ERRATUM TO "POLYHEDRAL AND TROPICAL GEOMETRY OF FLAG POSITROIDS"

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All section/theorem numbers refer to the published version of the paper [BEW24]. We thank Jidong Wang for pointing out an error in the proof of Theorem 6.1, which renders the theorem invalid. This also invalidates the implication (e) $\implies$ (b) in Theorem A. No other statements are affected, and the error is local in the sense that we describe the following revisions in this document:

- Instead of proving the implications (d) $\implies$ (e) $\implies$ (b) in Theorem A, as was originally done, we prove the implication (d) $\implies$ (b) without passing through (e).
- We establish a modified version of Theorem 6.1 and a correspondingly modified statement (e′) of (e), and prove the equivalence (b) $\iff$ (e′).

1. The error

The error in the proof of Theorem 6.1 arises in the last two bullet points at the end of Section 6. The two cases considered by the two bullet points are duals of each other, so let us just address the last one, which reads:

- $(n+1) \notin S \cup ijk\ell$. *In this case, considering the minor* $\widetilde{\boldsymbol{\mu}}|S \cup ijk\ell(n+1)/S$ *and then applying Lemma 6.2 implies that the three-term Grassmann-Plücker relation is satisfied.*

However, Lemma 6.2 does not always apply here. The element $(n+1)$ can be a loop in the matroid $\widetilde{\underline{\boldsymbol{\mu}}}|S \cup ijk\ell(n+1)/S$, whereas the lemma requires that the "extra element" (labelled 5 in the lemma, and corresponding to $(n+1)$ here) is not a loop. Concretely, let us consider the flag matroid $(\underline{\mu}_1, \underline{\mu}_2)$ on $[5]$ whose sets of bases are

$$(\dagger) \qquad (\{12, 13, 14, 23, 24, 34\}, \{125, 135, 145, 235, 245, 345\}).$$

Note that $\underline{\mu}_1$ is a matroid in which 5 is a loop, while $\underline{\mu}_2$ is a matroid in which 5 is a coloop. Setting $S = \{5\}$ and $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$, we find that the positive-tropical three-term Grassmann–Plücker relations can fail to hold for μ_1 and μ_2 even if the positive-tropical three-term incidence-Plücker relations hold for (μ_1, μ_2).

For instance, define (μ_1, μ_2), with support $(\underline{\mu}_1, \underline{\mu}_2)$, by assigning the non-infinity values as follows, with the bases ordered as in (†):

$$(\{0, 1, 0, 0, 1, 0\}, \{0, 1, 0, 0, 1, 0\}).$$

One verifies that (μ_1, μ_2) satisfies every positive-tropical three-term incidence relations, but μ_1 and μ_2 both do not satisfy the positive-tropical three-term Grassmann–Plücker relations.

2. Proof of $(d) \implies (b)$

The proof of (d) $\Longrightarrow$ (b) given below is largely identical to the proof of (d) $\Longrightarrow$ (e) in [BEW24]. The portion that is substantially new is enclosed in symbols $\Diamond\Diamond\Diamond$.

Proof. First, assumption (d) implies that every edge of the subdivision $\mathcal{D}_{\boldsymbol{\mu}}$ of $P(\underline{\boldsymbol{\mu}})$ is a flag matroid polytope, i.e. it is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in [n]$ and its two vertices are equidistant from the origin. Hence the edges of $P(\underline{\boldsymbol{\mu}})$ have the same property, so $\underline{\boldsymbol{\mu}}$ is a flag matroid. By Proposition 3.9, to show (b) it now suffices to show that every positive-tropical three-term Plücker relation is satisfied.

