A Tale of Two Rings: MaTroCom Minicourse

Christopher Eur

These are lectures notes prepared for a mini-course at MaTroCom Workshop `https://sites.google.com/view/matrocomlondon/home` held in Queen Mary University of London. We assume some familiarity with polyhedra and an acquaintance with matroids. Statements involving algebraic geometry (toric varieties) are in a different color, which may be skipped.

Notation. Let $E = \{1, \ldots, n\}$ be a finite set of cardinality n. For a subset $S \subseteq E$, denote by $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ the sum of standard basis vectors in $\mathbb{R}^E$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product. Let $T = (\mathbb{C}^*)^E$ be the torus whose character lattice is $\mathbb{Z}^E$. A variety is reduced and irreducible.

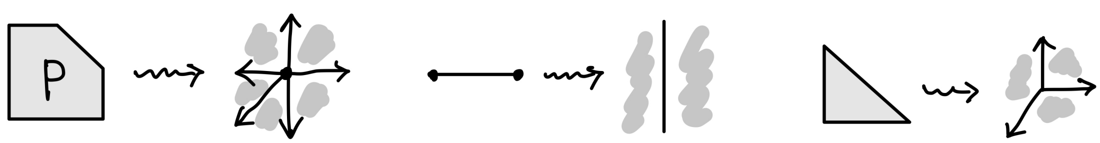
1 Overview

Let Σ be a unimodular projective fan in $\mathbb{R}^E$. *Projective* means that Σ is the (inner) normal fan Σ_P of a polytope P in $\mathbb{R}^E$. We allow $\dim P < n$, so the lineality space

$$\operatorname{lin}(\Sigma) = (\text{the minimal cone of } \Sigma) = \{u \in \mathbb{R}^E : \langle u, x \rangle = 0 \ \forall x \in P\}$$

may be nontrivial of dimension ℓ. *Unimodular* means that for any cone $\sigma \in \Sigma$, the *primitive ray vectors* of $\sigma / \operatorname{lin}(\Sigma)$ extends to a $\mathbb{Z}$-basis of $\mathbb{Z}^E / (\operatorname{lin}(\Sigma) \cap \mathbb{Z}^E)$. Let $\Sigma(d)$ be the set of d-dimensional cones of Σ.

Example 1.1.



Recall the dimension-reversing bijection between the faces of a polytope P and the cones of its normal fan Σ_P given by:

$$\Sigma_P \ni \sigma \leftrightarrow \operatorname{face}_\sigma(P) = \{p \in P : \langle p, v \rangle = \min_{q \in P} \langle q, v \rangle\} \text{ for any } v \text{ in the relative interior of } \sigma.$$

A lattice polytope Q is a *deformation* of Σ, denoted $Q \in \operatorname{Def}(\Sigma)$, if its normal fan Σ_Q coarsens Σ. For instance, deformations of Σ in Example 1.1 include rectangles, standard simplices, etc.

Let X_Σ be the smooth projective toric variety associated to $\Sigma / \operatorname{lin}(\Sigma)$, considered as a T-variety. Recall similarly the bijection between the cones of Σ and the torus-orbits of X_Σ. In particular, the maximal cones correspond to the torus-fixed points of X_Σ. A deformation Q of Σ corresponds to the base-point-free T-line bundle $\mathcal{L}_Q$ on X_Σ whose complete linear system gives a map $X_\Sigma \to \mathbb{P}^{|Q\cap\mathbb{Z}^E|-1}$ induced by the map of tori $t \mapsto (t^{\mathbf{m}})_{\in Q\cap\mathbb{Z}^E}$ [CLS11, Chapter 6].

We will learn about two well-studied rings $K(\Sigma)$ and $A^\bullet(\Sigma)$ attached to such Σ. In geometric terms, these are the Grothendieck K-ring of vector bundles and the Chow cohomology ring of the toric variety X_Σ. "GKM-varieties" is a good keyword for those wanting more geometric details. By associating to each matroid certain elements in these rings, one gains an insight into combinatorial properties of matroids via geometric methods.

2 K-rings and matroid polytopes

2.1 K-rings, polytope algebras, and "piecewise" Laurent polynomials

We first describe the ring denoted $K_T(\Sigma)$ and then describe $K(\Sigma)$ as its quotient. It has three different descriptions, whose equivalence is a consequence of some major theorems.

Definition 2.1. Let $K_T(\Sigma)$ be the Grothendieck K-ring of T-equivariant vector bundles on X_Σ, and let $K(\Sigma)$ be the non-equivariant K-ring. That is,

$$K_T(\Sigma) = \frac{\Big\{\sum_i a_i[\mathcal{E}_i]^T : a_i \in \mathbb{Z},\ \mathcal{E}_i \text{ a } T\text{-equivariant vector bundle on } X_\Sigma\Big\}}{\Big\langle [\mathcal{E}]^T = [\mathcal{E}']^T + [\mathcal{E}'']^T : \exists \text{ a } T\text{-equivariant SES } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0\Big\rangle}, \quad \text{and}$$
$$K(\Sigma) = \frac{\Big\{\sum_i a_i[\mathcal{E}_i] : a_i \in \mathbb{Z},\ \mathcal{E}_i \text{ a vector bundle on } X_\Sigma\Big\}}{\Big\langle [\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}''] : \exists \text{ a SES } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0\Big\rangle},$$

with the multiplication is given by tensor products.

The natural "forgetting T-equivariance" map $K_T(\Sigma) \to K(\Sigma)$ is a surjection [Mor93, Proposition 3]. This geometrically defined ring has the following combinatorial descriptions. For $\mathbf{m} \in \mathbb{Z}^E$, denote by $\mathbf{T}^{\mathbf{m}}$ the Laurent monomial $T_1^{m_1} \cdots T_n^{m_n} \in \mathbb{Z}[\mathbf{T}^\pm] = \mathbb{Z}[T_1^\pm, \dots, T_n^\pm]$.

