

The **shifted rank divisor** of M:

$$D_M := \sum_{F \in \overline{\mathscr{L}_M}} (\operatorname{rk}_M F) x_F$$

The **shifted rank volume** of M:

$$\operatorname{shRVol}(M) := \deg_M(D_M^d) = VP_M(t_F = \operatorname{rk}_M(F)).$$

Remark

Unrelated to:
- the Tutte polynomial
- volume of the matroid polytope

11. Shifted rank volume II

- uniform matroid $U_{r,n}$: n general vectors in r-space.

Theorem [E. '18]

For M a *realizable* matroid of rank $r = d + 1$ on n elements,

$$\mathrm{shRVol}(M) \leq \mathrm{shRVol}(U_{r,n}) = n^d, \text{ with equality iff } M = U_{r,n}.$$

Proof: Let $\pi : X_M \to \mathbb{P}^d$ be the wonderful compactification. Then $D_M = n\widetilde{H} - E$ where $\widetilde{H} = \pi^*(c_1(\mathscr{O}_{\mathbb{P}^d}(1)))$ the pullback of the hyperplane class, and E an effective divisor that is $E = 0$ iff $M = U_{r,n}$. Now, note $H^0(m(n\widetilde{H} - E)) \subset H^0(m(n\widetilde{H}))$ for any $m \in \mathbb{Z}_{\geq 0}$.

$\to$ counting sections of divisors in tropical setting?

Thanks

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Thank you for listening!