A Brief Glimpse of Topological Combinatorics

Christopher Eur

May 5, 2015

1 Introduction

Lovazs' proof of the Kneser Conjecture presents a beautiful application of Borsuk-Ulam Theorem, a purely topological result, to a combinatorial problem on finite graphs. Moreover, Kneser-Lovasz Theorem is far from being the only case in which a combinatorial problem admits as a solution that uses topological techniques. It is rather surprising that the study of continuous maps would yield elegant solutions to entirely discrete combinatorial problems. This survey seeks to give a brief glimpse of such beautiful topic of topological combinatorics.

We survey three kinds of combinatorial problems. The first problem concerns the game hex, and we discuss how one can apply Brouwer's Fixed Point Theorem to show that the game always has a winner. The second problem is the necklace problem, in which the ham-sandwich theorem provides a nice solution. Lastly, we give a short description of Lovasz's work on neighborhood complexes, which generalizes the Kneser conjecture and in fact gives a lower bound for the chromatic number of any graph.

2 The Game Hex

In the game of Hex, two players (red and green) take turns coloring one hexagon of a grid of hexagons. After all the hexagons are filled, red wins if there is a path of red hexagons connecting the top and the bottom, and green wins if there is a path of green hexagons connecting the left and the right. The figure below shows an empty board and a finished state in which green has won.

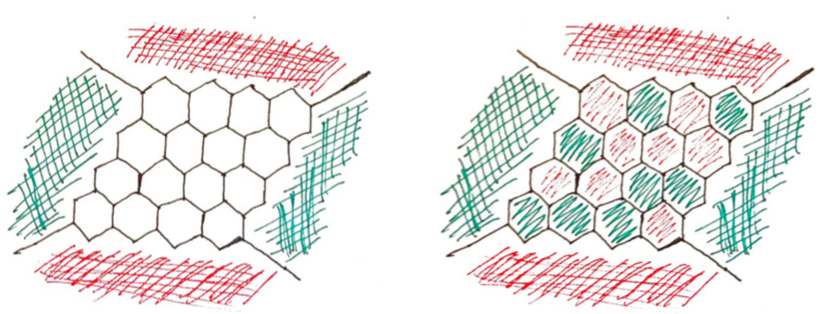


Figure 1: An empty board and a finished state with green victory

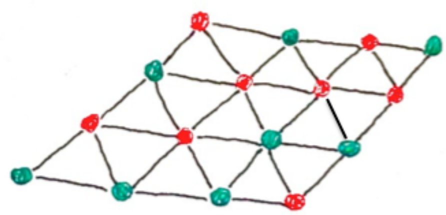
The question is whether the game Hex always has exactly one winner. If there is a winner, say red, then green cannot also have a path connecting the top to bottom since it must cross over the red path connecting the left to the right. Thus, the task is to show that there is at least one winner.

Proposition 2.1. *The game Hex always has a winner*

We prove this in two ways. First, for the proof using topology, we need Brouwer's Fixed Point Theorem, which we state here without proof:

Theorem 2.2 (Brouwer's Fixed Point Theorem)**.** *Let B^2 be a unit disk in $\mathbb{R}^2$, and $f : B^2 \to B^2$ a continuous map. Then there exists $x \in B^2$ such that $f(x) = x$.*

Proof of 2.1. [Kol10, 1.4] First, by making hexagons as vertices and common sides of hexagons as edges (i.e. by taking the dual of the graph) we can translate the hex game to coloring the vertices, in which case, green winning means that there is a path from the left to the right with all green vertices. For example, translation of the game we had in Figure 1 is now



In other words, our board is now a graph $G = (V, E)$, and the game is to color the vertices red or green. Now, suppose for a contradiction that there is a coloring such that there are no winners. Define

$$\begin{aligned} R_0 &= \{\text{red vertices reached from the bottom by a red path}\} \\ R_1 &= \{\text{red vertices not in } R_0\} \\ G_0 &= \{\text{green vertices reached from the left side by a green path}\} \\ G_1 &= \{\text{green vertices reached not in } G_0\} \end{aligned}$$