Theorem 2.2. The rings $K_T(\Sigma)$ and $K(\Sigma)$ have the following equivalent descriptions:

1. For a polytope $Q \subset \mathbb{R}^E$, let $1_Q : \mathbb{R}^E \to \mathbb{Z}$ be its indicator function given by $1_Q(x) = 1$ if $x \in Q$ and $1_Q(x) = 0$ otherwise. Then, we have by [EHL, Theorem A.10]

$$K_T(\Sigma) \simeq \mathbb{I}(\Sigma) = \text{the subgroup of } \mathbb{Z}^{(\mathbb{R}^E)} \text{ generated by } \{1_Q \mid Q \in \operatorname{Def}(\Sigma)\}, \quad \text{and}$$
$$K(\Sigma) \simeq \overline{\mathbb{I}}(\Sigma) = \mathbb{I}(\Sigma)/\operatorname{transl}(\Sigma)$$

   where $\operatorname{transl}(\Sigma)$ is the subgroup of $\mathbb{I}(\Sigma)$ generated by $\{1_Q - 1_{Q+u} \mid u \in \mathbb{Z}^m\}$. Multiplication in these rings are given by Minkowski sums of polytopes. Denote by $[Q]$ the class of 1_Q in $\overline{\mathbb{I}}(\Sigma)$. The ring $\overline{\mathbb{I}}(\Sigma)$ is also known as the *polytope algebra* [McM89].

2. For two maximal cones σ and σ' of Σ sharing a wall (i.e. a codimension 1 face), let $\mathbf{m}(\sigma, \sigma')$ be the primitive vector normal to $\sigma \cap \sigma'$. Then, we have by [Nie74, VV03]

$$K_T(\Sigma) \simeq LP(\Sigma) = \left\{ (f_\sigma)_{\sigma \in \Sigma_{\max}} \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}] \;\middle|\; \begin{array}{l} f_\sigma - f_{\sigma'} \equiv 0 \bmod (1 - \mathbf{T}^{\mathbf{m}(\sigma,\sigma')}) \\ \text{for any } \sigma \text{ and } \sigma' \text{ sharing a wall} \end{array} \right\}, \quad \text{and}$$
$$K(\Sigma) \simeq \overline{LP}(\Sigma) = LP(\Sigma)/I_K$$

where I_K is the ideal generated by $\{T_i - 1 : i \in E\}$ where T_i here is considered as an element $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}]$ by $f_\sigma = T_i$ for all σ.

Given $Q \in \mathrm{Def}(\Sigma)$, the claimed isomorphisms are given by $1_Q \mapsto (\mathbf{T}^{-\operatorname{face}_\sigma(Q)})_\sigma \in LP(\Sigma)$ and $1_Q \mapsto [\mathcal{L}_Q] \in K_T(\Sigma)$. The isomorphism $K_T(\Sigma) \simeq LP(\Sigma)$ is also described by $[\mathcal{E}]^T \mapsto \mathrm{Hilb}(\mathcal{E}|_{p_\sigma})_\sigma$, the restriction to the torus-fixed points.

These "K-rings" have distinguished maps to $\mathbb{Z}[\mathbf{T}^{\pm}]$ and $\mathbb{Z}$. Let $\chi^T : K_T(\Sigma) \to \mathbb{Z}[\mathbf{T}^{\pm}]$ and $\chi : K(\Sigma) \to \mathbb{Z}$ be the sheaf Euler characteristic maps. Combinatorial descriptions of these are:

Theorem 2.3. [CLS11, Ch. 9] For $1_Q \in \mathbb{I}(\Sigma)$, under the isomorphism $K_T(\Sigma) \simeq \mathbb{I}(\Sigma)$ we have

$$\chi^T(1_Q) = \sum_{\mathbf{m} \in Q \cap \mathbb{Z}^E} \mathbf{T}^{-\mathbf{m}} \quad \text{and} \quad \chi([Q])) = |Q \cap \mathbb{Z}^E|.$$

[Bri88, Ish90] For an element in $f \in K_T(\Sigma)$ given by $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}]$, we have

$$\chi^T(f) = \sum_{\sigma \in \Sigma_{\max}} \frac{f_\sigma}{\prod_{\substack{\mathbf{m} \text{ a primitive ray} \\ \text{generator of } \sigma^\vee}} (1 - \mathbf{T}^{-\mathbf{m}})} \quad \text{and} \quad \chi(f) = \chi^T(f)|_{T_1 = \cdots = T_n = 1}.$$

Exercise 2.4. Let Σ be the normal fan of the polytope P in the following. Verify the formula for χ^T in the case when $P = \mathrm{Conv}\{\mathbf{e}_1, \mathbf{e}_2\} \subset \mathbb{R}^2$ and $Q = \mathrm{Conv}\{2\mathbf{e}_1, 2\mathbf{e}_2\}$. If you'd like to get a feel of how quickly nontrivial it gets, try $P = Q = \mathrm{Conv}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathbb{R}^3$.

2.2 Permutohedral fan and base polytopes of matroids

Let $\mathfrak{S}_E$ be the permutation group of E. The *permutohedron* on E is the polytope

$$\Pi_E = \text{convex hull of } \{w \cdot (0, \ldots, n-1) : w \in \mathfrak{S}_E\}.$$

Let the *permutohedral fan* Σ_E be its normal fan in $\mathbb{R}^E$ with lineality space $\mathbb{R}\mathbf{e}_E$. It consists of the cones

$$\sigma_{\mathscr{C}} : \mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \ldots, \mathbf{e}_{S_k}\} + \mathbb{R}\mathbf{e}_E$$

for $\mathscr{C} = (\varnothing \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E)$ a nonempty proper chain of subsets of E. The *permutohedral variety* X_E is the toric variety of the fan $\Sigma_E/\mathbb{R}\mathbf{e}_E$, which contains the projectivized torus $\mathbb{P}T$ as its open dense torus, and is considered as a T-variety.

