A brief survey of A-resultants, A-discriminants, and A-determinants

Madeline Brandt, Aaron Brookner, Christopher Eur

November 27, 2017

Abstract

We give a minimalistic guide to understanding a central result of Part II of [GKZ94]—that the Chow polytope, the secondary polytope, and the Newton polytope of principal A-determinant all coincide. After giving preliminary definitions and theorems, we illustrate these for some families of examples.

Among central themes of *Discriminants, Resultants and Multidimensional Determinants* ([GKZ94]) is describing the structure of loci of non-generic behaviors: when do two polynomials share a root? When does a polynomial have repeated roots? When is a matrix not invertible? For polynomials of one variable, the classical answers to these questions are resultants, discriminants, and determinants, respectively.

A main goal of Part II of [GKZ94] is to generalize these notions through the lens of toric geometry. In Section §1, we give a brief summary of definitions and results regarding A-resultants, A-discriminants, and (principal) A-determinants, and in Section §2 we work out numerous examples illustrating concepts mentioned in §1. The codes used for computation in `Macaulay2` are described in Section §3, and are available for download on the third author's website.

Notation. Let $A = \{a_1, \dots, a_n\} \subset \mathbb{Z}^{k-1}$ be a finite subset of $\mathbb{Z}^{k-1}$; often we will write A as a $(k-1) \times n$ matrix. Let $X_A \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ be the projective toric variety defined by A; i.e. it is the closure of the set $\{(x^{a_1} : x^{a_2} : \dots : x^{a_n}) \mid x = (x_1, \dots, x_{k-1}) \in (\mathbb{C}^*)^{k-1}\}$. Denote by $\mathbb{C}^A := \{\sum_{i=1}^n c_i \underline{x}^{a_i}\}$ the Laurant polynomials in $\mathbb{C}[x_1^{\pm}, \dots, x_{k-1}^{\pm}]$ with supports in A. Denote by $Q = \mathrm{Conv}(A)$, and by $\mathrm{Vol}_\Gamma(Q)$ the volume of Q in Γ, the lattice of affine integer span of A where a unit $(\dim Q)$-simplex is normalized to have volume 1.

1 The theory

A central result, obtained by combining Theorem 8.3.1 and Theorem 10.1.4 of [GKZ94], is as follows.

Main Theorem. The Chow polytope $Ch(X_A)$ of X_A, the secondary polytope $\Sigma(A)$ of A, and the Newton polytope of the principal A-determinant $\mathrm{Newt}(E_A)$ coincide,

$$Ch(X_A) \simeq \Sigma(A) \simeq \mathrm{Newt}(E_A).$$

We now walk through the necessary definitions.

Given a Laurent polynomial $f = \sum_\alpha c_\alpha \underline{x}^\alpha \in \mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$, we define the **Newton polytope** of f to be the convex hull of the exponents of nonzero monomial terms of f:

$$\mathrm{Newt}(f) := \mathrm{Conv}(\alpha \mid c_\alpha \neq 0) \subset \mathbb{Z}^n.$$

More generally, suppose a torus $(\mathbb{C}^*)^m$ acts on a $\mathbb{C}$-vector space V. Then the characters $\chi : (\mathbb{C}^*)^m \to \mathbb{C}^*$ of $(\mathbb{C}^*)^m$ are just Laurent polynomials, and can be identified with $\mathbb{Z}^m$, so we have a decomposition $V = \oplus_{\chi \in \mathbb{Z}^m} V_\chi$ where $V_\chi = \{v \in V : t \cdot v = \chi(t)v\}$. We then define the **weight polytope** of $v = \sum_\chi v_\chi \in V$ to be

$$\mathrm{Wt}(v) := \mathrm{Conv}(\chi \mid v_\chi \neq 0) \subset \mathbb{Z}^m.$$

Example 1.1. (1) The weight polytope of a polynomial where we have the usual action of $(\mathbb{C}^*)^n$ on $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ is the Newton polytope.

(2) More generally, consider the action of $(\mathbb{C}^*)^n$ on $\mathrm{Gr}(k,n)$ induced by its action on $\mathbb{C}^n$, which gives an action of $(\mathbb{C}^*)^n$ on the Plücker coordinate ring B of $\mathrm{Gr}(k,n)$ by $t \cdot p_{i_1,\ldots,i_k} = t_{i_1}^{-1} \cdots t_{i_k}^{-1} p_{i_1,\ldots,i_k}$. Thus, a Plücker coordinate $p_I = p_{i_1,\ldots,i_k}$ has weight $-e_I := -\sum_{i \in I} e_i$. As multiplication by -1 and addition by all 1 vector are affine $\mathbb{Z}$-isomorphisms, we sometimes set p_I to have weight e_I, and if Plücker coordinates are indexed by complements then $p_I = [[n] \setminus I]$ is set to have weight $\sum_{j \notin I} e_j$.

