An Introduction to Toric Varieties

submitted by

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CHAPTER 1

Introduction

1.1 Introduction

Toric varieties form a small but wonderful class of algebraic varieties that is easy to work and compute with. They provide, as Fulton writes in [Ful93], "a remarkably fertile testing ground for general theories." When the study of toric varieties was first introduced in the early 1970s, it didn't take long for the subject to be recognized as a powerful technique in algebraic geometry. By the 1980s, the study of toric varieties was a rapidly growing field, producing results pertinent to algebraic geometry, commutative algebra, combinatorics, and convex geometry.

Loosely put, toric varieties are irreducible varieties containing an algebraic torus as an open dense subset. They can be defined by monomial mappings or equivalently by semigroups embedded in a lattice. Thus, one can expect an interaction between combinatorics on a lattice and toric varieties. What may be surprising is the extent of the deep connection that toric varieties have with the combinatorics of convex geometry such as cones, polytopes, and polyhedra. For example, Richard Stanley was able to give a succinct proof of the McMullen conjecture on the face numbers of a polytope using ideas from toric varieties. The applications of toric varieties have in fact reached diverse areas such as physics, coding theory, algebraic statistics, symplectic geometry, and integer programming.

This thesis aims to introduce the basic elements of toric varieties. The focus will be to illustrate how the combinatorial data of toric varieties provide ease in working and computing with toric varieties, while retaining a rich class of theory and examples. We start off with a careful study of affine toric varieties, how they are defined by semigroups and equivalently by monomial parameterizations. We will see that normal affine toric varieties are particularly nice in that they correspond to polyhedral cones. We then move on to construct abstract normal toric varieties by patching together affine normal toric varieties via data of a fan. For a rich set of examples of toric varieties, we then study projective toric varieties and discuss how they can be associated to polytopes. Our introduction of toric varieties culminates in two illustrative examples in which the combinatorics greatly informs the geometry of toric varieties. The first is the orbit-cone correspondence theorem, and the second is classification of all complete smooth normal toric surfaces.

1.2 A short note on notations

In this document, the rings under consideration will always be finitely generated $\mathbb{C}$-algebras, and hence a ring map $\phi : A \to B$ induces a map $\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ that always maps closed points to closed points. In this light, we will write "Spec" for "max Spec" unless otherwise noted. By a **lattice** we mean a finitely generated free abelian group. For $\mathbb{Z}^n$, we denote the standard basis by $\{e_1, \dots, e_n\}$. For a group G, we denote by $\mathbb{C}[G]$ its group algebra over $\mathbb{C}$.

CHAPTER 2

Affine Toric Varieties

In this chapter we introduce and describe affine toric varieties. As the name "toric variety" implies, affine toric varieties have two main structures, a torus and an algebraic variety. We thus start with careful a description of the algebraic torus. While we will eventually only consider *normal* toric varieties, it is not harder to discuss basic properties of affine toric varieties in the most general case. So, we then give the most general definition of affine toric varieties along with two main ways to construct them: by an affine semigroup and by a monomial parameterization. Then we describe affine semigroups come from rational polyhedral cones define normal affine toric varieties. We conclude with the study of toric morphisms, particularly toric morphisms of normal affine toric varieties.

2.1 The torus

The rich interaction between toric varieties and convex geometry starts at tori and the lattices associated to them. Hence, we first give a careful study of algebraic tori in this section in preparation for introducing affine toric varieties.

Definition 2.1.1. *An* ***affine algebraic group*** *is an affine variety V with a group structure, whose binary operation $V \times V \to V$ is given as a morphism of varieties. The set of* ***algebraic maps*** *of two algebraic groups V, W, denoted $\mathrm{Hom}_{alg}(V, W)$, is the set of group homomorphisms $\phi : V \to W$ that are also morphisms of affine varieties.*

The example most important and relevant to us is $(\mathbb{C}^*)^n \simeq \mathbb{C}^n - V(x_1x_2\cdots x_n) \simeq V(1 - x_1\cdots x_n y) \subset \mathbb{C}^{n+1}$. It is an affine variety, and its coordinate ring is indeed $\mathbb{C}[x_1, \ldots, x_n]_{x_1\cdots x_n} = \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}] \simeq \mathbb{C}[\mathbb{Z}^n]$. The group operation $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ by coordinate-wise multiplication is given by $\mathbb{C}$-algebra homomorphism $\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \to \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}] \otimes \mathbb{C}[y_1^{\pm}, \ldots, y_n^{\pm}]$ defined by $t_i \mapsto x_i \otimes y_i$. This algebraic group $(\mathbb{C}^*)^n$ is the archetype of tori, for which we now give a formal definition.

Definition 2.1.2. *A* ***torus*** *is an affine variety isomorphic to $(\mathbb{C}^*)^n$ for some n, whose group structure is inherited from that of $(\mathbb{C}^*)^n$ via the isomorphism.*

Example 2.1.3. Let $V = V(x^2 - y) \subset \mathbb{C}^2$, and consider $V_{xy} = V \cap (\mathbb{C}^*)^2$. Since V_{xy} is the graph of the map $\mathbb{C}^* \to \mathbb{C}$ given by $t \mapsto t^2$, the morphism $\mathbb{C}^* \to V_{xy}$ given by $t \mapsto (t, t^2)$ is indeed bijective, and this isomorphism gives V_{xy} the structure of $\mathbb{C}^*$ by $(a, a^2) \cdot (b, b^2) = (ab, (ab)^2)$.

It is an important feature that algebraic maps between tori have a particularly nice form:

Proposition 2.1.4. *A map $\phi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ is **algebraic** if and only if the corresponding map of coordinate rings $\phi^* : \mathbb{C}[y_1^\pm, \ldots, y_m^\pm] \to \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ is given by $y_i \mapsto x^{\alpha_i}$ for $\alpha_i \in \mathbb{Z}^n$ $(i = 1, \ldots, m)$. In other words, algebraic maps $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ correspond bijectively to maps of lattices $\mathbb{Z}^m \to \mathbb{Z}^n$.*

Proof. Since y_i is a unit in $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm]$, it must map to a unit in $\mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$, which means that $\phi^*(y_i)$ is a monomial cx^{α_i}. Now, that ϕ is a group homomorphism implies that $\phi(1, \ldots, 1) = (1, \ldots, 1)$, and thus $c = 1$. The latter statement follows immediately from the first. □

Define Tori to be the category whose objects are tori and morphisms are algebraic maps, and define $\mathsf{Lattice}$ to be the category of (finitely generated) lattices. The proposition above implies that $\mathrm{Hom}_{alg}((\mathbb{C}^*)^n, \mathbb{C}^*) \simeq \mathbb{Z}^n$, and hence one can check that $\mathrm{Hom}_{alg}(-, \mathbb{C}^*)$ is a contravariant functor $\mathsf{Tori} \to \mathsf{Lattice}$. We likewise have a covariant functor $\mathrm{Hom}_{alg}(\mathbb{C}^*, -) : \mathsf{Tori} \to \mathsf{Lattice}$.

Note that the two functors are in fact category equivalences, as implied by the second part of above proposition. This will be handy, as we will often use the fact that a contravariant category equivalence reverses monomorphisms and epimorphisms. These two functors are the starting points of interaction between toric varieties and rational convex geometry, and are given names as follows:

Definition 2.1.5. *A **character** of a torus T is a map $\chi \in \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$, and $M := \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ is called the **character lattice** of T. Likewise, a **one-parameter subgroup** of T is a map $\lambda \in \mathrm{Hom}_{alg}(\mathbb{C}^*, T)$, and $N := \mathrm{Hom}_{alg}(\mathbb{C}^*, T)$ is called the **lattice of one-parameter subgroups**.*

We will reserve letters M and N as the character lattice and one-parameter subgroup of a torus. $\mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \simeq \mathbb{C}[\mathbb{Z}^n]$ is naturally the coordinate ring of $(\mathbb{C}^*)^n$ because χ^{e_i} as an element of $\mathbb{Z}^n \simeq \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*)$ is the ith coordinate map of $(\mathbb{C}^*)^n$. Hence, in a likewise manner the coordinate ring of a torus T with character lattice M is naturally $\mathbb{C}[M]$. One can check that the group multiplication map $T \times T \to T$ is given by map of rings $\mathbb{C}[M] \to \mathbb{C}[M] \otimes \mathbb{C}[M]$, $\chi^m \mapsto \chi^m \otimes \chi^m$.

Note that there is a natural bilinear pairing $M \times N \to \mathbb{Z}$ given via the composition map $M \times N = \mathrm{Hom}_{alg}(T, \mathbb{C}^*) \times \mathrm{Hom}_{alg}(\mathbb{C}^*, T) \to \mathrm{Hom}_{alg}(\mathbb{C}^*, \mathbb{C}^*) \simeq \mathbb{Z}$. Choosing an isomorphism $T \simeq (\mathbb{C}^*)^n$ gives a natural dual bases for M, N as $\mathbb{Z}^n$, and under this choice of bases the map $M \times N \to \mathbb{Z}$ is the usual Euclidean inner product. Thus, the pairing is perfect. Moreover, because $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[M] \to \mathbb{C}$ bijectively correspond to group homomorphisms $M \to \mathbb{C}^*$, we now have a canonical isomorphism $T \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Thus, we denote a torus with lattice of one-parameter subgroups N as T_N and its coordinate ring $\mathbb{C}[M]$.

2.2 Semigroups and affine toric varieties

Here we give the definition of affine toric varieties. We show how an affine semigroup defines an affine toric variety and how a monomial parameterization equivalently defines an affine toric variety. We then finish with the discussion toric morphisms.

Definition 2.2.1. *An **affine toric variety** V is an irreducible affine variety such that (i) it contains a torus T as a Zariski open subset and (ii) the action of T on itself extends to an action $T \times V \to V$ given as a morphism of varieties.*

When the context is clear, we denote by M and N the character lattice and lattice of one-parameter subgroups of the torus T_N of the affine toric variety.

Here are two simple examples of affine toric varieties:

Example 2.2.2.

- $\mathbb{C}^n$ is an obvious example, containing $(\mathbb{C}^*)^n$ as the torus and $(\mathbb{C}^*)^n \times \mathbb{C}^n \to \mathbb{C}^n$ given by $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[y_1^{\pm}, \ldots, y_n^{\pm}] \otimes \mathbb{C}[z_1, \ldots, z_n],\ x_i \mapsto y_i \otimes z_i$.
- Consider $V := V(x^2 - y) \subset \mathbb{C}^2$, which as we have seen in Example 2.1.3 contains the torus $\{(t, t^2) \mid t \neq 0\} = V_{xy} \simeq \mathbb{C}^*$ as an open subset. Moreover, the torus action easily extends to V. It is the regular multiplication map: for $(a, b) \in V$, $(t, t^2) \cdot (a, b) = (ta, t^2 b)$.

Our first main result is that an affine semigroup defines an affine toric variety. An **affine semigroup** (or just **semigroup** if there is no ambiguity) is a finitely generated commutative semigroup with an identity that can be embedded into a lattice. Given a subset $\mathscr{A} \subset M$ of some lattice M, we denote by $\mathbb{N}\mathscr{A}$ the semigroup generated by $\mathscr{A}$ and by $\mathbb{Z}\mathscr{A}$ the sublattice of M generated by $\mathscr{A}$. Given a semigroup S, we define $\mathbb{C}[S]$, the **$\mathbb{C}$-algebra of a semigroup** S, as

$$\mathbb{C}[S] := \{\sum_{m \in S} a_m \chi^m \mid a_m \in \mathbb{C},\ a_m = 0 \text{ for all but finitely many } m\}$$

with the multiplication given by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. For example, we have $\mathbb{C}[\mathbb{N}^n] \simeq \mathbb{C}[x_1, \ldots, x_n]$ via $\chi^{e_i} \simeq x_i$. Likewise, we have $\mathbb{C}[\mathbb{Z}^n] \simeq \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}] \simeq \mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n}$.

We now show that an affine variety defined as $\operatorname{Spec} \mathbb{C}[S]$ for some affine semigroup S is in fact an affine toric variety:

Theorem 2.2.3. *Let S be an affine semigroup. Then* $\operatorname{Spec} \mathbb{C}[S]$ *is an affine toric variety with the torus* $\operatorname{Spec} \mathbb{C}[\mathbb{Z}S]$.