Exercise 2.5. Show that the span of a codimension 1 cone $\sigma \in \Sigma_E(n-1)$ is the hyperplane $\{x_i = x_j\} \subset \mathbb{R}^E$ for some $i \neq j \in E$.

Proposition 2.6. [Pos09, ACEP20] A lattice polytope $Q \subset \mathbb{R}^E$ is a deformation of Σ_E if and only if each edge of Q is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$. Deformations of Σ_E are also known as (integral) *generalized permutohedra.*

Matroids finally enter into our picture as follows.

Theorem 2.7. [GGMS87] For a collection $\mathcal{B} \subseteq 2^E$ of subsets of E, the polytope

$$\text{convex hull of } \{\mathbf{e}_B : B \in \mathcal{B}\} \subset \mathbb{R}^E.$$

is a generalized permutohedron if and only if $\mathcal{B}$ is the set of basis of a matroid M on E.

For a matroid M on E, we call the polytope in the theorem the *base polytope* of M, denoted $P(\mathrm{M})$. The theorem implies that the base polytopes of matroids are exactly the generalized permutohedra contained in the unit cube $[0,1]^E$.

Remark 2.8. When a matroid M of rank r has a realization by $L \subseteq \mathbb{C}^E$, that is, a point in the Grassmannian $Gr(r;E)$ with the usual T-action, we have that $\overline{T \cdot L}$ is isomorphic to the toric variety of the base polytope. The line bundle $\mathcal{L}_{P(M)}$ is the pullback of $\mathcal{O}(1)$ on $Gr(r;E)$ along the composition $X_E \to X_{P(\mathrm{M})} \to Gr(r;E)$.

Exercise 2.9. Deduce the greedy algorithm property of matroids from the fact that base polytopes of matroids are generalized permutohedra.

The following notion of *valuativity* is a powerful tool in the study of matroid invariants [AFR10, DF10, AS22, BEST, FS].

Definition 2.10. For $0 \le r \le n$, define the (rank r) *valuative group* by

$$\mathrm{Val}_r(E) = \text{the subgroup of } \mathbb{I}(\Sigma_E) \text{ generated by } \{1_{P(\mathrm{M})} : \mathrm{M} \text{ a matroid on } E \text{ of rank } r\}.$$

A function f on the set of matroids on E with values in an abelian is *valuative* if it factors through $\bigoplus_{r=0}^n \mathrm{Val}_r(E)$.

An element $i \in E$ is a *loop* in a matroid M if i is in no basis of M, or equivalently $P(\mathrm{M}) \subset \{x_i = 0\}$. Dually, an element $i \in E$ is a *coloop* if i is in every basis of M, or equivalently $P(\mathrm{M}) \subset \{x_i = 1\}$. Let $\mathrm{Val}_r^\circ(E)$ be the subgroup of $\mathrm{Val}_r(E)$ generated by the loopless matroids.

Exercise 2.11. Let $E = \{1,2,3,4\}$. Compute that the ranks of the groups $\mathrm{Val}_r^\circ(E)$ for $r = 0, \dots, 4$ are $0, 1, 11, 11, 1$. Compare this to the h-vector of the simple polytope Π_E (i.e. the h-vector of the simplicial complex which is the full barycentric subdivision of the boundary of the 3-simplex—that is, take stellar subdivisions of all facets, and then of all original codimension 1 faces, and so forth).

Theorem 2.12. We have an isomorphism

$$\bigoplus_{r=1}^{n} \mathrm{Val}_r^\circ(E) \xrightarrow{\sim} \overline{\mathbb{I}}(\Sigma_E) \quad \text{given by} \quad 1_{P(\mathrm{M})} \mapsto [P(\mathrm{M})].$$

This is a special case of [EL, Theorem 6.1]. Similar statements can be found in [EHL, Proposition 7.4] and [EFLS, Corollary 2.16].

Corollary 2.13. For any valuative function f on the set of loopless matroids with values in $\mathbb{Z}$, there exists loopless matroids $\mathrm{M}_1, \dots, \mathrm{M}_k$ and integers $a_1, \dots, a_k$ such that

$$f(\mathrm{M}) = \chi\big([P(\mathrm{M})] \cdot (a_1[P(\mathrm{M}_1)] + \cdots + a_k[P(\mathrm{M}_k)])\big) \quad \text{for all loopless matroid M on } E.$$

Proof. Combine the theorem with the fact [AP15, Theorem 6.1] that the bilinear map $K(\Sigma) \times K(\Sigma) \to \mathbb{Z}$ defined by $(a, b) \mapsto \chi(a \cdot b)$, is non-degenerate. □

In fact, any valuative function on the loopless matroids extends to all matroids by sending matroids with loops to zero. That is, the conclusion of the corollary holds for all valuative function on matroids that is zero on matroids with loops.

Sketch of the proof of Theorem 2.12. The key fact is that the intersection of an (integral) generalized permutohedra with a coordinate half-space is an (integral) generalized permutohedra. Then, one considers tiling $\mathbb{R}^E$ by integral translates of unit cubes, and observes that base polytopes of matroids are translates of each other if and only if the matroids differ by converting some loops to coloops. □

Exercise 2.14. Finish the proof of the theorem, given the key fact.