Moreover, define e_1 to be rightward shift by one vertex (parallel to the top and bottom sides) and e_2 to be upward shift (parallel to the left and right sides). Then define $f : V(G) \to V(G)$ as follows:

$$f(v) := \begin{cases} v + e_2 & v \in R_0 \\ v - e_2 & v \in R_1 \\ v + e_1 & v \in G_0 \\ v - e_1 & v \in G_1 \end{cases}$$

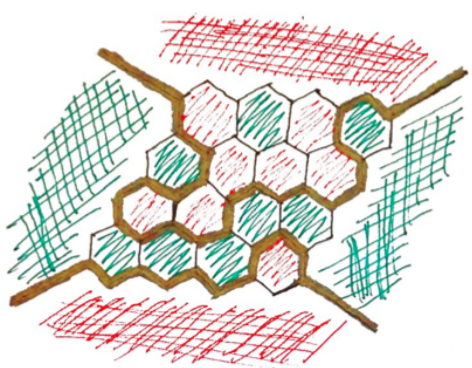
Since we assumed that there are no winners, the map f is well-defined (nothing goes off the board). Now considering any triangle in the board G with vertices v_1, v_2, v_3 as the convex hull of the vertices, each point x in the triangle can be written uniquely as $x = \sum x_i v_i$ where $x_i \geq 0$ and $\sum x_i = 1$. So, extend the map f linearly on each triangle by defining $f(x) := \sum x_i f(v_i)$; it is not hard to see that this is continuous. Since the filled G (as union of triangles) is homeomorphic to B^2, f is a map $B^2 \to B^2$, and thus there exists a fixed point $x \in G$. Let $x = \sum_i x_i v_i$ for a triangle v_1, v_2, v_3, and define $\epsilon_i \in \{\pm e_1, \pm e_2\}$ so that $f(v_i) = v_i + \epsilon_i$. Then, $f(x) = x$ implies that $\sum x_i(v_i + \epsilon_i) = \sum x_i v_i$ and thus

$$\sum x_i \epsilon_i = 0$$

Now, WLOG say $x_1 > 0$ and $\epsilon_1 = e_1$. Then one of the other must be $-e_1$, say ϵ_2. This means that for v_1, v_2, ones belongs to G_0 and the the other to G_1, but this is impossible since v_1, v_2 are on a same triangle. □

Unlike some of the other theorems we explore in the later sections of this article, the hex problem also has a clever combinatorial proof of Proposition 2.1.

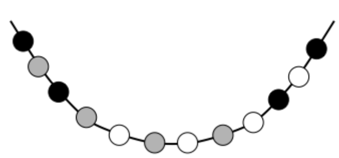
Combinatorial Proof of 2.1. Consider a path along the edges of the hexagon as follows: start at the left bottom corner. When the path meets a green hexagon, turn right, if red then turn left. See the following for an example (the bottom yellow path):



One can see that everywhere on this path, oriented so that it starts from the left bottom corner, green tiles are always on the left and red tiles are always on the right. One can check that this path must end in either the right bottom corner (in which case green wins) or the left top corner (in which the red wins). □

3 The Necklace theorem

Consider an open necklace that has n beads of d different types, with even number of each type.