Proof. Let $V = \operatorname{Spec} \mathbb{C}[S]$ and $T = \operatorname{Spec} \mathbb{C}[\mathbb{Z}S]$. We first check that $T \hookrightarrow V$ is an embedding of an open dense subset. Since S is an affine semigroup, it is embedded into a lattice M', and has a finite generating set $\mathscr{A} = \{m_1, \ldots, m_s\} \subset M'$ (so $\mathbb{N}\mathscr{A} = S$ and $\mathbb{Z}\mathscr{A} = \mathbb{Z}S$). Note that $\mathbb{C}[\mathbb{N}\mathscr{A}]_{\chi^{m_1} \cdots \chi^{m_s}} \simeq \mathbb{C}[\mathbb{Z}\mathscr{A}]$, and so $\mathbb{C}[\mathbb{N}\mathscr{A}] \hookrightarrow \mathbb{C}[\mathbb{Z}\mathscr{A}]$ is given by a localization. Moreover, a $\mathbb{C}$-algebra homomorphism out of $\mathbb{C}[\mathbb{Z}\mathscr{A}]$ is completely determined by the values that $\chi^{m_1}, \ldots, \chi^{m_s}$ take, and hence $\mathbb{C}[\mathbb{N}\mathscr{A}] \hookrightarrow \mathbb{C}[\mathbb{Z}\mathscr{A}]$ is an epimorphism. Thus, since $\mathbb{C}[S]$ is clearly an integral domain, the map $\mathbb{C}[S] \hookrightarrow \mathbb{C}[\mathbb{Z}S]$ induces an inclusion $T \hookrightarrow V$ as an open dense subset.

To show that the action of T on itself extends to V. Consider the map $\phi : \mathbb{C}[S] \to \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S]$ given by $\chi^m \mapsto \chi^m \otimes \chi^m$, which gives us a map $\phi^* : T \times V \to V$. Since the map $\widetilde{\phi} : \mathbb{C}[\mathbb{Z}S] \to \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[\mathbb{Z}S],\ \chi^m \mapsto \chi^m \otimes \chi^m$ induces the torus action $\widetilde{\phi}^* : T \times T \to T$, we see from the following commuting diagram that ϕ^* is indeed an extension of the torus action on itself:

$$\begin{tikzcd}
\mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S] \arrow[d, hook] & \mathbb{C}[S] \arrow[l, "\phi"'] \arrow[d, hook] & \overset{\text{Spec}}{\rightsquigarrow} & T \times V \arrow[r, "\phi^*"] & V \\
\mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[\mathbb{Z}S] & \mathbb{C}[\mathbb{Z}S] \arrow[l, "\widetilde{\phi}"] & & T \times T \arrow[u, hook] \arrow[r, "\widetilde{\phi}^*"'] & T \arrow[u, hook]
\end{tikzcd}$$

Lastly, we check that ϕ^* is a group action. Let $\widetilde{1}$ be the identity element of T. Then $\widetilde{1} \cdot x = x$ follows from the commutativity of the following diagram

$$\begin{tikzcd}
\widetilde{1} \times V \arrow[r, hook] & T \times V \arrow[r] & V \\
\widetilde{1} \times T \arrow[u, hook] \arrow[r, hook] & T \times T \arrow[u, hook] \arrow[r] & T \arrow[u, hook]
\end{tikzcd}$$

since in the bottom row $T \simeq \widetilde{1} \times T \to T$ is the identity map and T is an open dense subset of V, implying that $\widetilde{1} \times V \to V$ is also the identity map by [Vak15, 10.2.2]. Moreover, for $a, b \in T$, that $(ab) \cdot x = a \cdot (b \cdot x)$ follows from applying Spec to the commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S] & \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S] \arrow[l, "\mathrm{Id} \otimes \phi"'] \\
\mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S] \arrow[u, "\widetilde{\phi} \otimes \mathrm{Id}"] & \mathbb{C}[S] \arrow[l, "\phi"] \arrow[u, "\phi"']
\end{tikzcd}$$

where $\mathrm{Id} \otimes \phi : \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S] \to \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S]$ is defined by $\chi^m \otimes \chi^{m'} \mapsto \chi^m \otimes \chi^{m'} \otimes \chi^{m'}$ and $\widetilde{\phi} \otimes \mathrm{Id} : \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S] \to \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S]$ is defined by $\chi^m \otimes \chi^{m'} \mapsto \chi^m \otimes \chi^m \otimes \chi^{m'}$. □

Remark 2.2.4. In fact, all affine toric varieties arise from affine semigroups; for proof, see [CLS11, 1.1.17]

The proof for Theorem 2.2.3 provides a general and useful way to construct affine toric varieties. We start with a lattice M' and select a finite set $\{m_1, \ldots, m_s\}$ so as to make a semigroup $\mathbb{N}\mathscr{A}$, and this then defines the affine toric variety $\operatorname{Spec} \mathbb{C}[\mathbb{N}\mathscr{A}]$. The following proposition gives us a way to embed $\operatorname{Spec} \mathbb{C}[\mathbb{N}\mathscr{A}]$ in $\mathbb{C}^s$.

Proposition 2.2.5. *Let $T_{N'}$ be a torus with character lattice M', and let $\mathscr{A} = \{m_1, \ldots, m_s\} \subset M'$. Let L be the kernel of the map $\mathbb{Z}^s \to M'$ defined by $e_i \mapsto m_i$. Then there is an isomorphism $i : \mathbb{C}[x_1, \ldots, x_s]/I_L \xrightarrow{\sim} \mathbb{C}[\mathbb{N}\mathscr{A}]$ via $x_i \mapsto \chi^{m_i}$ where $I_L \subset \mathbb{C}[x_1, \ldots, x_s]$ is an ideal defined as*

$$I_L := \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s,\ \alpha - \beta \in L \rangle$$

In other words, $\operatorname{Spec} \mathbb{C}[\mathbb{N}\mathscr{A}] \simeq V(I_L) \subset \mathbb{C}^s$.

Proof. It suffices to show that I_L is the kernel of the map $\phi : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}[\mathbb{N}\mathscr{A}]$ given as $x_i \mapsto \chi^{m_i}$. First, $I_L \subset \ker \phi$ not very difficult: if $\alpha = (\alpha_1, \ldots, \alpha_s), \beta = (\beta_1, \ldots, \beta_s)$ such that $\alpha - \beta \in L$, then $\sum \alpha_i m_i = \sum \beta_i m_i$, and so $\phi(x^\alpha - x^\beta) = \chi^{\sum \alpha_i m_i} - \chi^{\sum \beta_i m_i} = 0$.

Now suppose for contradiction that $I_L \subsetneq \ker \phi$. Pick a monomial order on $\mathbb{C}[x_1, \ldots, x_n]$ and an isomorphism $T_{N'} \simeq (\mathbb{C}^*)^n$ so that $M' = \mathbb{Z}^n$ with $m_1, \ldots, m_s$ now considered as elements in $\mathbb{Z}^n$. Since $I_L \subsetneq \ker \phi$, there exists a nonzero $f \in \ker \phi \backslash I_L$ with minimal multi-degree among nonzero elements in $\ker \phi \backslash I_L$. Rescaling f if necessary, let x^α be its leading term.

Now, define $\widetilde{\phi} : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}[\mathbb{Z}^n]$ by $\widetilde{\phi} := j \circ \phi$ where $j : \mathbb{C}[\mathbb{N}\mathscr{A}] \hookrightarrow \mathbb{C}[M'] = \mathbb{C}[\mathbb{Z}^n]$. Since $f \in \ker \widetilde{\phi}$, the map $f^* : (\mathbb{C}^*)^n \to \mathbb{C}^s$ which is given by $t \mapsto f(\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) = f(t^{m_1}, \ldots, t^{m_s})$ is identically zero for all $t \in (\mathbb{C}^*)^n$, and thus the Laurent polynomial $f(t^{m_1}, \ldots, t^{m_s})$ itself is zero. Thus, in order to cancel out the term coming from x^α, f must have at least one monomial x^β ($\beta < \alpha$) such that $\prod_i (t^{m_i})^{\alpha_i} = \prod_i (t^{m_i})^{\beta_i}$, implying $\sum \alpha_i m_i = \sum \beta_i m_i$. Hence, $x^\alpha - x^\beta \in I_L \subset \ker \phi$, and so we have $f - x^\alpha + x^\beta \in \ker \phi \backslash I_L$, a polynomial of multi-degree lower than that of f, which contradicts our minimality condition on f. □

Remark 2.2.6. The previous proposition implies that given an affine toric variety $\operatorname{Spec} \mathbb{C}[S]$ for some affine semigroup S, selecting any finite subset $\mathscr{A} = \{m_1, \ldots, m_s\}$ such that $\mathbb{N}\mathscr{A} = S$ gives us an embedding $\operatorname{Spec} \mathbb{C}[S] \hookrightarrow \mathbb{C}^s$. In fact, we can even let $\mathscr{A}$ to be a multiset. Moreover, with respect to this embedding $\operatorname{Spec} \mathbb{C}[\mathbb{N}\mathscr{A}] \hookrightarrow \mathbb{C}^s$, we note that $\chi^{m_1}, \ldots, \chi^{m_s}$ are in fact the maps into each coordinates; in other words, if π_i is the projection $\mathbb{C}^s \to \mathbb{C}$ onto the ith coordinate, then χ^{m_i} is the map $\operatorname{Spec} \mathbb{C}[\mathbb{N}\mathscr{A}] \hookrightarrow \mathbb{C}^s \xrightarrow{\pi_i} \mathbb{C}$.

Since $\chi^m \in \mathbb{C}[\mathbb{N}\mathscr{A}]$ as a map $\operatorname{Spec}\mathbb{C}[\mathbb{N}\mathscr{A}] \to \mathbb{C}$ is the unique extension of the map $\chi^m : \operatorname{Spec}\mathbb{C}[\mathbb{Z}\mathscr{A}] \to \mathbb{C}^*$, the previous proposition suggests that one may be able to describe the embedding $\operatorname{Spec}\mathbb{C}[\mathbb{N}\mathscr{A}] \hookrightarrow \mathbb{C}^s$ in terms of a parameterization by χ^{m_i}'s as functions from the torus $\operatorname{Spec}\mathbb{C}[\mathbb{Z}\mathscr{A}]$. We describe such parameterization more precisely in the next proposition:

Proposition 2.2.7. *Let $T_{N'}$ be a torus with character lattice M', and let $\{m_1, \ldots, m_s\} \subset M'$, and L be the kernel of $\mathbb{Z}^s \to M'$, $e_i \mapsto m_i$. The affine toric variety $\operatorname{Spec}\mathbb{C}[\mathbb{N}\mathscr{A}] \simeq V(I_L) \subset \mathbb{C}^s$ is the Zariski closure of the map $\Phi : T_{N'} \to \mathbb{C}^s$ defined as $\Phi(t) = (\chi^{m_1}(t), \ldots, \chi^{m_s}(t))$.*

Proof. Denote $M := \mathbb{Z}\mathscr{A}$, $T_N := \operatorname{Spec}\mathbb{C}[\mathbb{Z}\mathscr{A}]$, $V := \operatorname{Spec}\mathbb{C}[\mathbb{N}\mathscr{A}]$. Note that the map $\Phi : T_{N'} \to \mathbb{C}^s$ is given by the map $\varphi : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}[M']$, $x_i \mapsto \chi^{m_i}$. Now, applying Spec to the following commuting diagram:

$$\begin{tikzcd}
\mathbb{C}[M] \arrow[dr, hook] & \mathbb{C}[V] \arrow[l, hook'] \arrow[d, hook] & \mathbb{C}[x_1, \ldots, x_s] \arrow[l, two heads, "i"'] \arrow[dl, "\varphi"] \\
 & \mathbb{C}[M'] &
\end{tikzcd}$$

we obtain

$$\begin{tikzcd}
T_N \arrow[r, hook] & V \arrow[r, hook, "i"] & \mathbb{C}^s \\
 & T_{N'} \arrow[ul] \arrow[u] \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

Hence, $\overline{\operatorname{Im}(\Phi)} = \overline{T_N} = V$ since $i : V \hookrightarrow \mathbb{C}^s$ is a closed embedding. $\square$

Remark 2.2.8. Via $T_{N'} \simeq (\mathbb{C}^*)^n$ and $M' \simeq \mathbb{Z}^n$, the above Proposition 2.2.7 implies that affine toric varieties are ones parameterized by Laurent monomials.