3 Chow cohomology and Bergman fans

By *primitive ray generators* $\mathcal{R}(\Sigma)$ of a fan Σ with possibly nontrivial lineality space, we mean a set of vectors in $\mathbb{Z}^E$ whose images in $\mathbb{R}^E/\operatorname{lin}(\Sigma)$ are the primitive ray generators of $\Sigma/\operatorname{lin}(\Sigma)$. Fix such a choice, and denote $u_\rho \in \mathcal{R}(\Sigma)$ for each $\rho \in \Sigma(\ell+1)$. Let us also fix a $\mathbb{Z}$-basis $(u_1, \dots, u_\ell)$ of $\operatorname{lin}(\Sigma) \cap \mathbb{Z}^E$.

Example 3.1. For the permutohedral fan Σ_E, our choice of $\mathcal{R}(\Sigma_E)$ will always be $\{\mathbf{e}_S : \varnothing \subsetneq S \subsetneq E\}$, and we fix $\{\mathbf{e}_E\}$ as the basis of $\operatorname{lin}(\Sigma_E)$.

3.1 Cohomology rings, Minkowski weights, and piecewise polynomials

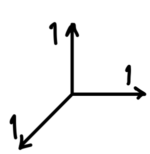
Like the K-ring case, we describe (torus-equivariant) Chow cohomology rings in three different ways. We will need the following notion of Minkowski weights (a.k.a. tropical cycles).

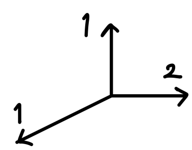
Definition 3.2. A d-dimensional *Minkowski weight* on Σ is a function $w : \Sigma(d) \to \mathbb{Z}$ such that for any $\tau \in \Sigma(d-1)$, it satisfies

$$\sum_{\sigma \supset \tau} w(\sigma) u_{\sigma\setminus\tau} \in \operatorname{span}_{\mathbb{R}}(\tau).$$

where $u_{\sigma\setminus\tau}$ denotes the unique primitive ray generator in σ not in τ. The group of d-dimensional Minkowski weights is denoted $\mathrm{MW}_d(\Sigma)$.

Example 3.3.





Example 3.4. As Σ is a complete (i.e. its support is all of $\mathbb{R}^E$), the constant function $\Sigma(n) \to \mathbb{Z}$ sending $\sigma \mapsto 1$ for all $\sigma \in \Sigma(n)$ is the Minkowski weight of weight n up to scaling.

Exercise 3.5. Show that: For a deformation Q of Σ, the function $w_Q : \Sigma(n-1) \to \mathbb{Z}$ given by

$$w_Q(\sigma) = \text{(lattice) length of } \operatorname{face}_\sigma(Q)$$

is a Minkowski weight.

Remark 3.6. Given a *very affine variety*, i.e. a subvariety X of $(\mathbb{C}^*)^E$, its *tropicalization* $\operatorname{trop}(X)$ is a pure-dimensional polyhedral complex in $\mathbb{R}^E$ with appropriate weights on the maximal cells. It defines a Minkowski weight on any Σ containing a subfan whose support equals the support of $\operatorname{trop}(X)$. Moreover, under the isomorphism in Theorem 3.7, this Minkowski weight equals the Chow homology class $[\overline{X}]$ of closure of X in X_Σ. See [MS15, Ch. 6].

Let $A_T^\bullet(\Sigma)$ and $A^\bullet(\Sigma)$ be the *T-equivariant Chow cohomology ring* and *Chow cohomology ring* of X_Σ, respectively. These geometric rings have the following combinatorial descriptions. For $\mathbf{m} \in \mathbb{Z}^E$, denote by $\mathbf{m} \cdot \mathbf{t}$ the polynomial $m_1 t_1 + \cdots + m_n t_n \in \mathbb{Z}[\mathbf{t}] = \mathbb{Z}[t_1, \ldots, t_n]$.

Theorem 3.7. The rings $A_T^\bullet(\Sigma)$ and $A^\bullet(\Sigma)$ have the following equivalent descriptions:

1. [CLS11, Theorems 12.4.4 & 12.4.14] Let $[\ell] = \{1, \ldots, \ell\}$, indexing the fixed basis $\{u_1, \ldots, u_\ell\}$ of $\operatorname{lin}(\Sigma) \cap \mathbb{Z}^E$. As standard graded polynomial rings we have

$$A_T^\bullet(\Sigma) = \frac{\mathbb{Z}[x_\rho : \rho \in \Sigma(\ell+1) \cup [\ell]]}{\left\langle \prod_{\rho \in S} x_\rho : S \subseteq \Sigma(\ell+1) \cup [\ell] \text{ such that } \{u_\rho\}_{\rho \in S} \text{ not in a common cone of } \Sigma \right\rangle}, \quad \text{and}$$

$$A^\bullet(\Sigma) = \frac{A_T^\bullet(\Sigma)}{\left\langle \sum_{\rho \in \Sigma(\ell+1) \cup [\ell]} \langle u_\rho, m \rangle x_\rho : m \in \mathbb{Z}^E \right\rangle}.$$

2. [FS97] For $0 \le i \le n - \ell$, denote by $\operatorname{MW}^i(\Sigma) = \operatorname{MW}_{n-\ell-i}(\Sigma)$. We have

$$A^\bullet(\Sigma) \simeq \operatorname{MW}^\bullet(\Sigma)$$

where the multiplication in MW is given by *stable intersections* [FS97].

3. Recall the notation that $\mathbf{m}(\sigma, \sigma')$ denotes the primitive vector normal to $\sigma \cap \sigma'$ for two maximal cones σ and σ' of Σ sharing a wall. Then, we have [Bri96, Bri97]

$$A_T^\bullet(\Sigma) \simeq PP(\Sigma) = \left\{ (f_\sigma)_{\sigma \in \Sigma_{\max}} \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{t}] \;\middle|\; \begin{array}{l} f_\sigma - f_{\sigma'} \equiv 0 \bmod \mathbf{m}(\sigma, \sigma') \cdot \mathbf{T} \\ \text{for any } \sigma \text{ and } \sigma' \text{ sharing a wall} \end{array} \right\}, \quad \text{and}$$

$$A^\bullet(\Sigma) \simeq \overline{PP}(\Sigma) = PP(\Sigma)/I_A$$

where I_A is the ideal generated by $\{t_i\}$ where t_i here is considered as an element $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{t}]$ by $f_\sigma = t_i$ for all σ.