We wish to divide the beads of each type evenly by cutting the necklace into pieces and distributing the pieces between two people. The task is to use the minimum number of cuts. One bad scenario is the necklace where each type of beads is clustered together, in which case one needs d cuts at least. The claim is that d is really the worst case:

Theorem 3.1 (Necklace theorem)**.** *Every open necklace with d kinds of beads can be divided between two people using no more than d cuts.*

For the proof of Theorem 3.1, we will use a discrete version of the ham-sandwich theorem. The Ham-sandwich theorem is a common application of the Borsuk-Ulam theorem. The original formulation is that given a slice of ham and a slice cheese on a board, there exists a straight line that cuts both into equal portions. Among many generalizations, the most relevant one is as follows:

Theorem 3.2 (Finite Ham-sandwich)**.** *[Mat03, 3.1.3]. Let $A_1, A_2, \ldots, A_d \in \mathbb{R}^d$ be disjoint finite point sets in general position (such that no more than d points of $A_1 \cup \cdots \cup A_d$ are contained in any hyperplane). Then there exists a hyperplane h that bisects each A_i, such that there are exactly $\lfloor \frac{1}{2}|A_i| \rfloor$ points from A_i in each of the open half-spaces defined by h.*

Before using this theorem for the necklace problem, we need one small lemma concerning the moment curve.

Lemma 3.3. *A* ***moment curve*** *γ in $\mathbb{R}^d$ is defined as $\gamma = \{(t, t^2, \ldots, t^d) \in \mathbb{R}^d \mid t \in \mathbb{R}\}$. No hyperplane $h \subset \mathbb{R}^d$ intersects the moment curve γ in $\mathbb{R}^d$ in more than d points.*

Proof. The intersection of hyperplane h defined by $a_1x_1 + a_2x_2 + \cdots a_dx_d = b$ and γ is given by the solutions to the polynomial $a_1t + a_2t^2 + \cdots a_dt^d - b = 0$, which as at most d distinct roots. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We consider beads to the put on the moment curve $\gamma(t) = (t, t^2, \ldots, t^d)$ in $\mathbb{R}^d$ where the kth bead is placed at $\gamma(k) = (k, k^2, \ldots, k^d)$. It is not hard to check that the points $\{\gamma(k)\}_{k=1,\ldots,n}$ are in general position. Now, define

$$A_i = \{\gamma(k) \mid k\text{th bead is of type } i,\ k = 1, \ldots, n\}$$

By the Finite Ham-sandwich theorem, there exists a hyperplane h that divides bisects each A_i. Moreover, by Lemma 3.3, this hyperplane h meets the curve γ at most d points.

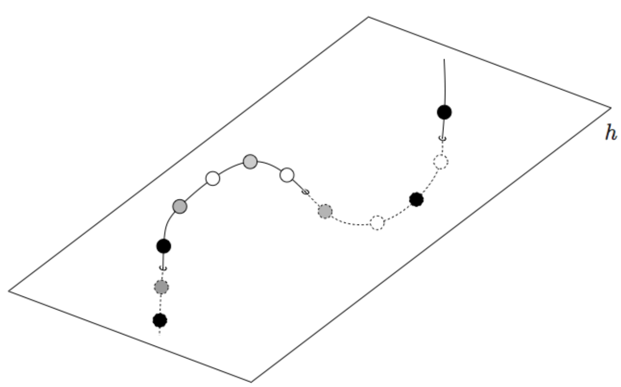


Figure 2: Illustration of the proof (figure from [Mat03])

□

4 Neighborhood complex

In this section, we present a generalization of Lovasz's result on the Kneser Conjecture without proof. First, we need define some notions.

Definition 4.1. *Given a finite simple graph $G = (V, E)$, its* ***neighborhood complex*** $\mathcal{N}(G)$ *is defined as the simplicial complex with vertex set V and simplicies given by subsets $A \subset V$ such that all vertices in A have a common neighbor.*

Example 4.2. The following figures show an example of the neighborhood complex of a graph:

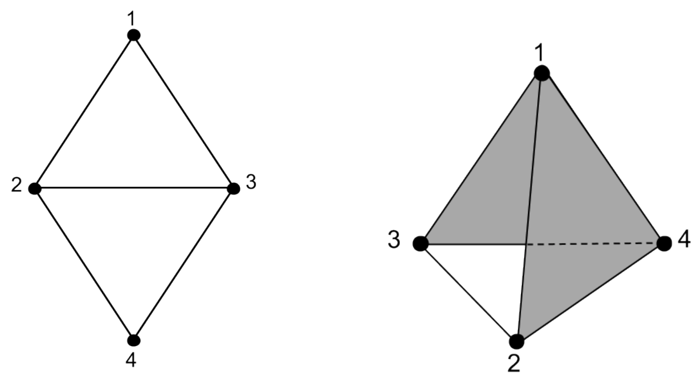


Figure 3: The graph G and its neighborhood complex $\mathcal{N}(G)$

Example 4.3. One can check that for a cycle C_n, the neighborhood complex $\mathcal{N}(C_n)$ is a disjoint union of two cycles if n is even, and $\mathcal{N}(C_n)$ is C_n if odd. Also, the neighborhood complex of a complete graph K_n is the boundary complex of the standard $(n-1)$-simplex, and hence $\mathcal{N}(K_n) \simeq \mathbb{S}^{n-2}$.

Definition 4.4. *A topological space X is k-**connected** if for all $\ell = 0, 1, \ldots, k$, any continuous map $f : \mathbb{S}^\ell \to X$ extends to a continuous map $\widetilde{f} : \mathbb{B}^{\ell+1} \to X$ (i.e. $\widetilde{f}|_{\mathbb{S}^\ell} = f$). The largest k such that X is k-connected is called the **connectivity** of X, denoted $\operatorname{conn}(X)$ (and $\operatorname{conn}(X) = -1$ when X is disconnected).*

Example 4.5. 0-connected means that the space is path-connected. In Example 4.2, $\mathcal{N}(G)$ is 0-connected but not 1-connected. If n is even, $\mathcal{N}(C_n)$ is -1-connected, whereas it is 0-connected if n is odd.

Proposition 4.6. *[Lon13, B.27]* $\operatorname{conn}(\mathbb{S}^n) = n - 1$.

We are now ready to state the main theorem:

Theorem 4.7 (Lovasz)**.** *[Lon13, 2.3] Let $G = (V, E)$ be a finite simple graph. If the neighborhood complex $\mathcal{N}(G)$ of G is k-connected, then the graph has chromatic number at least $k + 3$. In other words,*

$$\chi(G) \geq \operatorname{conn}(\mathcal{N}(G)) + 3$$

Example 4.8. Let's do some sanity checks. With this theorem, Example 4.5 implies that the chromatic number of even cycles is (at least) 2, whereas it is (at least) 3 for odd cycles. Moreover, in combination with Proposition 4.6, we have that the chromatic number of K_n is (at least) n, as expected. One can also check that the graph is Example 4.2 has chromatic number 3, and applying the theorem says the same since its neighborhood complex is 0-connected (but not 1-connected).

Theorem 4.7 is remarkable in that it tells us a lower bound for the chromatic number of any graph. In fact, this is how Lovasz first proved Kneser conjecture by proving the following proposition:

Proposition 4.9. *[Lon13, 2.4] The neighborhood complex $\mathcal{N}(KG_{n,k})$ of the Kneser graph is homotopy equivalent to a wedge of spheres of dimension $n-2k$. In particular,* $\operatorname{conn}(\mathcal{N}(KG_{n,k})) = n-2k-1$.

Combined with Proposition 4.6, this shows that $\chi(KG_{n,k}) \geq n-2k+2$.

References

[Kol10] **S. Kolins**. "Topological Methods in Combinatorics." Course notes, Spring 2010. http://www.math.cornell.edu/ eranevo/homepage/TopMethNotes-1Sam.pdf

[Lon13] **M. de Longueville**. *A Course in Topological Combinatorics*. Universitext. Springer, New York. 2013.

[Mat03] **J. Matousek**. *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*. Written in cooperation with Anders Bjorner and Gunter M. Ziegler. Universitext. Springer-Verlag, Berlin, 2003.

[PR08] **S. Payne, E. Robeva**. "Artificial Intelligence for Bidding Hex." arXiv:0812.3677. 2008.