We fix here some notations for future reference. Given a torus $T_{N'}$ with character lattice M' and $\mathscr{A} = \{m_1, \ldots, m_s\} \subset M'$, denote by $Y_{\mathscr{A}}$ the affine toric variety $\operatorname{Spec}\mathbb{C}[\mathbb{N}\mathscr{A}]$, and denote by $\Phi_{\mathscr{A}} : T_{N'} \to \mathbb{C}^s$ the map $\Phi_{\mathscr{A}}(t) = (\chi^{m_1}(t), \ldots, \chi^{m_s}(t))$. Moreover, let L denote the kernel of the map $\mathbb{Z}^s \to M', e_i \mapsto m_i$, and I_L the ideal as given in Proposition 2.2.5. Lastly, when $T_{N'} = (\mathbb{C}^*)^n$ and $M' = \mathbb{Z}^n$ so that $\mathscr{A} = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^n$, we denote by A the $n \times s$ matrix whose columns are $m_1, \ldots, m_s$.

We conclude with concrete examples of the constructions introduced so far:

Example 2.2.9. Let's consider different embeddings of $\operatorname{Spec}\mathbb{C}[\mathbb{N}^2] \simeq \mathbb{C}^2$ into affine spaces by selecting different set of lattice points that generate $\mathbb{N}^2$. Let $\mathscr{A} := \{e_1, e_2\}$ and $\mathscr{A}' := \{e_1, e_2, e_1 + e_2\} \subset \mathbb{Z}^2$. For both cases, we have $\mathbb{N}\mathscr{A} = \mathbb{N}\mathscr{A}' = \mathbb{N}^2$ as semigroups, so $Y_{\mathscr{A}} \simeq Y_{\mathscr{A}'}$. But if we are following the embedding laid out in Proposition 2.2.5, for $\mathscr{A}$ we get $\mathbb{C}^2$, whereas for $\mathscr{A}'$ we get $V(xy - z) \subset \mathbb{C}^3$ since $\ker A'$ has basis $(1, 1, -1)$. Indeed, $\mathbb{C}^2 \simeq V(xy - z)$ via $(x, y) \mapsto (x, y, xy)$.

Example 2.2.10. Taking $\mathscr{A} = \{2, 3\} \in \mathbb{Z}$, the kernel of the matrix $A = [2\ 3]$ has basis $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and hence $\operatorname{Spec}\mathbb{C}[\mathbb{N}\mathscr{A}] \simeq V(x^3 - y^2) \subset \mathbb{C}^2$.

Example 2.2.11 (Singular quadratic cone)**.** The singular quadratic cone $C = V(y^2 - xz)$ defined equivalently by parameterization $C = \{(s, st, st^2) \in \mathbb{C}^3 \mid s, t \in \mathbb{C}\}$ is an affine toric variety. One sees this by noting that $C \cap (\mathbb{C}^*)^3 = \{(s, st, st^2) \mid s, t, \in \mathbb{C}^*\}$ is the image of the map $\Phi_{\mathscr{A}}$ where $\mathscr{A} =$

$\{(1,0),(1,1),(1,2)\} \subset \mathbb{Z}^2$. Since $L = \ker A = \operatorname{span}((-1,2,-1))$, we have that $V(I_L) = V(y^2 - xz)$ is an affine toric variety.

Example 2.2.12. Consider the determinantal variety $V(xy - zw) \subset \mathbb{C}^4$. Since the kernel L of the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ is spanned by $(1,1,-1,-1)$, we have that $V(I_L) = V(xy - zw)$ is an affine toric variety that is parameterized by $\Phi: \ (\mathbb{C}^*)^3 \to \mathbb{C}^4, \ (t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$.

2.3 Cones and normal affine toric varieties

We now turn to the *normal* affine toric varieties. We will see in the coming sections the particularly rich interactions between general normal toric varieties and combinatorial structures of lattices. Here we describe the starting point of such interactions, which is the relationship between affine normal toric varieties and rational polyhedral cones.

Let M, N be dual lattices. A **rational polyhedral cone** $\sigma \subset N_{\mathbb{R}}$ is a set of the form

$$\sigma = \operatorname{Cone}(n_1, \dots, n_k) := \{\lambda_1 n_1 + \dots + \lambda_k n_k \in N_{\mathbb{R}} \mid \lambda_i \geq 0\}$$

for some $n_1, \dots, n_k \in N$. The dimension of a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is defined as the dimension of the smallest linear subspace that σ spans in $N_{\mathbb{R}}$. Given a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, its **dual cone** $\sigma^\vee \subset M_{\mathbb{R}}$ defined as

$$\sigma^\vee := \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$$

is also a rational polyhedral cone. A rational polyhedral cone σ is **strongly convex** if σ contains no positive-dimensional subspace of $N_{\mathbb{R}}$, or equivalently, if $\sigma^\vee \subset M_{\mathbb{R}}$ is full dimensional. For example, the two cones drawn below are strongly convex rational polyhedral cones that are duals of each other:

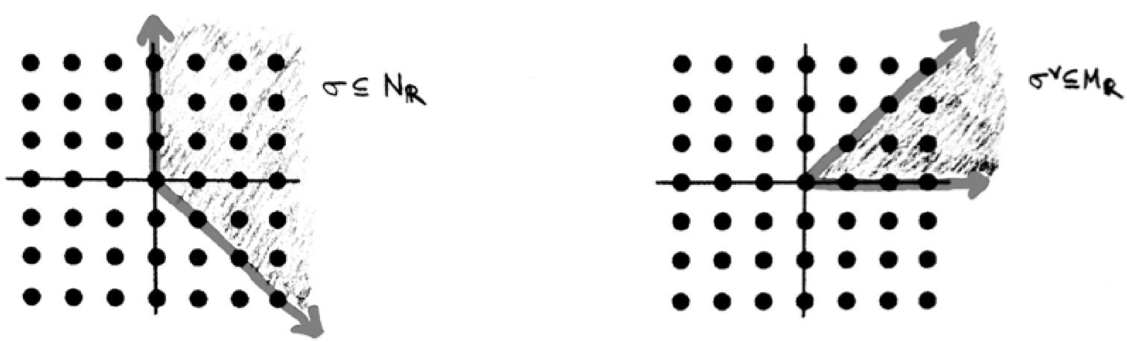


Figure 2.1: $\sigma = \operatorname{Cone}(e_1 - e_2, e_2)$ and its dual $\sigma^\vee = \operatorname{Cone}(e_1, e_1 + e_2)$

As in the figure above, when $M \simeq \mathbb{Z}^n \simeq N$, as long as there is no ambiguity we will use $\{e_1, \dots, e_n\}$ as basis for both M and N, although they are technically dual spaces.

Gordan's Lemma ([CLS11, Proposition 1.2.17]) tells us that given a rational polyhedral cone $\tau \subset M_{\mathbb{R}}$, its lattice points $\tau \cap M$ form an affine semigroup. Thus, given a rational polyhedral cone

$\sigma \subset N_{\mathbb{R}}$, let the affine semigroup associated to σ be $S_\sigma := \sigma^\vee \cap M$. That we associate to the cone $\sigma \subset N_{\mathbb{R}}$ the semigroup defined by its *dual* $\sigma^\vee \subset M_{\mathbb{R}}$ may seem odd for now, but the reason will be clear later. We can now define the affine toric variety associated to a rational polyhedral cone:

Theorem 2.3.1. *Let $\sigma \subset N_{\mathbb{R}}$ be a rational polyhedral cone, and define $S_\sigma := \sigma^\vee \cap M$ to be the semigroup associated to it. Then*

$$U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$$

is an affine toric variety, and moreover, the torus of U_σ is $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ if and only if σ is strongly convex.

Proof. That U_σ is an affine toric variety follows from the fact that S_σ is an affine semigroup by Gordan's Lemma. Now, T_N is the torus of U_σ if and only if $\mathbb{Z}S_\sigma = M$. One can check that if $km \in \mathbb{Z}S_\sigma$ for some $k \in \mathbb{N}$ and $m \in M$, then $m \in \mathbb{Z}S_\sigma$. Thus, $\mathbb{Z}S_\sigma = M$ if and only if $\operatorname{rank} \mathbb{Z}S_\sigma = \operatorname{rank} M$, which is equivalent to $\sigma^\vee \subset M_{\mathbb{R}}$ being full dimensional. $\square$

Example 2.3.2. Let $M \simeq \mathbb{Z}^n \simeq N$. Since $\sigma^\vee = \operatorname{Cone}(e_1, \dots, e_n) \subset M_{\mathbb{R}}$ is the dual of $\sigma = \operatorname{Cone}(e_1, \dots, e_n) \subset N_{\mathbb{R}}$, we have $\mathbb{N}^n = \sigma^\vee \cap \mathbb{Z}^n$, so that $S_\sigma = \mathbb{N}^n$ and thus $U_\sigma = \mathbb{C}^n$.

Example 2.3.3 (Singular Quadratic Cone)**.** As seen in Example 2.2.11, the semigroup $\mathbb{N}\mathscr{A}$ where $\mathscr{A} = \{(1,0),(1,1),(1,2)\}$ defines the singular quadratic cone $C = V(y^2 - xz)$. Since $\mathbb{N}\mathscr{A} = \operatorname{Cone}((1,0),(1,2)) \cap \mathbb{Z}^n$, we have that $C = U_\sigma$ where $\sigma = \operatorname{Cone}((2,-1),(0,1))$, as shown in the figure below.

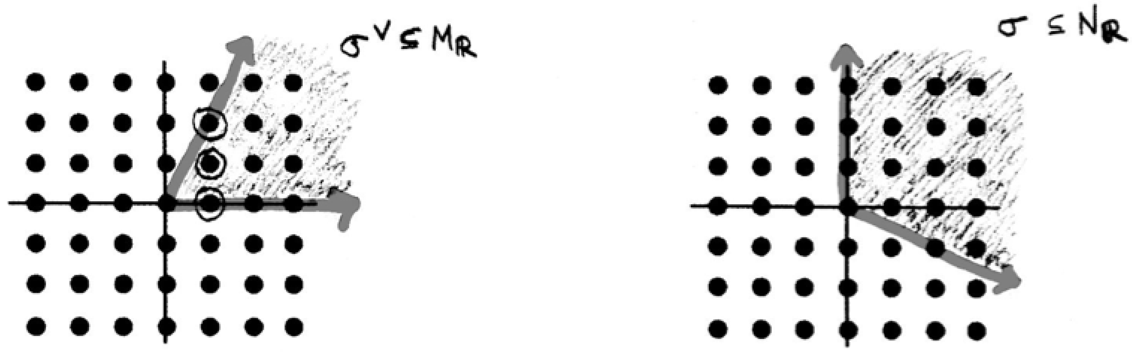


Figure 2.2: Cones associated to the singular quadratic cone

Example 2.3.4. Let $\sigma = \operatorname{Cone}(e_1, e_2, e_1+e_3, e_2+e_3)$. One can check that $\sigma^\vee = \operatorname{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3)$ and that $\sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}\{e_1, e_2, e_3, e_1 + e_2 - e_3\}$. Thus, as mentioned in Example 2.2.12, we have $U_\sigma = V(xy - zw)$.

The important fact is that every normal affine toric variety arises as U_σ for some rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$. The key idea is that S_σ is saturated: an affine semigroup S is **saturated** if for all $k \in \mathbb{N}$ and $m \in \mathbb{Z}S$, $km \in S$ implies $m \in S$. The following theorem tells us that every normal affine toric variety comes from a saturated semigroup and that every saturated semigroups arise from rational polyhedral cones, and hence, every normal affine toric variety arises from rational polyhedral cones.

Theorem 2.3.5. *[CLS11, Proposition 1.3.5] Let V be an affine toric variety with torus T_N (and character lattice M). Then the following are equivalent:*

1. *V is normal,*

2. *$V = \operatorname{Spec} \mathbb{C}[S]$ where $S \subset M = \mathbb{Z}S$ is saturated,*

3. *$V = \operatorname{Spec} \mathbb{C}[S_\sigma]$ for some $\sigma \subset N_\mathbb{R}$ a strongly convex rational polyhedral cone.*

Example 2.3.6. As seen in Example 2.2.10, the affine toric variety $V(x^3 - y^2)$ is defined by the semigroup $\mathbb{N}\{2,3\}$ which is not saturated in $\mathbb{Z}\{2,3\} = \mathbb{Z}$. Hence, it is a non-normal affine toric variety. Indeed, the affine toric varieties given in Example 2.3.3 and Example 2.3.4 are both normal.

A **face** of a rational polyhedral cone $\sigma \subset N_\mathbb{R}$ is $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$, where $H_m := \{u \in N_\mathbb{R} \mid \langle m, u\rangle = 0\}$. It is denoted $\tau \preceq \sigma$, and by $\tau \prec \sigma$ we mean that $\tau \neq \sigma$. For example, in Figure 2.1, $\operatorname{Cone}(e_2) \prec \sigma$ since $\operatorname{Cone}(e_2) = \sigma \cap H_{e_1}$.