Given a deformation Q, one obtains

$$D_Q = \sum_{\rho \in \Sigma(\ell+1) \cup [\ell]} - \min_{m \in Q} \langle u_\rho, m \rangle x_\rho \in A_T^\bullet(\Sigma) \quad \text{and} \quad (-\operatorname{face}_\sigma(Q) \cdot \mathbf{t})_\sigma \in PP(\Sigma).$$

Under the claimed isomorphism $A_T^\bullet(\Sigma) \simeq PP(\Sigma)$, these two elements agree, and their image in $A^\bullet(\Sigma)$ is the Minkowski weight w_Q in Exercise 3.5 under the isomorphism $A^\bullet(\Sigma) \simeq \operatorname{MW}^\bullet(\Sigma)$.

Example 3.8. For $\Sigma = \Sigma_E$, for reach "ray" $\rho_S = \mathbb{R}_{\geq 0}\mathbf{e}_S + \mathbb{R}\mathbf{e}_E$ of Σ_E, let us denote x_{ρ_S} by x_S. Then, we compute that

$$A^\bullet_T(\Sigma_E) = \frac{\mathbb{Z}[x_S : \varnothing \subsetneq S \subseteq E]}{\langle x_S x_{S'} : S, S' \text{ incomparable}\rangle} \quad \text{and} \quad A^\bullet(\Sigma_E) = \frac{A^\bullet_T(\Sigma_E)}{\langle \sum_{S \ni i} x_S : i \in E\rangle}.$$

Theorem 3.9. [Ful93, Ch. 5] There is an isomorphism, called the *degree map* and denoted $\deg_\Sigma : A^{n-\ell}(\Sigma) \xrightarrow{\sim} \mathbb{Z}$, such that $\deg_\Sigma(\prod_{\rho \in \sigma(\ell+1)} x_\rho) = 1$ for all $\sigma \in \Sigma_{\max}$. The degree map is characterized by

$$\deg_\Sigma(D_Q^{n-\ell}) = \mathrm{Volume}(Q) \quad \text{for any } Q \in \mathrm{Def}(\Sigma),$$

and in terms of the description of $A^\bullet(\Sigma)$ as $\overline{PP}(\Sigma)$, one has

$$\deg_\Sigma((f_\sigma)_\sigma) = \sum_{\sigma \in \Sigma_{\max}} \frac{f_\sigma}{\prod_{\substack{\mathbf{m} \text{ a primitive ray} \\ \text{generator of } \sigma^\vee}} (\mathbf{m} \cdot \mathbf{t})}$$

Here, Volume is defined by declaring that the volume of a full-dimensional unit simplex in $\mathrm{lin}(\Sigma)^\perp = \{x \in \mathbb{R}^E : \langle x, y\rangle = 0 \text{ for all } y \in \mathrm{lin}(\Sigma)\}$ is equal to 1.

3.2 Bergman fans of matroids

Definition 3.10. A *flat* of a matroid M on E is a subset $F \subseteq E$ such that

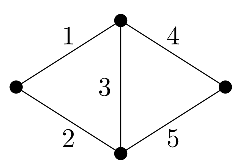
$$\max\{|(F \cup i) \cap B| : B \text{ a basis of M}\} > \max\{|F \cap B| : B \text{ a basis of M}\} \quad \text{for all } i \in E \setminus F.$$

Or equivalently, a flat is a subset of E that is maximal for its rank.

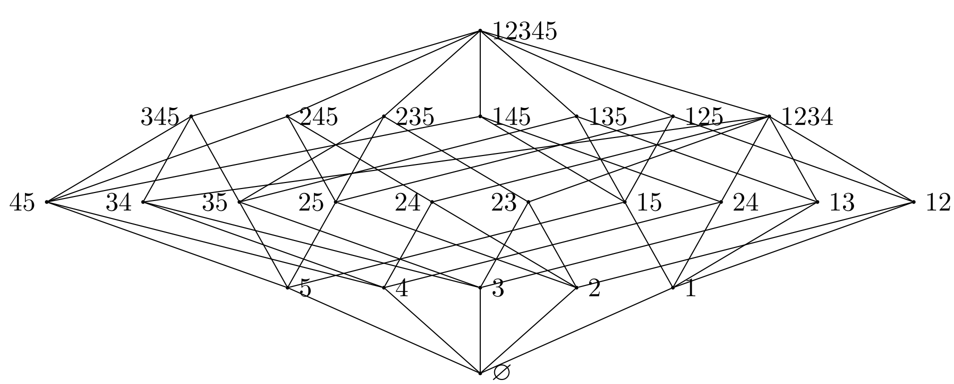
Note that $\varnothing$ is a flat of M if and only if M is loopless. The set of flats of a matroid M forms a poset under inclusion. This poset is a *lattice* with meet and join defined by

$$F \wedge F' = F \cap F' \quad \text{and} \quad F \vee F' = \text{the smallest flat containing } F \cup F'.$$

Example 3.11. Let M be the graphical matroid of the graph



The lattice of flats of M is as follows.