We start with the case $a = b$, where $\boldsymbol{\mu}$ is just (μ). We need check the validity of the three-term positive-tropical Grassmann-Plücker relations, say for an arbitrary choice of $S \in \binom{[n]}{a-2}$ and $\{i < j < k < \ell\} \subseteq [n] \setminus S$. If S is not independent in the matroid $\underline{\mu}$, then every term in the three-term relation involving S and $ijk\ell$ is ∞, so we may assume S is independent. Let $\mathscr{S}$ be a maximal chain $S_1 \subsetneq \cdots \subsetneq S_m$ of subsets of $[n]$ with the property that $S_{a-2} = S$ and $S_{a-1} = S \cup \{ijk\ell\}$. Then, Proposition 5.3 implies that for a vector $\mathbf{v}_{\mathscr{S}}$ in the relative interior of the cone $\mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \dots, \mathbf{e}_{S_m}\}$, we have

$$\operatorname{face}(P(\underline{\mu}), \mathbf{v}_{\mathscr{S}}) = P(\underline{\mu}^{\mathscr{S}}) \simeq P(\underline{\mu}|S \cup ijk\ell/S).$$

For the second identification, we have used that

(1) the matroid polytope of a direct sum of matroids is the product of the matroid polytopes;
(2) with the exception of $(S_{a-2}, S_{a-1}) = (S, S \cup ijk\ell)$, all other minors of the matroid $\underline{\mu}$ corresponding to (S_c, S_{c+1}) in the chain have their polytopes being a point because $|S_{c+1} \setminus S_c| = 1$.

Since S is assumed to be independent, the rank of the matroid minor $\underline{\mu}|S \cup ijk\ell/S$ is at most 2. If it is less than 2, then every term in the three-term relation involving S and $ijk\ell$ is ∞, so let us now treat the case when the rank is exactly 2. For a basis $\widehat{B}$ of $\underline{\mu}|S \cup ijk\ell/S$, let B be the basis of $\underline{\mu}$ such that the vertex $\mathbf{e}_B$ of $P(\underline{\mu})$ corresponds to the vertex $\mathbf{e}_{\widehat{B}}$ of $P(\underline{\mu}|S \cup ijk\ell/S)$ under the identification above. Identifying $[4] = \{1 < 2 < 3 < 4\}$ with $\{i < j < k < \ell\}$, we may thus consider "restricting" μ to the face $P(\underline{\mu}|S \cup ijk\ell/S)$ to obtain an element $\widehat{\mu} = \mu|S \cup ijk\ell/S \in \mathrm{Dr}_{2;4}$ defined by

$$\widehat{\mu}(\widehat{B}) = \begin{cases} \mu(B) & \text{if } \widehat{B} \text{ a basis of } \underline{\mu}|S \cup ijk\ell/S \\ \infty & \text{otherwise} \end{cases} \qquad \text{for } \widehat{B} \in \binom{[4]}{2}.$$

It is straightforward to check that for $\mathrm{Dr}_{2;4}$, the three-term positive-tropical Grassmann-Plücker relations are satisfied if and only if all 2-dimensional faces in the corresponding subdivision are positroid polytopes. Since the faces of the subdivision $\mathcal{D}_{\widehat{\mu}}$ of $P(\underline{\mu}|S \cup ijk\ell/S)$ are a subset of the faces of the subdivision $\mathcal{D}_{\mu}$, we have that μ satisfies the three-term tropical-positive Grassmann-Plücker relation involving $ijk\ell$ and S.

Let us now treat the case $a < b$.
$\Diamond\Diamond\Diamond$ That the three-term Grassmann-Plücker relations are satisfied for every μ_i where $i = a, \dots, b$ follows from our previous case of $a = b$ once we show the following claim:

> For a flag matroid $\underline{\boldsymbol{\mu}}$ with consecutive rank sequence $(a, \dots, b)$, if every face of $P(\underline{\boldsymbol{\mu}})$ of dimension at most 2 is a flag positroid polytope, then the same holds for every constituent matroid, i.e. for every $c = a, \dots, b$, every face of $P(\underline{\mu}_c)$ of dimension at most 2 is a positroid polytope .