We can now address why we defined the affine toric variety of a cone $\sigma \subset N_\mathbb{R}$ by the semigroup of its *dual.* The reason for doing so is that such association identifies the faces of σ to open subvarieties of U_σ. The next proposition provides the precise statement:

Proposition 2.3.7. *[CLS11, Proposition 1.3.16] Let $\sigma \subset N_\mathbb{R}$ be a strongly convex rational polyhedral cone, and let τ be a face of σ such that $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$. Then $S_\sigma + \mathbb{Z}m = S_\tau$, and thus $\mathbb{C}[S_\sigma]_{\chi^m} \simeq \mathbb{C}[S_\tau]$.*

Example 2.3.8. We consider some open subsets of the singular quadratic cone $C = V(y^2 - xz)$. Let our setting be as in the Example 2.3.3 so that $\mathbb{C}[x,y,z]/\langle y^2 - xz\rangle \simeq \mathbb{C}[S_\sigma]$ via $x, y, z \mapsto \chi^{e_1}, \chi^{e_1+e_2}, \chi^{e_1+2e_2}$ respectively. The ray $\tau = \operatorname{Cone}(e_2) = \sigma \cap H_{e_1} \subset N_\mathbb{R}$ is a face of σ, and one computes that $S_\tau = \mathbb{N}\{\pm(1,0),(1,1)\} = \mathbb{Z} \times \mathbb{N}$, so $U_\tau \simeq \mathbb{C}^* \times \mathbb{C}$. Indeed, we have that $\operatorname{Spec} \mathbb{C}[S_\sigma]_{\chi^{e_1}} \simeq \mathbb{C}^* \times \mathbb{C}$ since $V(y^2 - xz)_x = \{(x, y, y^2/x) \in \mathbb{C}^3 \mid x \neq 0\}$.

We finish with the criterion for when normal affine toric varieties are smooth.

Proposition 2.3.9. *[CLS11, Proposition 1.3.12] Let $\sigma \subset N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Then U_σ is smooth iff the minimal generating set of $\sigma \cap N$ can be extended to a basis of N.*

Example 2.3.10. Consider again the singular quadratic conic as in Example 2.3.3. It is not smooth since the minimal generating set of $\sigma \cap \mathbb{Z}^2$ is $\{(2,-1),(1,0),(0,1)\}$, which clearly does not extend to a basis of $\mathbb{Z}^2$. Indeed, $V(y^2 - xz)$ has a singularity at the origin. In contrast, $\tau = \operatorname{Cone}((0,1))$ is generated by $(0,1)$ which extends to basis of $\mathbb{Z}^2$, and so $U_\tau = \mathbb{C}^* \times \mathbb{C}$ is smooth as expected.

2.4 Morphisms of affine toric varieties

We conclude this chapter with a discussion of morphisms of affine toric varieties. As morphisms of tori correspond to group homomorphisms of their lattices (Proposition 2.1.4), it is natural to define morphisms of affine toric varieties to correspond to semigroup homomorphisms of their affine semigroups:

Definition 2.4.1. *Let $V_i = \operatorname{Spec} \mathbb{C}[S_i]$ be affine toric varieties $(i = 1, 2)$. Then a morphism $\phi : V_1 \to V_2$ is **toric** if the corresponding map $\phi^* : \mathbb{C}[S_2] \to \mathbb{C}[S_1]$ is induced by a semigroup homomorphism $\hat{\phi} : S_2 \to S_1$.*

Note that the definition is functorial, in the sense that if we have a chain of semigroup homomorphisms $S_3 \overset{\hat{\phi}}{\to} S_2 \overset{\hat{\psi}}{\to} S_1$, the composition of induced toric morphisms $V_1 \overset{\psi}{\to} V_2 \overset{\phi}{\to} V_3$ is the same as the toric morphism induced by $S_3 \overset{\hat{\psi}\circ\hat{\phi}}{\to} S_1$. Moreover, toric morphisms $\phi : V_1 \to V_2$ are defined exactly so that the morphism works nicely with the underlying maps of the tori $\phi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$. More precisely,

Proposition 2.4.2. *If T_{N_i} is the torus of the affine toric variety V_i $(i = 1, 2)$, then*

1. *A morphism $\phi : V_1 \to V_2$ is toric if and only if $\phi(T_{N_1}) \subset T_{N_2}$ and $\phi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism.*
2. *A toric morphism $\phi : V_1 \to V_2$ is equivariant, i.e. $\phi(t \cdot p) = \phi(t) \cdot \phi(p)$ for all $t \in T_{N_1}$ and $p \in V_1$.*

Proof. 1. Let $M_1 = \mathbb{Z}\mathscr{A}_1, M_2 = \mathbb{Z}\mathscr{A}_2$ be the character lattices of the tori T_{N_1}, T_{N_2}. A semigroup homomorphism $S_2 \to S_1$ uniquely extends to a group homomorphism $M_2 \to M_1$. Conversely, suppose a morphism $\phi : V_1 \to V_2$ induced by a map $\phi^* : \mathbb{C}[S_2] \to \mathbb{C}[S_1]$ is such that ϕ^* induces a group homomorphism $\hat{\phi}^* : M_2 \to M_1$. Then ϕ^* induces a semigroup homomorphism $\hat{\phi}^* : S_2 \to S_1$ since $\phi^*(\mathbb{C}[S_2]) \subset \mathbb{C}[S_1]$.

2. The diagram

$$\begin{tikzcd} T_{N_1} \times V_1 \arrow[r] \arrow[d, "\phi|_{T_N} \times \phi"'] & V_1 \arrow[d, "\phi"] \\ T_{N_2} \times V_2 \arrow[r] & V_2 \end{tikzcd}$$

commutes when V_i's are replaced with T_{N_i}'s in the diagram (since ϕ restricted to the tori is a group homomorphism), and $T_{N_i} \times T_{N_i}$ is open dense in $T_{N_i} \times V_i$. $\square$

Remark 2.4.3. Since a group homomorphism $M_2 \to M_1$ is in correspondence with its dual group homomorphism $N_1 \to N_2$, the proof for part 1. in fact shows that a toric morphism $\phi : V_1 \to V_2$ is equivalent to giving a group homomorphism $\hat{\phi}^* : M_2 \to M_1$, or equivalently $\hat{\phi} : N_1 \to N_2$, such that $\hat{\phi}^*(S_2) \subset S_1$.

The following is the translation of the above remark for the case of normal affine toric varieties:

Proposition 2.4.4. *Let $\sigma_i \subset (N_i)_\mathbb{R}$ be strongly convex rational polyhedral cones $(i = 1, 2)$. Then a homomorphism $\hat{\phi} : N_1 \to N_2$ induces a toric morphism $\phi : U_{\sigma_1} \to U_{\sigma_2}$ extending $\phi : T_{N_1} \to T_{N_2}$ if and only if $\hat{\phi}_\mathbb{R}(\sigma_1) \subset \sigma_2$.*

Proof. Let $\hat{\phi}^* : M_2 \to M_1$ be the dual homomorphism, so $\phi = \operatorname{Spec} \hat{\phi}^*$ when $\hat{\phi}^*$ is considered as a map $\mathbb{C}[M_2] \to \mathbb{C}[M_1]$. One can check that $\hat{\phi}_\mathbb{R}(\sigma_1) \subset \sigma_2$ if and only if $\hat{\phi}^*_\mathbb{R}(\sigma_2^\vee) \subset \sigma_1^\vee$, which is exactly when $\hat{\phi}^*|_{\sigma_2^\vee} : \sigma_2^\vee \to \sigma_1^\vee$ is a semigroup homomorphism. Now use Remark 2.4.3. $\square$

Remark 2.4.5. A particularly nice case of the above Proposition 2.4.4 is when $N_1 = N_2 = N$ and $\widetilde{\phi} = \mathrm{Id}_N$. For strongly convex rational polyhedral cones $\sigma_1, \sigma_2 \subset N_\mathbb{R}$, if $\sigma_1 \subset \sigma_2$, the inclusion a canonical toric morphism $U_{\sigma_1} \to U_{\sigma_2}$. Moreover, by the functoriality of the definition of toric morphisms, we have that if $\sigma_1 \subset \sigma_2 \subset \sigma_3$, the composition of maps $U_{\sigma_1} \to U_{\sigma_2} \to U_{\sigma_3}$ is the same as the map $U_{\sigma_1} \to U_{\sigma_3}$ given by $\sigma_1 \subset \sigma_3$. In this view, Proposition 2.3.7 is stating that an inclusion of a face of a strongly convex polyhedral cone $\tau \preceq \sigma$ canonically induces a toric morphism $U_\tau \hookrightarrow U_\sigma$ that is an open embedding.

Example 2.4.6. Consider the chain of inclusions of cones $\mathrm{Cone}(e_1, e_2) \subset \mathrm{Cone}(2e_1 - e_2, e_2) \subset \mathrm{Cone}(e_1 - e_2, e_2) \subset \mathbb{R}^2 = N_{\mathbb{R}}$ which induces morphisms $\mathbb{C}^2 \xrightarrow{\phi} V(y^2 - xz) \xrightarrow{\psi} \mathbb{C}^2$, where ϕ is the parameterization map $\phi(s,t) = (s, st, st^2)$ and ψ is the projection map $\psi(x, y, z) = (x, y)$. The composed map $(\psi \circ \phi)(s, t) = (s, st)$ is indeed the map induced by $\mathrm{Cone}(e_1, e_2) \subset \mathrm{Cone}(e_1 - e_2, e_2)$.

CHAPTER 3

Normal Toric Varieties

The definition of a general toric variety is a straightforward generalization of the affine toric varieties:

Definition 3.0.7. *A **toric variety** is an irreducible variety X such that (i) X contains a torus T_N as a Zariski open subset and (ii) the action of T_N on itself extends to an action $T_N \times X \to X$ given as morphism of varieties.*

While non-normal toric varieties exist (see Example 3.2.6), we focus exclusively on the normal ones, which are obtained by patching together affine toric varieties with a data of a fan. Historically, the study of toric varieties included only normal ones, as normal toric varieties provide a rich interaction between their geometric structures and combinatorial structures that define them.

We start by defining the normal toric variety associated to a fan, and give some examples. We then study the toric varieties defined by the normal fans of polytopes, which will turn out to be projective normal toric varieties. These projective normal toric varieties provide rich examples of toric varieties that are relatively easy to compute and work with. We finish with two illustrative examples in which the combinatorics informs the geometry of toric varieties, which are the orbit-cone correspondence and the classification of complete smooth normal toric surfaces.

3.1 Fans and normal toric varieties

We start with how a fan defines a normal toric variety and consider some examples. As in the case of affine normal toric varieties (Proposition 2.4.4), toric morphisms of general normal toric varieties correspond particular subset of maps of lattices.

Definition 3.1.1. *A **fan** Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subset N_{\mathbb{R}}$ such that*

1. *Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.*
2. *For every $\sigma \in \Sigma$, each face of σ is also in Σ.*
3. *For every $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of each.*

Given a fan $\Sigma \subset N_{\mathbb{R}}$, each of the cones $\sigma \in \Sigma$ is a strongly convex polyhedral cone, which defines a normal affine toric variety U_σ. Moreover, given $\sigma_1, \sigma_2 \in \Sigma$, the affine toric varieties U_{σ_1} and U_{σ_2} each contain an open subset isomorphic to $U_{\sigma_1 \cap \sigma_2}$ that can be canonically identified via the canonical embeddings $U_{\sigma_1} \overset{i}{\hookleftarrow} U_{\sigma_1 \cap \sigma_2} \overset{j}{\hookrightarrow} U_{\sigma_2}$ induced by $\sigma_1 \succeq \sigma_1 \cap \sigma_2 \preceq \sigma_2$ (Remark 2.4.5). The important result is that the maps g_{σ_2, σ_1} glue together the affine toric varieties into a toric variety.