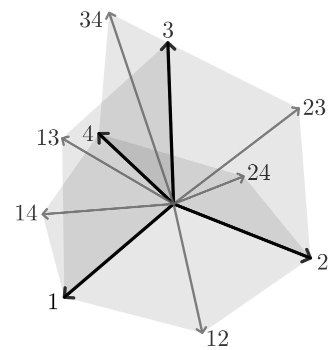
Definition 3.12. The Bergman fan of a loopless matroid M is a subfan Σ_{M} of Σ_E consisting of cones

$$\sigma_{\mathscr{F}} = \mathrm{cone}\{\mathbf{e}_{F_1}, \dots, \mathbf{e}_{F_k}\} + \mathbb{R}\mathbf{e}_E$$

for every chain $\mathscr{F} = \{\varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\}$ of nonempty proper flats of M.

Example 3.13. When $\mathrm{M} = U_{n,n}$ (sometimes called the *Boolean matroid* on E), we have that $\Sigma_{\mathrm{M}} = \Sigma_E$.

Example 3.14. The following is the Bergman of $U_{3,4}$ with the lineality space $\mathbb{R}\mathbf{e}_E$ quotiented out.



Proposition 3.15. Let M be a loopless matroid of rank r on E.

(a) [AHK18, Proposition 5.2] The function

$$w_{\mathrm{M}} : \Sigma_E(r) \to \mathbb{Z} \quad \text{defined by} \quad w_{\mathrm{M}}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in \Sigma_{\mathrm{M}} \\ 0 & \text{otherwise} \end{cases}$$

is a Minkowski weight, called the *Bergman class* of a (loopless) matroid M.

(b) [BEST, Theorem 7.6] For $\sigma \in \Sigma_E(n)$, denote by $B_{\overline{\sigma}}(\mathrm{M}^\perp)$ the basis of the dual matroid $\mathrm{M}^\perp$ such that $\mathbf{e}_{B_{\overline{\sigma}}(\mathrm{M}^\perp)} = \mathrm{face}_\sigma(\mathrm{M}^\perp)$. Define an element w_{M}^T in $PP(\Sigma_E)$ by

$$(w_{\mathrm{M}})^T_\sigma = \prod_{i \in B_{\overline{\sigma}}(\mathrm{M}^\perp)} -t_i = (-1)^{n-r} t^{\mathbf{e}_{B_{\overline{\sigma}}(\mathrm{M}^\perp)}} \quad \text{for every } \sigma \in \Sigma(n).$$

Then, its image in $\overline{PP}(\Sigma_E)$, under the isomorphism $\overline{PP}(\Sigma_E) \simeq \mathrm{MW}^\bullet(\Sigma)$, equals w_{M}.

When M has a loop, by definition we have $w_{\mathrm{M}} = 0$. Note that $(w_{\mathrm{M}})^T_\sigma$ as defined in part (b) makes sense even when M has a loop $k \in E$, but its image in $\overline{PP}(\Sigma_E)$ is zero because it is divisible by the global t_k.

Exercise 3.16. If M is a loopless matroid of rank $n-1$ on E, then $w_{\mathrm{M}} = D_{P(\mathrm{M}^\perp)}$.

Remark 3.17. Per Remark 3.6, one has the following motivation for the Bergman fan Σ_{M} and its Minkowski weight w_{M}. When M has a realization by a linear subspace $L \subseteq \mathbb{C}^E$, its tropicalization is the Bergman fan M. The closure of $\mathbb{P}L \cap \mathbb{P}T$ inside the permutohedral toric variety X_E is known as the "wonderful compactification," and its homology class as a subvariety of X_E equals the class w_{M}.

The following is a key application of the Hodge theory of matroids developed in [AHK18].

Theorem 3.18. Let Q_1 and Q_2 be deformations of Σ_E, and M a loopless matroid of rank r. Then,

$$\big(\deg_{\Sigma_E}(w_{\mathrm{M}}\cdot D_{Q_1}^{r-1}), \deg_{\Sigma_E}(w_{\mathrm{M}}\cdot D_{Q_1}^{r-2}D_{Q_2}), \dots, \deg_{\Sigma_E}(w_{\mathrm{M}}\cdot D_{Q_1}^{r-1-i}D_{Q_2}^{i}), \dots, \deg_{\Sigma_E}(w_{\mathrm{M}}\cdot D_{Q_2}^{r-1})\big)$$

is a log-concave nonnegative sequence with no internal zeros.

Setting Q_1 and Q_2 to be the standard and the opposite simplex, the authors of [AHK18] resolved the long-standing conjecture on a log-concavity of the coefficients of $T_{\mathrm{M}}(1+x,0)$ and $T_{\mathrm{M}}(1+x,1)$.

4 One ring to rule them all..

4.1 Exceptional Hirzebruch-Riemann-Roch

The classical Hirzebruch-Riemann-Roch theorem states that for a smooth projective variety X, one has an isomorphism $ch : K(X)\otimes\mathbb{Q} \xrightarrow{\sim} A^\bullet(X)\otimes\mathbb{Q}$ such that

$$\chi(\mathcal{E}) = \deg_X\big(ch(\mathcal{E})\cdot Td(X)\big)$$

where $Td(X)\in A^\bullet(X)$ is the Todd class of X. The Todd class is often quite hard to get a good enough handle for combinatorial purposes. For the permutohedral fan Σ_E however, we have the following "exceptional" Hirzebruch-Riemman-Roch-type formula.

Theorem 4.1. [BEST, Theorem D] There exists a unique ring isomorphism $\phi : K(\Sigma_E)\xrightarrow{\sim} A^\bullet(\Sigma_E)$ such that, for any (not necessarily loopless) matroid M,

$$\phi([P(\mathrm{M}^\perp)]) = \xi_0 + \cdots + \xi_{n-r}$$

with $\xi_i\in A^i(\Sigma_E)$ for all $i=0,\dots,n-1$ and $\xi_{n-r}=w_{\mathrm{M}}$. Moreover, for $\mathcal{E}\in K(X_E)$, one has

$$\chi(\mathcal{E}) = \deg_{\Sigma_E}\big(\phi(\mathcal{E})\cdot(1+D_\nabla+\cdots+D_\nabla^{n-1})\big)$$

where $\nabla = P(U_{n-1,n}) = \mathrm{Conv}\{\mathbf{e}_{E\setminus i} : i\in E\}$ is the opposite standard simplex.