To prove the claim, suppose for some $a \le c \le b$ that a 2-dimensional face Q of $P(\underline{\mu}_c)$ is not a positroid polytope. Our goal is to use Q to find a 2-dimensional face of $P(\underline{\boldsymbol{\mu}})$ that is not a flag positroid polytope. By [ŁPW23, Theorem 3.9], a 2-dimensional matroid polytope which is not a positroid polytope has vertices of the form $\mathbf{e}_{Sij}, \mathbf{e}_{Sk\ell}, \mathbf{e}_{Si\ell}, \mathbf{e}_{Sjk}$, where $S \subset [n]$ with $|S| = c-2$ and $\{i < j < k < \ell\} \subset [n] \setminus S$; thus $Q = \operatorname{conv}(\mathbf{e}_{Sij}, \mathbf{e}_{Sk\ell}, \mathbf{e}_{Si\ell}, \mathbf{e}_{Sjk})$ for such $\{S, i, j, k, l\}$[1]. Note that this 2-face Q is the Minkowski sum of $\mathbf{e}_S$ with the product $\operatorname{conv}(\mathbf{e}_i, \mathbf{e}_k) \times \operatorname{conv}(\mathbf{e}_j, \mathbf{e}_\ell)$.

Let $\mathscr{S}$ be a maximal chain $S_1 \subsetneq \cdots \subsetneq S_m$ of subsets of $[n]$ with the property that $S_{c-1} = S$, $S_c = S \cup ik$, and $S_{c+1} = S \cup ijk\ell$. Then, Proposition 5.3 implies that for a vector $v_{\mathscr{S}}$ in the relative interior of the cone $\mathbb{R}_{\ge 0}\{\mathbf{e}_{S_1}, \dots, \mathbf{e}_{S_m}\}$, we have

$$\operatorname{face}(P(\underline{\boldsymbol{\mu}}), \mathbf{v}_{\mathscr{S}}) = P(\underline{\boldsymbol{\mu}}^{\mathscr{S}}) \simeq P(\underline{\boldsymbol{\mu}}|S_{c+1}/S_c) \times P(\underline{\boldsymbol{\mu}}|S_c/S_{c-1}).$$

For the second identification, we have used that

(1) the matroid polytope of a direct sum of matroids is the product of the matroid polytopes;
(2) with the exception of $(S_{c-1}, S_c) = (S, S \cup ik)$ and $(S_c, S_{c+1}) = (S \cup ik, S \cup ijkl)$, all other minors of the constituent matroids of $\underline{\boldsymbol{\mu}}$ corresponding to (S_d, S_{d+1}) in the chain have their polytopes being a point because $|S_{d+1} \setminus S_d| = 1$.

Note that the polytope $P(\underline{\boldsymbol{\mu}}|S_{c+1}/S_c) \times P(\underline{\boldsymbol{\mu}}|S_c/S_{c-1})$ is at most 2-dimensional since $\underline{\boldsymbol{\mu}}|S_{c+1}/S_c$ and $\underline{\boldsymbol{\mu}}|S_c/S_{c-1}$ are flag matroids on ground sets $\{j, \ell\}$ and $\{i, k\}$, respectively. The polytope has Q as a Minkowski summand, and thus in particular is not a flag positroid polytope. $\Diamond\Diamond\Diamond$

Lastly, we check the validity of the three-term positive-tropical incidence-Plücker relations, say for an arbitrary choice of $S \subset [n]$ with $a - 1 \le |S| \le b - 2$ and $\{i < j < k\} \subseteq [n] \setminus S$. We may assume that S has rank $|S|$ in the matroid $\mu_{|S|+1}$, since otherwise every term in the three-term positive-tropical incidence relation is ∞, so that the relation is vacuously satisfied. Let $\mathscr{S}$ be a maximal chain $S_1 \subsetneq \cdots \subsetneq S_m$ of subsets of $[n]$ with the property that $S_c = S$ and $S_{c+1} = S \cup ijk$ for $c = |S|$. Then, Proposition 5.3 implies that for a vector $\mathbf{v}_{\mathscr{S}}$ in the relative interior of the cone $\mathbb{R}_{\ge 0}\{\mathbf{e}_{S_1}, \dots, \mathbf{e}_{S_m}\}$, we have

$$\operatorname{face}(P(\underline{\boldsymbol{\mu}}), \mathbf{v}_{\mathscr{S}}) = P(\underline{\boldsymbol{\mu}}^{\mathscr{S}}) \simeq P(\underline{\boldsymbol{\mu}}|S \cup ijk/S).$$