Theorem 3.1.2. *Let $\Sigma \subset N_{\mathbb{R}}$ be a fan. Then the morphisms g_{σ_2,σ_1} glue together the finite collection of affine toric varieties $\{U_\sigma\}_{\sigma\in\Sigma}$ to define a variety X_Σ. More importantly, X_Σ is a separated normal toric variety.*

Proof. We need first check that the morphisms g_{σ_2,σ_1} satisfy the cocycle condition: if $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ then $g_{\sigma_3,\sigma_2} \circ g_{\sigma_2,\sigma_1}$ coincides with g_{σ_3,σ_1} on the appropriate domain. Since the maps $U_{\sigma_1} \overset{i}{\hookleftarrow} U_{\sigma_1\cap\sigma_2} \overset{j}{\hookrightarrow} U_{\sigma_2}$ are canonically induced by the inclusion maps $\sigma_1 \hookleftarrow \sigma_1 \cap \sigma_2 \hookrightarrow \sigma_1$ (cf. Remark 2.4.5), the following commuting diagram

$$\begin{tikzcd}
\sigma_1 & & \sigma_1\cap\sigma_2 \arrow[ll, hook'] \arrow[rr, hook] & & \sigma_2 \\
& & \sigma_1\cap\sigma_2\cap\sigma_3 \arrow[u, hook] \arrow[dll, hook] \arrow[drr, hook] & & \\
\sigma_1\cap\sigma_3 \arrow[uu, hook] \arrow[rr, hook] & & \sigma_3 & & \sigma_2\cap\sigma_3 \arrow[uu, hook] \arrow[ll, hook']
\end{tikzcd}
\quad \text{induces} \quad
\begin{tikzcd}
U_{\sigma_1} & & U_{\sigma_1\cap\sigma_2} \arrow[ll, hook'] \arrow[rr, hook] & & U_{\sigma_2} \\
& & U_{\sigma_1\cap\sigma_2\cap\sigma_3} \arrow[u, hook] \arrow[dll, hook] \arrow[drr, hook] & & \\
U_{\sigma_1\cap\sigma_3} \arrow[uu, hook] \arrow[rr, hook] & & U_{\sigma_3} & & U_{\sigma_2\cap\sigma_3} \arrow[uu, hook] \arrow[ll, hook']
\end{tikzcd}$$

which implies that $g_{\sigma_3,\sigma_2} \circ g_{\sigma_2,\sigma_1}$ agrees with g_{σ_3,σ_1} on $U_{\sigma_!\cap\sigma_2\cap\sigma_3}$. Moreover, since every cone in Σ is strongly convex, the origin $\{0\} \subset N$ is a face of each. Thus, for all $\sigma \in \Sigma$ we have torus actions $T_N \times U_\sigma \to U_\sigma$. For any $\sigma_1, \sigma_2 \in \Sigma$, the T_N-actions on U_{σ_1} and U_{σ_2} are compatible since the maps $U_{\sigma_1} \hookleftarrow U_{\sigma_1\cap\sigma_2} \hookrightarrow U_{\sigma_2}$ are toric morphisms and are thus equivariant by Proposition 2.4.2. Moreover, since the affine varieties $\{U_\sigma\}_{\sigma\in\Sigma}$ defining X_Σ are normal, so is X_Σ. Lastly, X_Σ irreducible since it is clearly connected and covered by open irreducible subsets. Thus, we have thus shown that X_Σ is a normal toric variety. For the proof that X_Σ is separated, see [CLS11, Theorem 3.1.5].

Remark 3.1.3. In fact every normal separated toric variety arises in this way, proof of which is outside the scope of this article.

Example 3.1.4. Consider the fan Σ in $\mathbb{R}$ drawn below that is defined as the collection of two opposites rays $\sigma_1 = \mathrm{Cone}(1), \sigma_2 = \mathrm{Cone}(-1)$ and the origin 0. Setting $U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x]$ and $U_{\sigma_2} = \operatorname{Spec}\mathbb{C}[x^{-1}]$, we have that g_{σ_2,σ_1} is given by $\mathbb{C}[x^{-1}]_{x^{-1}} \xrightarrow{\sim} \mathbb{C}[x]_x$ via $x^{-1} \mapsto x^{-1} = 1/x$. In other words, $g_{\sigma_2,\sigma_1} : U_{\sigma_1} = \mathbb{C} \to U_{\sigma_2} = \mathbb{C}$ is given as $t \mapsto 1/t$. This is exactly the description of $\mathbb{P}^1$ and its two distinguished affine patches. Hence we have $X_\Sigma \simeq \mathbb{P}^1$.

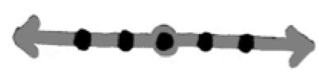


Figure 3.1: The fan corresponding to $\mathbb{P}^1$

Example 3.1.5 (Blow-up $\mathrm{Bl}_0(\mathbb{C}^2)$)**.** Consider the fan $\Sigma \subset N_{\mathbb{R}}$ given by cones $\sigma_1 = \mathrm{Cone}(e_2, e_1 + e_2)$, $\sigma_2 = \mathrm{Cone}(e_1, e_1 + e_2)$ and their faces. The fan Σ and the dual cones of σ_1, σ_2 are as drawn below; note that $S_{\sigma_1} + \mathbb{Z}(e_2 - e_1) = S_{\sigma_1\cap\sigma_2} = S_{\sigma_2} + \mathbb{Z}(e_1 - e_2)$. With $U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x, y/x] \simeq \mathbb{C}^2$ and $U_{\sigma_2} = \operatorname{Spec}\mathbb{C}[y, x/y] \simeq \mathbb{C}^2$, one checks that g_{σ_2,σ_1} is given by $\mathbb{C}[x, y/x]_{y/x} \to \mathbb{C}[y, x/y]_{x/y}$ via $x \mapsto y(x/y)$, $y/x \mapsto x/y$. This is exactly the description of the patching of two affine pieces in the blow-up of $\mathbb{C}^2$ at the origin, and hence $X_\Sigma \simeq \mathrm{Bl}_0(\mathbb{C}^2)$. Thus, we have shown that if $\{v_0, v_1\}$ is a basis of N, then splitting $\sigma = \mathrm{Cone}(v_0, v_1)$ into a fan Σ of two cones $\mathrm{Cone}(v_0, v_0 + v_1)$, $\mathrm{Cone}(v_0 + v_1, v_1)$ corresponds to blowing-up $U_\sigma \simeq \mathbb{C}^2$ at the origin.

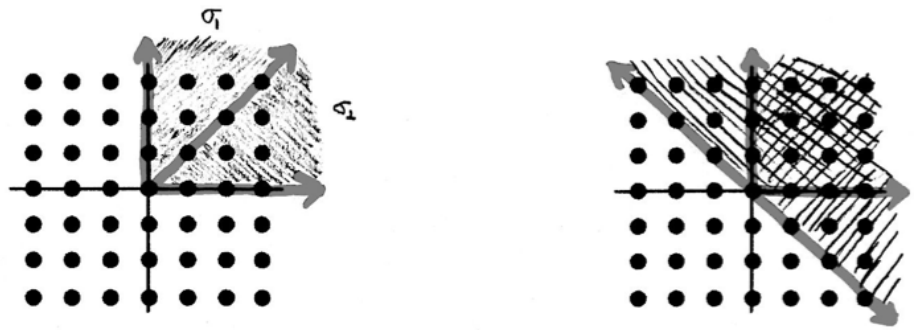


Figure 3.2: The fan corresponding to the blow-up $\mathrm{Bl}_0(\mathbb{C}^2)$

The criterion for when an affine normal toric variety is smooth generalizes immediately to a normal toric variety X_Σ:

Proposition 3.1.6. *For $\Sigma \subset N_\mathbb{R}$ a fan, X_Σ is a smooth variety if and only if every cone $\sigma \in \Sigma$ is smooth.*

Proof. A variety is smooth if and only if it is covered by smooth affine open patches, which by Proposition 2.3.9 occurs exactly when every cone $\sigma \in \Sigma$ is smooth. $\square$

The definition of toric morphism also generalizes naturally (cf. Proposition 2.4.2).

Definition 3.1.7. *Let X_{Σ_i} be normal toric varieties of fans $\Sigma_i \subset (N_i)_\mathbb{R}$ for $i = 1, 2$. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is **toric** if $\phi(T_{N_1}) \subset T_{N_2}$ and $\phi|_{T_{N_1}}$ is a group homomorphism.*

The equivariance of toric morphism follows naturally from Proposition 2.4.2, and moreover Remark 2.4.3 likewise generalizes naturally to the following statement:

Proposition 3.1.8. *[CLS11, Proposition 3.3.4] Let X_{Σ_i} be as above. We say that a map of lattices $\hat{\phi} : N_1 \to N_2$ is **compatible** with fans Σ_1, Σ_2 if for every $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ such that $\hat{\phi}_\mathbb{R}(\sigma_1) \subset \sigma_2$. The compatible maps $\hat{\phi} : N_1 \to N_2$ induce toric morphisms $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ and conversely.*

Example 3.1.9. In the Example 3.1.5, the identity map is compatible with fans Σ and σ, and the induced map $\phi : X_\Sigma \to U_\sigma$ is the canonical blow-down map $\mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$.

3.2 Polytopes and projective normal toric varieties

In this section, we discuss the projective normal toric varieties, which turns out to arise from lattice polytopes. A **lattice polytope** $P \subset M_\mathbb{R}$ is a set of the form

$$P = \mathrm{Conv}(m_1, \ldots, m_s) := \left\{ \lambda_1 n_1 + \cdots + \lambda_s m_s \in M_\mathbb{R} \mid \lambda_i \geq 0 \text{ for all } i = 1, \ldots, s \text{ and } \sum_{i=1}^{s} \lambda_i = 1 \right\}$$

for some $m_1, \ldots, m_s \in M$. The dimension of P is the dimension of the smallest affine subspace of $M_\mathbb{R}$ containing P. For positive integers k, denote by kP the k-fold Minkowski sum of P, which is also a lattice polytope. For a lattice point $p \in P \cap M$, denote by $P \cap M - p$ the set $\{m - p \mid m \in P \cap M\}$. Then p is a **vertex** of P if the cone $\sigma_p \subset M_{\mathbb{R}}$ defined as

$$\sigma_p := \text{Cone}(P \cap M - p)$$

is strongly convex. Given a full dimensional lattice polytope $P \subset M_{\mathbb{R}}$, its **normal fan** Σ_P is a fan in $N_{\mathbb{R}}$ obtained by collecting together $\sigma_p^\vee$ and all its faces for every very vertex p of P. Thus, a lattice polytope defines a normal toric variety.

Since the normal fan of P and kP (for $k \in \mathbb{N}$) is the same, we can canonically associate a normal toric variety to a polytope up to scalar multiplication. In this section we show that these normal toric varieties defined by polytopes are in fact projective normal toric varieties.

We start off with a more general construction of projective toric varieties. Similar to the affine case, we can construct a projective toric variety by selecting finite set $\mathscr{A}$ of lattice points in M and considering the monomial maps it defines. We will then show that when $\mathscr{A} = P \cap M$ for a very ample polytope $P \subset M_{\mathbb{R}}$, the two constructions — one by monomial maps and the other by the normal fan of P— define the same variety. One advantage of considering the polytope is that instead of considering the affine patches and their gluing maps, we can compute the ideal defining the projective toric variety directly from the points of $P \cap M$.

We note first the torus that is naturally contained as an open dense subset in the projective space $\mathbb{P}^n$, which is

$$T_{\mathbb{P}^n} := \{(a_0 : \ldots : a_n) \in \mathbb{P}^n \mid a_0 a_1 \cdots a_n \neq 0\} \simeq \{(1 : t_1 : \ldots : t_n) \mid t_1, \ldots, t_n \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^n.$$

Recall that we constructed affine toric varieties by considering the image of a map of tori. We proceed likewise for the projective case.

Definition 3.2.1. *Let M be a character lattice of a torus T_N, and let $\mathscr{A} = \{m_1, \ldots, m_s\} \subset M$. Define a map $\widetilde{\Phi}_{\mathscr{A}} : T_N \to \mathbb{P}^{s-1}$ to be the composition $T_N \overset{\Phi_{\mathscr{A}}}{\to} (\mathbb{C}^*)^s \overset{\pi}{\to} T_{\mathbb{P}^{s-1}} \hookrightarrow \mathbb{P}^{s-1}$, and define $X_{\mathscr{A}} := \overline{\text{Im}(\widetilde{\Phi}_{\mathscr{A}})}$ to be the closure of the image of the map.*

Proposition 3.2.2. *Let T_N be a torus with lattice M, and let $\mathscr{A} = \{m_1, \ldots, m_s\} \subset M$. Then $X_{\mathscr{A}}$ as defined above is a toric variety.*

Proof. we will work in the affine patches U_i of $\mathbb{P}^{s-1}$ for $i = 1, \ldots, s$. Since $T_{\mathbb{P}^{s-1}}$ is contained in any open affine piece U_i, under the canonical isomorphism $U_i \simeq \mathbb{C}^s$ the map $\widetilde{\Phi}_{\mathscr{A}} : T_N \to T_{\mathbb{P}^{s-1}} \subset U_i \simeq \mathbb{C}^s$ is given as $t \mapsto (\chi^{m_1 - m_i}(t), \ldots, \chi^{m_s - m_i}(t))$ (where the term $\chi^{m_i - m_i}(t) = 1$ is omitted). Hence, if we denote $\mathscr{A}_i = \{m_1 - m_i, \ldots, m_s - m_i\}$, then the closure of $\widetilde{\Phi}_{\mathscr{A}}$ in U_i is $Y_{\mathscr{A}_i}$, an affine toric variety.