Sketch of proof. To define the ring map, we use the description of K as the ring $\overline{LP}$ and of $A^\bullet$ as $\overline{PP}$. That is, given a tuple of Laurent polynomials $(f_\sigma(T_1,\dots,T_n))_\sigma\in\prod_{\sigma\in\Sigma_E(n)}\mathbb{Z}[T_1^{\pm},\dots,T_n^{\pm}]$, we map

$$\big(f_\sigma(T_1,\dots,T_n)\big)_\sigma \mapsto \left(f_\sigma\Big(\frac{1}{1-t_1},\dots,\frac{1}{1-t_n}\Big)\right)_\sigma,$$

which is obviously an invertible operation. Ignoring the issue that the image may not be a polynomial but a rational function, we see that this map is well-defined, because if a Laurent polynomial $f(T_1,\dots,T_n)\equiv 0 \bmod 1-\mathbf{T}^{\mathbf{e}_i-\mathbf{e}_j}$, then it is divisible by $T_i^{-1}-T_j^{-1}$, and hence $f(\frac{1}{1-t_1},\dots,\frac{1}{1-t_n})$ is divisible by $t_j-t_i=(\mathbf{e}_j-\mathbf{e}_i)\cdot\mathbf{t}$. That this ring map has the claimed characteristic property follows from Proposition 3.15.(b). The last property follows by writing out how this map behaves with respect to the formulas in Theorem 2.3 and Theorem 3.9. □

Corollary 4.2. The assignment $\mathrm{M}\mapsto w_{\mathrm{M}}$ defines an isomorphism $\bigoplus_{r=1}^n \mathrm{Val}_r^\circ(E)\to\bigoplus_{r=1}^n A^{n-r}(\Sigma_E)$ of graded abelian groups.

Proof. Let $\text{inv} : \bar{\mathbb{I}}(\Sigma_E) \to \bar{\mathbb{I}}(\Sigma_E)$ be an involution given by $[Q] \mapsto [-Q]$. Composing the isomorphism in Theorem 2.12 with inv and ϕ, we obtain an invertible map $\bigoplus_{r=1}^n \text{Val}_r^\circ(E) \to \bigoplus_{r=1}^n A^{n-1-r}(\Sigma_E)$ that is not a graded map, but is lower-triangular. □

Corollary 4.3. The numbers $\text{rank}\,\text{Val}_r^\circ(E)$ are symmetric, and $\sum_r \text{rank}\,\text{Val}_r^\circ(E) = n!$.

Proof. The symmetry follows from Poincaré duality and Corollary 4.2. That the sum equals $n!$ follows from the general fact that the topological Euler characteristic of a smooth projective toric variety associated to Σ equals $|\Sigma(n)|$. In our case, $\chi_{top}(X_E) = |\Sigma_E(n)| = n!$. □

We remark that it is known that $\text{rank}\,A^{n-1-i}(\Sigma_E) = i$-th Eulerian number on n, which equals to h-vector of the permutohedral complex.

Corollary 4.4. For any valuative function f on the set of loopless matroids with values in $\mathbb{Z}$, there exists loopless matroids $\mathrm{M}_1, \dots, \mathrm{M}_k$ and integers $a_1, \dots, a_k$ such that

$$f(\mathrm{M}) = \deg_{\Sigma_E}\left(w_{\mathrm{M}} \cdot (a_1 w_{\mathrm{M}_1} + \cdots + a_k w_{\mathrm{M}_k})\right) \quad \text{for all loopless matroid M on } E.$$

Proof. Poincaré duality and Corollary 4.2. □

For a matroid M and M', their *matroid intersection* $\mathrm{M} \wedge \mathrm{M}'$ is a matroid whose bases are the minimal elements of $\{B \cap B' : B \text{ a basis of } \mathrm{M} \text{ and } B' \text{ a basis of } \mathrm{M}'\}$. Using Theorem 4.1, one recovers the following [Spe08, Proposition 4.4].

Corollary 4.5. For loopless matroids M_1 and M_2, we have that

$$w_{\mathrm{M}_1} \cdot w_{\mathrm{M}_2} = \begin{cases} w_{\mathrm{M}_1 \wedge \mathrm{M}_2} & \text{if } \mathrm{M}_1 \wedge \mathrm{M}_2 \text{ is loopless} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4.6. Prove the corollary in the following steps. Let r and r' be the ranks of loopless matroid M_1 and M_2, respectively.

(1) Consider applying ϕ from Theorem 4.1 to the product $[P(\mathrm{M}_1^\perp)] \cdot [P(\mathrm{M}_2^\perp)] = [P(\mathrm{M}_1^\perp) + P(\mathrm{M}_2^\perp)]$.

(2) Tiling by boolean cubes to subdivide $P(\mathrm{M}_1^\perp) + P(\mathrm{M}_2^\perp)$, show the corollary when the condition "if $\mathrm{M}_1 \wedge \mathrm{M}_2$ is loopless" is replaced by "$(n-r) + (n-r') = n - \text{rank}(\mathrm{M}_1 \wedge \mathrm{M}_2)$."

(3) Show that if $\mathrm{M}_1 \wedge \mathrm{M}_2$ is loopless then $(n-r) + (n-r') = n - \text{rank}(\mathrm{M}_1 \wedge \mathrm{M}_2)$.

[Ham17, BES] showed that $A^\bullet(X_E)$ is generated as an abelian group by

$$\{w_{\mathrm{M}} : \mathrm{M} \text{ a matroid realizable over any infinite field}\}.$$

In fact, they showed that the loopless Schubert matroids form a basis. Applying this to Corollary 4.2, one recovers a result of [DF10] that the loopless Schubert matroids form a basis for the valuative group. The following observation is often useful.