For the second identification, we have used that

(1) the matroid polytope of a direct sum of matroids is the product of the matroid polytopes;
(2) with the exception of $(S_c, S_{c+1}) = (S, S \cup ijk)$, all other minors of the constituent matroids of $\underline{\boldsymbol{\mu}}$ corresponding to (S_d, S_{d+1}) in the chain have their polytopes being a point because $|S_{d+1} \setminus S_d| = 1$.

[1] One may also deduce this independently of [ŁPW23] by using the argument given in the first third of this proof of (d) $\Longrightarrow$ (b) concerning the $a = b$ case.

Note that the polytope $P(\underline{\boldsymbol{\mu}}|S\cup ijk/S)$ is at most 2-dimensional since it is a flag matroid polytope on 3 elements. Similarly to the $a = b$ case, we may "restrict" $\underline{\boldsymbol{\mu}}$ to the face $P(\underline{\boldsymbol{\mu}}|S \cup ijk/S)$ to obtain an element $\widehat{\boldsymbol{\mu}} = \boldsymbol{\mu}|S \cup ijk/S \in \mathrm{FlDr}_{\widehat{\boldsymbol{r}};3}$. We may assume that $\widehat{\boldsymbol{r}} = (1,2)$ since otherwise every term in the three-term incidence relation of the pair (S, ijk) is ∞. For FlDr_3, it is straightforward to verify that the unique three-term positive-tropical incidence relation involving S and ijk is satisfied if and only if the subdivision $\mathcal{D}_{\widehat{\boldsymbol{\mu}}}$ consists only of flag positroid polytopes. Since the faces of the subdivision $\mathcal{D}_{\widehat{\boldsymbol{\mu}}}$ are a subset of the faces of the subdivision $\mathcal{D}_{\boldsymbol{\mu}}$, we have that $\boldsymbol{\mu}$ satisfies the three-term incidence relation involving S and $\{i,j,k\}$. ☐

3. Modification of Theorem 6.1 and statement (e)

We now state a revised version of Theorem 6.1.

Theorem 6.1$^{\text{rev}}$. Suppose $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r}}\right) \times \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r+1}}\right)$ satisfies every three-term positive-tropical incidence relation, and suppose that the support $\underline{\boldsymbol{\mu}}$ is a flag matroid. Then, we have $\boldsymbol{\mu} \in \mathrm{FlDr}^{\geq 0}_{r,r+1;n}$ if either of the following (incomparable) conditions hold:

(i) The support $\underline{\boldsymbol{\mu}}$ consists of uniform matroids.
(ii) Either $\mu_1 \in \mathrm{Dr}^{\geq 0}_{r;n}$ or $\mu_2 \in \mathrm{Dr}^{\geq 0}_{r+1;n}$.

Correspondingly, we modify the statement (e) in Theorem A to the following

(e′) The support $\underline{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ is a flag matroid, $\boldsymbol{\mu}$ satisfies every three-term positive-tropical incidence relation, and either $\underline{\boldsymbol{\mu}}$ consists of uniform matroids or $\mu_i \in \mathrm{Dr}^{\geq 0}_{i;n}$ for at least one $a \leq i \leq b$.

Proof of (b) $\iff$ (e′). By Proposition 3.9, the implication (b) $\implies$ (e′) is immediate. The converse follows from Theorem 6.1$^{\text{rev}}$ and Proposition 3.9. ☐

Proof of Theorem 6.1^{rev}. The proof is identical to the original proof of Theorem 6.1 in [BEW24], with the difference that we need to use hypotheses (i) and (ii) to justify the conclusions of the last two bullet points in the proof of Theorem 6.1, reproduced below:

- $(n+1) \in S$. *In this case, considering the minor* $\widetilde{\boldsymbol{\mu}}|S \cup ijk\ell/(S \setminus (n+1))$ *and then applying Corollary 6.4 implies that the three-term Grassmann-Plücker relation is satisfied.*
- $(n+1) \notin S \cup ijk\ell$. *In this case, considering the minor* $\widetilde{\boldsymbol{\mu}}|S \cup ijk\ell(n+1)/S$ *and then applying Lemma 6.2 implies that the three-term Grassmann-Plücker relation is satisfied.*

Under condition (i), i.e. when the support $\underline{\boldsymbol{\mu}}$ consists of uniform matroids, the element $(n+1)$ is not a coloop in the minor $\widetilde{\boldsymbol{\mu}}|S \cup ijk\ell/(S \setminus (n+1))$, and is not a loop in the minor $\widetilde{\boldsymbol{\mu}}|S \cup ijk\ell(n+1)/S$. Hence, both Corollary 6.4 and Lemma 6.2 apply respectively.

Now suppose condition (ii) holds. We verify that in the cases where Corollary 6.4 and Lemma 6.2 do not apply, the theorem still holds. Let us consider the second bullet point, and suppose that $(n+1)$ is a loop in the minor $\widetilde{\boldsymbol{\mu}}|S \cup ijk\ell(n+1)/S$, i.e. where Lemma 6.2 does not apply; the argument for the first bullet point is similar by matroid duality. In this case, since $(n+1)$ is not a loop in the matroid $\underline{\widetilde{\boldsymbol{\mu}}}$, $(n+1)$ belongs to the closure (also called *span*) in $\underline{\widetilde{\boldsymbol{\mu}}}$ of S. Since S is also independent, there is an element $s \in S$ such that $(S \setminus s) \cup (n+1)$ is independent and has the same closure as S in $\underline{\widetilde{\boldsymbol{\mu}}}$. Let $S' = S \setminus s$. For any $a, b \in \{i, j, k, \ell\}$, by our choice of $s \in S$, we have that Sab is a basis of $\underline{\widetilde{\boldsymbol{\mu}}}$ if and only if $S'ab(n+1)$ is a basis of $\underline{\widetilde{\boldsymbol{\mu}}}$. Moreover, for any $a, b, c \in \{i, j, k, \ell\}$ such that the values involved below are finite, we claim

$$\widetilde{\boldsymbol{\mu}}(Sab) - \widetilde{\boldsymbol{\mu}}(Sac) = \widetilde{\boldsymbol{\mu}}(S'ab(n+1)) - \widetilde{\boldsymbol{\mu}}(S'ac(n+1)).$$

Note that using the definition of $\widetilde{\boldsymbol{\mu}}$, the above claim can be equivalently written as

$$\mu_2(Sab) - \mu_2(Sac) = \mu_1(S'ab) - \mu_1(S'ac).$$

From the claim, we conclude as follows. Let $\overline{\mu}_1$ be the projection of μ_1 to the coordinates labelled by $S'xy$ where $x \neq y \in \{i, j, k, \ell\}$, and let $\overline{\mu}_2$ be the projection of μ_2 to the coordinates labelled by Sxy where $x \neq y \in \{i, j, k, \ell\}$. Then, as elements of $\mathbb{P}(\mathbb{T}^{\binom{\{i,j,k,\ell\}}{2}})$, the two tropical vectors $\overline{\mu}_1$ and $\overline{\mu}_2$ are equal. Hence, the claim implies that if one of μ_1 or μ_2 satisfies the three-term Grassmann-Plücker relations on these coordinates, then so does the other.

The claim follows from the validity of three-term tropical incidence relations, which is implied by the validity of three-term positive-tropical incidence relations. Namely, we have that the minimum is achieved at least twice in

$$\{\mu_1(S'ab) + \mu_2(S'asc), \mu_1(S'as) + \mu_2(S'abc), \mu_1(S'ac) + \mu_2(S'asb)\},$$

from which the claim follows because $Sa(n+1)$ is not a basis of $\underline{\widetilde{\boldsymbol{\mu}}}$, forcing $\mu_1(S'as) = \infty$. $\square$

References

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