Since closure of a set X in $\mathbb{P}^{s-1}$ is the same as taking the union of the closures of X is each U_i, we have that $X_{\mathscr{A}}$ is the projective variety obtained by patching together the distinguished affine pieces $Y_{\mathscr{A}'}$, which are affine toric varieties. Since a torus is irreducible, its image $\text{Im}(\widetilde{\Phi}_{\mathscr{A}})$ and hence the closure $X_{\mathscr{A}}$ are irreducible. Moreover, note that $\mathbb{Z}\mathscr{A}_i$'s for all $i = 1, \ldots, s$ are isomorphic in that $\mathbb{Z}\mathscr{A}_i \simeq \mathbb{Z}'\mathscr{A} := \{a_1 m_1 + \cdots + a_s m_s \in \mathbb{Z}\mathscr{A} \mid a_1 + \cdots + a_m = 0\}$. $X_{\mathscr{A}}$ thus contains the torus $\text{Spec}\, \mathbb{C}[\mathbb{Z}'\mathscr{A}]$ as an open dense subset whose action on itself extends to $X_{\mathscr{A}}$ by working in the affine patches. $\square$

We state as a separate lemma an important fact that was stated as part of the proof just given. In fact, we also give a further improvement of the result.

Lemma 3.2.3. *[CLS11, Proposition 2.1.9] The i^{th} distinguished affine piece $(X_{\mathscr{A}})_{x_i}$ of the projective toric variety $X_{\mathscr{A}}$ is $Y_{\mathscr{A}_i}$ where $\mathscr{A}_i = \{m_1 - m_i, \ldots, m_s - m_i\}$. Moreover, $X_{\mathscr{A}}$ is efficiently covered by ones that form vertices of* Conv$(\mathscr{A})$*; more precisely,*

$$X_{\mathscr{A}} = \bigcup_{i \in I} Y_{\mathscr{A}_i}$$

where $I = \{i \in \{1, 2, \ldots, s\} \mid m_i \text{ is a vertex of } \operatorname{Conv}(\mathscr{A}) \subset M_{\mathbb{R}}\}$.

The advantage of this construction is that one can directly compute the defining ideal $I(X_{\mathscr{A}})$. The next proposition tells us how to do so.

Proposition 3.2.4. *[CLS11, Proposition 2.1.4] Let T_N, M, $\mathscr{A}$ be given as above. Recall that I_L is the lattice ideal of $L = \ker(\mathbb{Z}^s \to M)$. Then $Y_{\mathscr{A}} \subset \mathbb{C}^s$ is the affine cone of $X_{\mathscr{A}} \subset \mathbb{P}^{s-1}$ if and only if there exists $u \in N$ and $k > 0$ in $\mathbb{N}$ such that $\langle m_i, u \rangle = k$ for all $i = 1, \ldots, s$*

Remark 3.2.5. Using Proposition 3.2.4 we now note how to compute the ideal $I(X_{\mathscr{A}})$. Let $T_N = (\mathbb{C}^*)^n$, $M = \mathbb{Z}^n$, and $\mathscr{A} = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^n$. We note that in this case the statement 2. of Proposition 3.2.4 is equivalent to stating that $(1, \ldots, 1)$ is in the row span of the matrix A formed by columns m_i. If $(1, \ldots, 1)$ is not in the row span of A, we can simply add it to A as the last row to obtain a new matrix A', which this is equivalent to setting $\mathscr{A}' := \{m_1 + e_{n+1}, \ldots, m_s + e_{n+1}\} \subset \mathbb{Z}^{n+1}$. We see that $X_{\mathscr{A}} = X_{\mathscr{A}'}$ since $\{(t^{m_1} : \ldots : t^{m_s}) \in \mathbb{P}^{s-1} \mid \ t \in (\mathbb{C}^*)^n\} = \{(st^{m_1} : \ldots : st^{m_s}) \in \mathbb{P}^{s-1} \mid \ t \in (\mathbb{C}^*)^n, \ s \in \mathbb{C}^*\}$. This allows us to easily construct and compute the ideals of projective (even non-normal) toric varieties.

Example 3.2.6. For $\mathscr{A} = \{(2,1), (3,1), (0,1)\} \subset \mathbb{Z}^2$, the matrix A already has a row $(1,1,1)$ and $\ker A = \operatorname{span}((3,-2,-1))$. Thus, we have $X_{\mathscr{A}} = V(x^3 - y^2 z)$. This is a non-normal toric variety because $(X_{\mathscr{A}})_z \simeq V(x^3 - y^2)$ is not normal (cf. Example 2.3.6).

Example 3.2.7. Let $\mathscr{A} = \{(0,0), (1,0), (1,1), (1,2)\} \subset \mathbb{Z}^2$. Since $(1,1,1,1)$ is not in the row span of A, we add it to obtain $A' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Since $\ker A' = \operatorname{span}((0,1,-2,1))$, we have $X_{\mathscr{A}} = V(y^2 - xz) \subset \mathbb{C}^4$ (with coordinates (w,x,y,z)). As expected from Proposition 3.2.3, the semigroup $\mathbb{N}\mathscr{A}_1 = \operatorname{Cone}((1,0),(1,1),(1,2))$ gives the singular quadratic cone $V(y^2 - xz) \subset \mathbb{C}^3$, which is the affine patch U_w of $X_{\mathscr{A}}$. Likewise, $\mathbb{N}\mathscr{A}_3 = \operatorname{Cone}(-e_1, \pm e_2)$ gives $\mathbb{C} \times \mathbb{C}^*$, which is indeed $U_y \simeq V(1 - xz) \subset \mathbb{C}^3$ (with coordinates (w,x,z)).

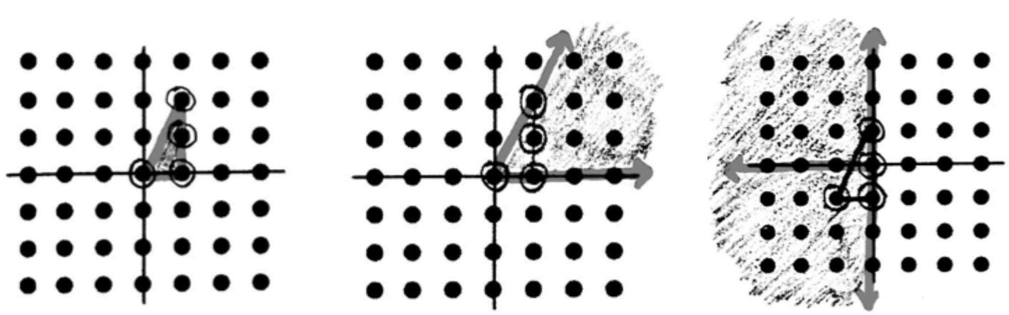


Figure 3.3: Lattice points $\mathscr{A} = \{(0,0),(1,0),(1,1),(1,2)\}$ and the semigroups $\mathbb{N}\mathscr{A}_1$ and $\mathbb{N}\mathscr{A}_3$

We now turn to the case when $\mathscr{A} \subset M$ is given as $P \cap M$, the lattice points of some lattice polytope $P \subset M_{\mathbb{R}}$. From the discussion so far, one might guess that the projective variety associated to a lattice polytope P to be $X_{P\cap M}$. However, there are two problems with such naive approach. The first is that if P is not full dimensional then the normal fan Σ_P has cones that are not strongly convex, so we need restrict ourselves to full dimensional lattice polytopes. The second issue is more subtle; there are cases when the variety $X_{P\cap M}$ does not match the variety X_{Σ_P} defined by the normal fan Σ_P of P. Consider the following example:

Example 3.2.8. Consider the polytope $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 2e_3) \subset \mathbb{R}^3$. One can compute that $X_{P\cap\mathbb{Z}^3} \simeq \mathbb{P}^3$. However, Σ_P contains a cone σ that is the dual of $\sigma^\vee = \sigma_0 = \mathrm{Cone}(e_1, e_2, e_1 + e_2 + 2e_3)$, and $S_\sigma = \mathbb{N}\{e_1, e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3\}$ so $U_\sigma \simeq V(w^2 - xyz) \subset \mathbb{C}^4$, which is not smooth (has a singularity at the origin).

In the previous example, the main problem is that there is a vertex $p \in P$ ($p = 0$ in the example) such that $\mathbb{N}(P\cap M - p) \neq S_{\sigma_p^\vee} = \mathrm{Cone}(P\cap M - m)\cap M$. We can remedy this problem by considering **very ample** lattice polytopes $P \subset M_{\mathbb{R}}$, which are lattice polytopes such that $\mathbb{N}(P \cap M - p) = \mathrm{Cone}(P \cap M - p) \cap M$ for every vertex $p \in P$. Note that when P is full dimensional, being very ample equivalently means that $\mathbb{N}(P \cap M - p)$ is saturated in M for every vertex $p \in P$. The issue that arose in the previous example does not occur when P is very ample:

Theorem 3.2.9. *Let P be a full dimensional very ample polytope in $M_{\mathbb{R}}$. Then $X_{P\cap M} \simeq X_{\Sigma_P}$.*

Proof. Let $\mathscr{A} = \{m_1, \ldots, m_s\}$ be the vertices of P, and let $\sigma_1, \ldots, \sigma_s$ be the corresponding cones in the normal fan Σ_P of P. Note that since P is very ample, $\mathbb{N}\mathscr{A}_i = M \cap \mathrm{Cone}(\mathscr{A}_i) = M \cap \sigma_i^\vee$ for all $i = 1, \ldots, s$. Thus, we have $Y_{\mathscr{A}_i} \simeq U_{\sigma_i}$ for all i.

By Lemma 3.2.3 we know that $X_{P\cap M}$ is covered by affine pieces $Y_{\mathscr{A}_i}$ ($i = 1, \ldots, s$), whereas X_{Σ_P} is covered by U_{σ_i}. Since $Y_{\mathscr{A}_i} \simeq U_{\sigma_i}$, it only remains to show that the gluing in respective cases match each other, but this follows immediately from the properties of taking a normal fan of a polytope shown in [CLS11, Proposition 2.3.13]. $\square$

Example 3.2.10. For a simple example of very ample polytopes, consider a n-simplex defined as $\Delta_n := \mathrm{Conv}(0, e_1, \ldots, e_n)$. For all i, the set $\Delta_n \cap \mathbb{Z}^n - e_i$ forms a basis of $\mathbb{Z}^n$, and hence $\mathbb{N}(\Delta_2 \cap \mathbb{Z}^n - e_i)$ is indeed saturated in $\mathbb{Z}^n$. Thus, Δ_n is very ample. Moreover, one can compute easily that $X_{\Delta_n} \simeq \mathbb{P}^n$. For Δ_1, its normal fan Σ_{Δ_1} is the fan given in Example 3.1.4, which as we saw gives $X_{\Sigma_{\Delta_1}} \simeq \mathbb{P}^1$. For Δ_2, its normal fan Σ_{Δ_2} is drawn below, and one indeed checks that $\sigma_1, \sigma_2, \sigma_3$ all define $\mathbb{C}^2$ and the gluing maps match that of distinguished affine patches of $\mathbb{P}^2$.