Corollary 4.7. $\bigoplus_{r=1}^n \text{Val}_r^\circ(E)$ is generated by a set of loopless matroids realizable over any infinite field. Hence, if two valuative invariants f and g on loopless matroids agree for all realizable matroids, then they agree for all matroids.

4.2 Tautological classes

For a matroid M and $\sigma \in \Sigma_E(n)$, denote by $B_\sigma(\mathrm{M})$ the basis of M such that $\mathbf{e}_{B_\sigma(\mathrm{M})} = \mathrm{face}_{-\sigma}(\mathrm{M})$.

Definition 4.8. For a matroid M, define two elements $[\mathcal{S}_\mathrm{M}]^T$ and $[\mathcal{Q}_\mathrm{M}]^T$ in $K_T(X_E) \simeq LP(\Sigma_E)$ by

$$[\mathcal{S}_\mathrm{M}]_\sigma = \sum_{i \in B_\sigma(\mathrm{M})} T_i^{-1} \quad \text{and} \quad [\mathcal{Q}_\mathrm{M}]_\sigma = \sum_{i \in E \setminus B_{\sigma(\mathrm{M})}} T_i^{-1},$$

called the *tautological K-classes* of the matroid M. Define the *tautological Chern classes* $c_i^T(\mathcal{S}_\mathrm{M})$ and $c_i^T(\mathcal{Q}_\mathrm{M})$ of M to be elements in $A^\bullet_T(X_E) \simeq PP(\Sigma_E)$ by

$$c_i^T(\mathcal{S}_\mathrm{M}) = \mathrm{Elem}_i\{t_i : i \in B_\sigma(\mathrm{M})\} \quad \text{and} \quad c_i^T(\mathcal{Q}_\mathrm{M}) = \mathrm{Elem}_i\{t_i : i \in E \setminus B_\sigma(\mathrm{M})\}$$

where Elem_i denotes the i-th elementary symmetric polynomial.

We denote the images of these in $K(X_E)$ or $A^\bullet(X_E)$ (respectively) by removing the superscript ^{T}.

When M is realized by a linear subspace $L \subseteq \mathbb{C}^E$, these come from the following geometric construction. Let T act on $\mathbb{C}^E$ via the inverse standard action: $(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1^{-1} x_1, \dots, t_n^{-1} x_n)$. Then, one can consider the T-equivariant subbundle $\mathcal{S}_L$ of $X_E \times \mathbb{C}^E$ defined by $\mathcal{S}_L|_{\bar{t} \in \mathbb{P}T} = t^{-1} L$ for any $t \in T$. One defines a T-equivariant quotient bundle $\mathcal{Q}_L$ similarly, fitting into the short exact sequence

$$0 \to \mathcal{S}_L \to \mathcal{O}_{X_E}^{\oplus E} \to \mathcal{Q}_L \to 0.$$

Alternatively, per Remark 2.8, with the action of T on $Gr(r; E)$ via its inverse standard action on $\mathbb{C}^E$, we have that $\mathcal{S}_L$ and $\mathcal{Q}_L$ are the pullbacks of the tautological subbundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$ on the Grassmannian along the map $X_E \to X_{-P(\mathrm{M})} \to \overline{T \cdot L} \hookrightarrow Gr(r; E)$.

This geometric perspective explains Proposition 3.15.(b) as follows. Consider the constant $\mathbf{1}$ section of the trivial bundle $\mathcal{O}_{X_E}^{\oplus E}$, which defines the section $s_\mathbf{1}$ of the bundle $\mathcal{Q}_L$. The vanishing loci of this section is exactly the wonderful compactification of the realization L, since $\infty \in t^{-1} L$ iff $\bar{t} \in \mathbb{P}L \cap \mathbb{P}T$ for all $t \in T$.

Theorem 4.9. [BEST, Theorems A & B] Let $\Delta = \mathrm{Conv}\{\mathbf{e}_i : i \in E\}$ and $\nabla = \mathrm{Conv}\{\mathbf{e}_{E \setminus i} : i \in E\}$ be the standard and the opposite simplex in $\mathbb{R}^E$, respectively. Then, for a matroid M on E of rank r, we have

$$\sum_{i,j,k,\ell = n-1} \deg_{\Sigma_E} \big(D_\Delta^i D_\nabla^j c_k(\mathcal{S}_\mathrm{M}) c_\ell(\mathcal{Q}_\mathrm{M})\big) x^i y^j (-z)^k w^\ell$$
$$= (x+y)^{-1} (y+z)^r (x+w)^{n-r} \mathrm{T}_\mathrm{M}\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big),$$

and this 4-variable transformation of the Tutte polynomial T_M satisfies a "multivariate log-concavity" property (i.e. is a denormalized Lorentzian polynomial in the sense of [BH20]).

The first part of the theorem is proved by showing that the tautological Chern classes of matroids satisfy a deletion-contraction relation. It recovers several previous geometric formulas about the Tutte polynomial of a matroid, including those of [FS12, HK12, LdMRS20, BKP22, CF22]. The second part of the theorem uses the Hodge theory of matroids [AHK18, ADH22] in a critical way, combined with tools provided by valuativity.

The technique of studying the K-ring and the Chow cohomology ring has been fruitful in many areas of research in matroids. In particular, the study of the interplay between them, in the form of the framework of tautological classes of matroids, have given rise to many new developments, including the K-theory of matroids [LLPP], the stellahedral geometry of matroids [EHL], the Gromov-Witten theory of matroids [RU22], intersection theory of polymatroids [EL], and the tropical geometry of "type B" generalizations of matroids known as delta-matroids [EFLS].

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