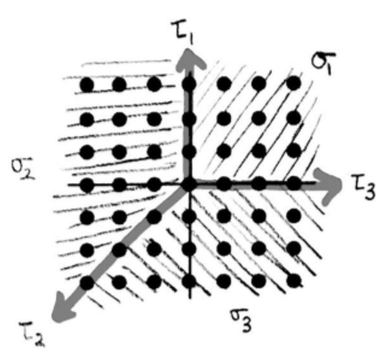


Figure 3.4: Normal fan Σ_{Δ_2} of Δ_2

For any lattice polytope P, there is an integer $N > 0$ such that kP is very ample for any integer $k \geq N$. Since the normal fan Σ_P of P is the same as that of kP, Theorem 3.2.9 implies that we can canonically associate a projective normal toric variety to a full dimensional lattice polytope P by defining it as $X_{kP\cap M}$ for any k such that kP is very-ample. In other words, just as selecting any $\mathscr{A} = \{m_1, \ldots, m_s\}$ such that $\mathbb{N}\mathscr{A} = S$ gives an embedding $\operatorname{Spec}\mathbb{C}[S] \hookrightarrow \mathbb{C}^s$, selecting any k such that kP is very ample gives an embedding $X_{\Sigma_P} \hookrightarrow \mathbb{P}^{\#|kP\cap M|-1}$ as $X_{kP\cap M} \subset \mathbb{P}^{\#|kP\cap M|-1}$. One immediate question that one may ask is how these different embeddings are related. When P is very ample, the isomorphism between $X_{P\cap M}$ and $X_{kP\cap M}$ can in fact be related by Veronese maps.

Recall that the Veronese map of degree k, denoted $\nu_k : \mathbb{P}^s \to \mathbb{P}^N$ with $N = \binom{s+d}{d} - 1$, is defined as $(x_0 : \ldots : x_s) \mapsto (x^I)_{I\in\mathcal{I}}$ where $\mathcal{I} := \{(i_0, \ldots, i_s) \in \mathbb{N}^{s+1} \mid \sum_l i_l = k\}$. In other words, ν_k sends $(x_0 : \ldots : x_s)$ to all possible monomials of degree d. Note that ν_k is a closed embedding that is isomorphism onto its image.

Indeed, the number $\#|kP\cap M|$ may not equal $\binom{\#|P\cap M|-1+k}{k}$, but we can modify the embedding of $X_{kP\cap M}$ into a projective space so that the isomorphism $X_{P\cap M} \simeq X_{kP\cap M}$ is given exactly by the degree k Veronese map ν_k. We first note that if $\mathscr{A}' \subset M$ is a finite *multi*set and $\mathscr{A}$ is the set defined by $\mathscr{A}'$, then $X_{\mathscr{A}'} \simeq X_{\mathscr{A}}$. Now, consider the multiset $\{kP \cap M\}$ defined as

$$\{kP\cap M\} := \{\{ \sum_{p_l\in P\cap M} i_l p_l \mid i_l \text{ integers such that } i_l \geq 0 \text{ and } \sum_{p_l\in P\cap M} i_l = k \;\}\}$$

which is the multiset obtained by k-fold Minkowski sums of $P \cap M$. When P is very ample, lattice points of kP are k-fold sums of lattice points of P; in other words, $kP \cap M$ is the set defined by $\{kP \cap M\}$. Thus, we have $X_{kP\cap M} \simeq X_{\{kP\cap M\}} \subset \mathbb{P}^{\#|\{kP\cap M\}|-1}$ with $\#|\{kP\cap M\}| = \binom{\#|P\cap M|-1+k}{k}$. We are now ready to relate projective toric varieties of P and kP by the Veronese map:

Proposition 3.2.11. *Let $P \subset M_{\mathbb{R}}$ be a full dimensional, very ample lattice polytope. Then for any integer $k > 0$, the isomorphism $X_{P\cap M} \simeq X_{\{kP\cap M\}}$ is given by restricting the degree k Veronese map ν_k. In particular, the isomorphism $X_{\Delta_n} \simeq X_{k\Delta_n}$ is the degree k Veronese map $\nu_k : \mathbb{P}^n \to \mathbb{P}^{\binom{n+k}{k}-1}$.*

Proof. We see this immediately by looking at the parameterizations. Let $\mathscr{A} = P\cap M = \{m_0, \ldots, m_s\}$ (so $s = \#|P\cap M| - 1$). Then the toric variety $X_{P\cap M}$ is parameterized by $\Phi_{\mathscr{A}} : T_N \to \mathbb{P}^s,\ t \mapsto (\chi^{m_0}(t) : \ldots : \chi^{m_s}(t))$, whereas $X_{\{kP\cap M\}}$ is parameterized by $\Phi_{\{kP\cap M\}} :\ T_N \to \mathbb{P}^{\binom{s+k}{k}-1},\ t \mapsto (\chi^{I\cdot\vec{m}}(t))_{I\in\mathcal{I}}$ where $\mathcal{I} := \{(i_0, \ldots, i_s) \in \mathbb{N}^{s+1} \mid \sum_l i_l = k\}$ and $\vec{m} = (m_0, \ldots, m_s)$. Lastly, one can check that the multiset $\{k\Delta_n\}$ has no repeated elements, so $\{k\Delta_n\} = k\Delta_n$. Thus, combining with Example 3.2.10 we have that $X_{\Delta_n} \xrightarrow{\sim} X_{k\Delta_n}$ is exactly the Veronese map $\nu_k : \mathbb{P}^n \to \mathbb{P}^{\binom{n+k}{k}-1}$. □

Example 3.2.12 (Rational Normal Curve)**.** The rational normal curve C_d of degree d is a curve in $\mathbb{P}^d$ defined by a parameterization $C_d := \{(s^d : s^{d-1}t : \ldots : st^{d-1} : t^d) \in \mathbb{P}^d \mid s, t \in \mathbb{C} \text{ not both zero}\}$. Equivalently, since the matrix $A = \begin{bmatrix} d & d-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & d-1 & d \end{bmatrix}$ can be obtained by adding the row $(d, \ldots, d)$ to the matrix $[0\ 1\ \cdots\ d-1\ d]$ and doing an row operation, C_d is the projective toric variety of the polytope $\operatorname{Conv}(0, d) = d\Delta_1 \subset M_{\mathbb{R}} = \mathbb{R}$. Indeed, as Proposition 3.2.11 suggests, the rational normal curve C_d is equivalently characterized as the image of Veronese embedding $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$.

We conclude with an important class of examples of a projective normal toric varieties, the Hirzebruch surfaces $\mathscr{H}_r$.

Example 3.2.13. For $1 \leq a \leq b \in \mathbb{N}$, consider lattice polytopes $P_{a,b} = \mathrm{Conv}(0, ae_1, , e_2, be_1 + e_2) \subset M_{\mathbb{R}} = \mathbb{R}^2$, which is very ample. Note that the normal fan of $P_{a,b}$ depends only on the difference $r = b - a$. We define Hirzebruch surfaces $\mathscr{H}_r$ to be the normal toric variety given by the normal fan of $P_{a,b}$ where $b - a = r$. By looking at the polytope $P_{a,b}$ and applying Lemma 3.2.3, or by looking at the normal fan and applying Proposition 3.1.6, it is easy to see that $\mathscr{H}_r$ is obtained by piecing together four patches of $\mathbb{C}^2$ and is hence smooth. Drawn below is the polytope $P_{1,4}$ and its normal fan which corresponds to $\mathscr{H}_3$.

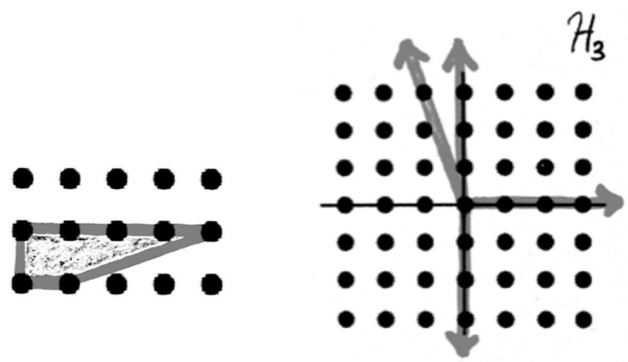


Figure 3.5: The polytope $P_{1,4}$ and its normal fan

3.3 The orbit-cone correspondence

As the first example in which the combinatorial data defining the normal toric variety richly informs its geometry, we show that the cones of a fan corresponds to torus-orbits of the normal toric variety that the fan defines. We will see that there are really two key ideas are in play: points correspond to semigroup homomorphisms, and $\tau \preceq \sigma$ induces a *toric* morphism $U_\tau \hookrightarrow U_\sigma$.

First, we note that given S an affine semigroup, the points of $\mathrm{Spec}\,\mathbb{C}[S]$ correspond to semigroup homomorphisms $S \to \mathbb{C}$. Indeed, points of $\mathrm{Spec}\,\mathbb{C}[S]$ correspond to $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[S] \to \mathbb{C}$, which in turn corresponds to semigroup homomorphisms $S \to \mathbb{C}$. Furthermore, this identification works nicely with the torus action:

Lemma 3.3.1. *Let $p \in \mathrm{Spec}\,\mathbb{C}[S]$ and $\gamma : S \to \mathbb{C}$ be its corresponding semigroup homomorphism. For $t \in \mathrm{Spec}\,\mathbb{C}[\mathbb{Z}S]$, the semigroup homomorphism for the point $t \cdot p$ is given by $m \mapsto \chi^m(t)\gamma(m)$.*

Proof. Let $\mathscr{A} = \{m_1, \ldots, m_s\}$ generate S so that we embed $V = \mathrm{Spec}\,\mathbb{C}[S] \hookrightarrow \mathbb{C}^s$. In this embedding, the torus action map $T_N \times V \to V$ given by $\mathbb{C}[S] \to \mathbb{C}[M] \otimes \mathbb{C}[S]$, $\chi^m \mapsto \chi^m \otimes \chi^m$ becomes $\mathbb{C}[x_1, \ldots, x_s]/I_L \to \mathbb{C}[t_1^{\pm}, \ldots, z_s^{\pm}]/I_L \otimes \mathbb{C}[y_1, \ldots, t_s]/I_L$, $x_i \mapsto t_i y_i$. In other words, if a point $p \in V$ corresponds to $\gamma : S \to \mathbb{C}$, then $t \cdot p$ is just given by the regular multiplication $(\chi^{m_1}(t)\gamma(m_1), \ldots, \chi^{m_s}(t)\gamma(m_s))$, and hence the semigroup homomorphism for $t \cdot p$ is given by $m \mapsto \chi^m(t)\gamma(m)$. $\square$

The second ingredient to the orbit-cone correspondence is that if τ is a face of a strongly convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$, then the inclusion $\tau \preceq \sigma$ induces a *toric* morphism $U_\tau \hookrightarrow U_\sigma$. Since both U_τ and U_σ have T_N as the torus, the equivariance of toric morphism (Proposition 2.4.2) implies that U_τ and $U_\sigma - U_\tau$ are T_N-invariant subsets of U_σ. Moreover, if we had chain of inclusion of faces

$\sigma_1 \preceq \sigma_2 \preceq \cdots \preceq \sigma_n$ then we have a chain of T_N-invariant subsets $U_{\sigma_1} \subset \cdots \subset U_{\sigma_n}$. Therefore, given a toric variety X_Σ of a fan $\Sigma \subset N_\mathbb{R}$, one may expect the T_N-orbits to be of the form $U_\sigma - (\bigcup_{\tau \prec \sigma} U_\tau)$ (recall that $\tau \prec \sigma$ means τ is a proper face of σ). We now show in the following two lemmas that this expectation is true by using the correspondence between points and semigroup homomorphisms.

Lemma 3.3.2. *Let $\sigma \subset N_\mathbb{R}$ be strongly convex polyhedral cone. Then the set of points defined as*

$$O(\sigma) := U_\sigma - \left(\bigcup_{\tau \prec \sigma} U_\tau \right)$$

correspond bijectively to semigroup homomorphisms $\{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$.

Proof. For $\tau \prec \sigma$ a proper face, we have $S_\sigma \subset S_\tau$, so that the points of $U_\sigma - U_\tau$ correspond bijectively to semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}$ that does *not* extend to $\widetilde{\gamma} : S_\tau \to \mathbb{C}$. And $\gamma : S_\sigma \to \mathbb{C}$ does not extend to S_τ precisely when there is an element $m \in S_\sigma$ such that $\gamma(m) = 0$ but m is invertible in S_τ. In other words, $O(\sigma)$ exactly corresponds to semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}$ that has an element $m \in S_\sigma$ such that $\gamma(m) = 0$ but m is invertible in S_τ for any $\tau \prec \sigma$. Indeed, semigroup homomorphisms $\{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$ have this property: if $\tau = \sigma \cap H_m$ is a proper subspace then $m \in (\sigma^\vee \cap M) \setminus (\sigma^\perp \cap M)$ because $m \in \sigma^\perp \cap M$ implies $\sigma \cap H_m = \sigma$.

Conversely, suppose $\gamma : S_\sigma \to \mathbb{C}$ is such that $\gamma(m) \neq 0$ for some $m \in (\sigma^\vee \cap M) \setminus (\sigma^\perp \cap M)$. Then if $\{m_1, \ldots, m_k\} \subset M$ are the minimal ray generators of $\sigma^\vee$, then $m = a_1 m_1 + \cdots a_k m_k$ for some $a_1, \ldots, a_k \in \mathbb{N}$ not all zero. Since $\gamma(m) \neq 0$, this implies that $\gamma(m_i) \neq 0$ for all $a_i \neq 0$, and so there is $m_j \notin \sigma^\perp \cap M$ for some $j \in \{1, \ldots, k\}$ with $a_j \neq 0$ (if such i didn't exist then $m \in \sigma^\perp \cap M$). But this implies that γ extends to $S_\sigma + \mathbb{Z}(-m_i) = S_\tau$ where $\tau = \sigma \cap H_{m_i}$ is a proper face of σ. □

Note a distinguished point γ_σ in $O(\sigma) = U_\sigma - (\bigcup_{\tau \prec \sigma} U_\tau)$ given by

$$\gamma_\sigma(m) = \begin{cases} 1 & m \in \sigma^\perp \cap M \\ 0 & \text{otherwise} \end{cases}.$$

We now show that $O(\sigma)$ is in fact the orbit $T_N \cdot \gamma_\sigma$:

Lemma 3.3.3. *$T_N \cdot \gamma_\sigma = O(\sigma)$, and thus $O(\sigma)$ is a T_N orbit in U_σ.*

Proof. Let $\gamma \in O(\sigma)$; we need show that $t \cdot \gamma_\sigma = \gamma$ for some $t \in T_N$. Since $\sigma^\perp$ is a sub*space* of $M_\mathbb{R}$, the lattice points $\sigma^\perp \cap M$ form a sublattice of M, say of rank k. Let $\{m_1, \ldots, m_k\} \subset M$ be a $\mathbb{Z}$-basis of $\sigma^\perp \cap M$ and extend it to $\mathscr{A} = \{\pm m_1, \ldots, \pm m_k, m_{k+1}, \ldots, m_s\} \subset M$ so that $\mathbb{N}\mathscr{A} = \sigma^\vee \cap M$ (and $m_{k+1}, \ldots, m_s \notin \sigma^\perp$).

Now, by Lemma 3.3.2, the point γ corresponds to a k tuple $(\gamma(m_1), \ldots, \gamma(m_k)) \in (\mathbb{C}^*)^k$. Then by Lemma 3.3.1, this is equivalent to showing that such that there exists some $t \in T_N$ satisfying $\chi^{m_i}(t) = \gamma(m_i)$ for all $i = 1, \ldots, k$. But this is immediate once we choose an isomorphism $\mathbb{Z}\mathscr{A} \simeq \mathbb{Z}^k \times \mathbb{Z}^{\operatorname{rank} M - k}$ so that $m_i \mapsto e_i$. □

In the proof given above, we see that $O(\sigma)$ is in fact in bijection with $(\mathbb{C}^*)^k$. The following lemma provides an intrinsic description of this torus.

Lemma 3.3.4. *Let N_σ be sublattice of N spanned by $\sigma \cap N$, and let $N(\sigma) := N/N_\sigma$. Then the pairing $\sigma^\perp \cap M \times N(\sigma) \to \mathbb{Z}$ induced from $M \times N \to \mathbb{Z}$ is a perfect pairing, and thus we have*

$$O(\sigma) \simeq \operatorname{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)}.$$

Proof. It is not hard to show that the map $N(\sigma) \to \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{Z})$ defined as $\overline{n} \mapsto \langle \overline{n}, \cdot \rangle$ is both injective and surjective. The first part $O(\sigma) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}^*)$ follows easily from the identification of $O(\sigma)$ and $\{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^{\perp} \cap M\}$ (Lemma 3.3.2), and the perfect pairing induces $\mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}^*) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq T_{N(\sigma)}$. □

We are now ready to state and prove the main theorem of this section:

Theorem 3.3.5 (Orbit-Cone Correspondence)**.** *Let $\Sigma \subset N_{\mathbb{R}}$ be a fan and X_Σ its toric variety. Then*

1. *There is a bijective correspondence:*

$$\begin{array}{rcl} \{\text{cones in } \Sigma\} & \longleftrightarrow & \{T_N\text{-orbits in } X_\Sigma\} \\ \sigma & \longleftrightarrow & O(\sigma) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}^*) \end{array}$$

2. *The affine open subset U_σ is the union of the its orbits:*

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$$

3. $\dim O(\sigma) + \dim \sigma = \dim N_{\mathbb{R}}$

Proof. Statment 2. follows immediately from Lemma 3.3.2 and Lemma 3.3.3, and statement 3. follows from Lemma 3.3.4. For statement 1., let O be a T_N-orbit of X_Σ. We show that $O = O(\sigma)$ for some $\sigma \in \Sigma$. Since X_Σ is covered by T_N-invariant subsets $\{U_\sigma\}_{\sigma \in \Sigma}$, which are in turn covered by orbits $\{O(\tau)\}_{\tau \prec \sigma}$, we have $O = O(\sigma)$ for some σ since two orbits are either equal or disjoint. □

Example 3.3.6. Consider $X_{\Delta_2} \simeq \mathbb{P}^2$ (cf. Example 3.2.10). One can check without Theorem 3.3.5 that the $T_{\mathbb{P}^2}$-orbits of $\mathbb{P}^2$ are given as $\{(r : s : t)\}, \{(0 : s : t)\}, \{(r : 0 : t)\}, \{(r : s : 0)\}, \{(0 : 0 : t)\}, \{(r : 0 : 0)\}, \{(0 : s : 0)\}$. Indeed, by applying Theorem 3.3.5 one obtains the following diagram:

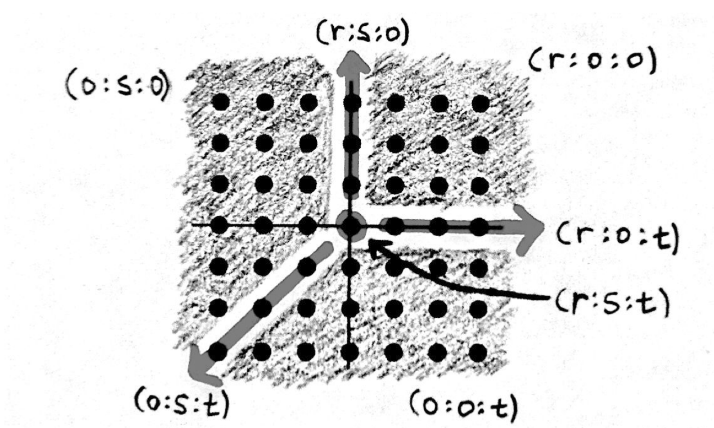
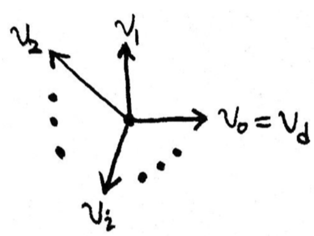


Figure 3.6: Orbits of $\mathbb{P}^2$ and the corresponding cones

3.4 Classification of complete smooth normal toric surfaces

In this section we classify all complete smooth normal toric surfaces as the second illustrative example of how the combinatorial structure of normal toric varieties informs their geometry. By combinatorial arguments about points in the lattice, we will find that every complete smooth normal toric surfaces arise from blow-ups of $\mathbb{P}^2$ or Hirzeburch surfaces $\mathscr{H}_r$.

One can show that the toric variety X_Σ of a fan $\Sigma \subset N_\mathbb{R}$ is complete if and only if the the support of Σ, defined $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma$, is equal to the whole $N_\mathbb{R}$ (for proof, see [Ful93, §2.4]). Combining this with Proposition 3.1.6, we have that complete smooth normal toric surfaces are given by a sequence of lattice points $v_0, v_1, \ldots, v_d = v_0$ in counterclockwise order in $N = \mathbb{Z}^2$ such that each consecutive pair of points $\{v_i, v_{i+1}\}$ forms a basis of $\mathbb{Z}^2$.



Thus, to classify all complete smooth normal toric surfaces, we consider what conditions the lattice points $v_0, \ldots, v_d = v_0$ need satisfy. Without loss of generality, we can assume $v_0 = e_1$ and $v_1 = e_2$. This forces $v_2 = -e_1 + ke_2$ for some $k \in \mathbb{Z}$. Thus, if $d = 3$ and 4, one can check that the only possible surfaces are $\mathbb{P}^2$ and $\mathscr{H}_r$, respectively. We can now state the main theorem:

Theorem 3.4.1. *Every complete smooth normal toric variety can be obtained by successive blow-ups starting either from $\mathbb{P}^2$ or $\mathscr{H}_r$.*

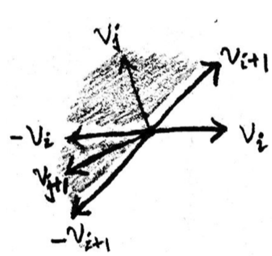
Proof. The statement holds for $d = 3$ and 4, and thus theorem follows from the remark at the end of Example 3.1.5 and the following claim:

Claim: For $d \geq 5$, there exists j such that $v_j = v_{j-1} + v_{j+1}$. $\square$

We now work towards the proof of the claim.

Lemma 3.4.2. *Let lattice points $v_0, \ldots, v_d = v_0 \in \mathbb{Z}^2$ be given as above. Then:*

(a) There cannot exist pairs $\{v_i, v_{i+1}\}$, $\{v_j, v_{j+1}\}$ such that v_j is in the angle strictly between v_{i+1} and $-v_i$ and v_{j+1} is in the angle strictly between $-v_i$ and v_{i+1}, as drawn below



(b) For every $1 \leq i \leq d$, there exists an integer a_i such that $a_i v_i = v_{i-1} + v_{i+1}$

Proof. Note that since $B_i = (v_i, v_{i+1}),\ B_j = (v_j, v_{j+1})$ are both bases of $\mathbb{Z}^2$ oriented the same way, if P is the change of basis matrix from B_i to B_j then $\det P = 1$. However, if v_j, v_{j+1} are positioned as stated in (a), then we have $\begin{pmatrix} v_i & v_{i+1} \end{pmatrix} \begin{bmatrix} -a & -c \\ b & -d \end{bmatrix} = \begin{pmatrix} v_j & v_{j+1} \end{pmatrix}$ for some positive integers a, b, c, d, and hence the determinant $ad + bc \geq 2$. This proves (a).

For (b), we note that the change of basis matrix P from (v_{i-1}, v_i) to (v_i, v_{i+1}) has $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as its first column, and hence the second column must be $\begin{bmatrix} -1 \\ a_i \end{bmatrix}$ for some $a_i \in \mathbb{Z}$. $\square$

*Proof of **Claim***. We first show that if $d \geq 4$, then there exist v_i, v_k such that $-v_i = v_k$. For the sake of contradiction, suppose not. Consider the maximal chain of lattice points that all lie in a same half-space. Without loss of generality, suppose the chain is $v_0, v_1, \ldots, v_k$ and $v_0 = e_1, v_1 = e_2$. Since $d \geq 4$, one can check that $k \geq 2$. Considering the pairs (v_i, v_{i+1}) and (v_k, v_{k+1}) for $0 \leq i \leq k-1$, Lemma 3.4.2(a) implies that v_{k+1} is not between $-v_i$ and $-v_{i+1}$. Since v_{k+1} cannot be an opposite of $v_0, \ldots, v_k$, we have that v_{k+1} lies between $-v_k$ and v_0, which contradicts the maximality of the chain.

Thus, for $d \geq 5$, we have lattice points v_i, v_k that are opposites, and moreover, one of two half-spaces divided by v_i, v_k has at least 4 lattice points. So, without loss of generality, let $v_i = v_0 = e_1$, $v_1 = e_2$, and $k \geq 3$. For $1 \leq j \leq k$, define $c_j := b_j + b'_j$ where $v_j = -b_j v_0 + b'_j v_1$. Note that $c_2 \geq 2$ and $c_1 = 1,\ c_k = 1$. Thus, there exists $2 \leq j' \leq k-1$ such that $c_{j'} > c_{j'+1}$ and $c_{j'} \geq c_{j'-1}$. From Lemma 3.4.2(b) we have that $a_{j'} c_{j'} = c_{j'-1} + c_{j'+1}$, and since $c_{j'} > c_{j'+1}$ and $c_{j'} \geq c_{j'-1}$, we have $a_{j'} = 1$ and thus $v_{j'} = v_{j'-1} + v_{j'+1}$, as desired. $\square$

Classifying higher dimensional smooth toric varieties is an active research problem.

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