Note. For this document, we'll set a Plücker coordinate p_I to have weight $\sum_{j \notin I} e_j$.

For $X \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ a projective variety of dimension $k-1$ and degree d, define the **associated hypersurface** of X by

$$\mathcal{Z}(X) := \{H \in \mathrm{Gr}(n-k,n) \mid H \cap X \neq \emptyset\}.$$

By [GKZ94, Proposition 3.2.2], $\mathcal{Z}(X)$ is a hypersurface of degree d in the (Plücker embedding) of $\mathrm{Gr}(n-k,n)$, and we call its defining equation the **Chow form** $R_X \in B_d$ of X, where B is the Plücker ring of the Grassmannian. Lastly, the **Chow polytope** $Ch(X)$ is the weight polytope of the Chow form R_X.

Example 1.2. If $X \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ is a hypersurface given by a homogeneous polynomial F, then $\mathcal{Z}(X)$ correspond to points on X, so that the Chow form is F. Thus, its Chow polytope is equivalent to the Newton polytope of F.

Example 1.3. If $L \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ is a linear subspace, say given by the row span of a full rank $k \times n$ matrix M, then $Ch(L)$ is the matroid polytope of the linear matroid of the columns of M (and indeed doesn't depend on the choice of M for L).

A **triangulation** T of (Q, A) (where $Q = \mathrm{Conv}(A)$) is a collection Σ of simplices with vertices in A such that the support $|\Sigma| = Q$ and any intersection of two simplices in Σ is in Σ and is a face of each. The **weight** or **characteristic function** $\omega_T \in \mathbb{Z}^A$ of a triangulation is defined as $\omega_T(a) := \sum_\sigma \mathrm{Vol}_\Gamma(\sigma)$, where the sum is over maximal simplices $\sigma \in T$ in which a is a vertex. The **secondary polytope** $\Sigma(A)$ of (Q, A) is the convex hull of the weights of triangulations:

$$\Sigma(A) := \mathrm{Conv}(\omega_T \mid T \text{ a triangulation of } (Q, A)) \subset \mathbb{Z}^A.$$

The dimension of $\Sigma(A)$ is $\#|A| - k$ (where $A \subset \mathbb{Z}^{k-1}$), and the vertices of $\Sigma(A)$ correspond to **coherent (regular)** triangulations, which are triangulations obtained by the projections of lower convex hull of lifts of A to $\mathbb{Z}^{k-1} \times \mathbb{R}$ via a function $\psi : A \to \mathbb{R}$ ([GKZ94, Theorem 7.1.7]).

Example 1.4. Let $A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The toric variety $X_A \subset \mathbb{P}_{\mathbb{C}}^3$ is a hypersurface $V(w^3 - xyz)$, which is an orbifold obtained as $\mathbb{P}_{\mathbb{C}}^2/(\mathbb{Z}/3)$. Its Chow form R_X is $w^3 - xyz$, so that the Chow polytope is $\mathrm{Conv}((0,3,3,3),(3,2,2,2))$. As expected from [GKZ94, Theorem 8.3.1], this is also the secondary polytope $\Sigma(A)$ (of dimension $4 - 3 = 1$), since (Q, A) has two triangulations of weights $(0,3,3,3)$ and $(3,2,2,2)$.

Figure 1: Chow polytope of X_A, which is a segment, is also its secondary polytope

Given $A_1, \ldots, A_k \subset \mathbb{Z}^{k-1}$ each $\mathbb{R}$-affinely generating $\mathbb{R}^{k-1}$ and together $\mathbb{Z}$-affinely generating $\mathbb{Z}^{k-1}$, we define the **mixed** $(A_1, \ldots, A_k)$**-resultant**, denoted $R_{A_1,\ldots,A_k}$, as the defining equation of the Zariski closure of

$$\nabla_{A_1,\ldots,A_k} := \{(f_1, \ldots, f_k) \in \prod_i \mathbb{C}^{A_i} \mid V(f_1, \ldots, f_k) \neq \emptyset \text{ in } (\mathbb{C}^*)^{k-1}\}.$$

[GKZ94, Theorem 8.1.1] guarantees that the closure $\overline{\nabla}_{A_1,\ldots,A_k}$ is irreducible and that $R_{A_1,\ldots,A_k}$ can be taken to be an irreducible polynomial over $\mathbb{Z}$. When $A_1 = \cdots = A_k = A$, we call $R_A = R_{A,\ldots,A}$ the A**-resultant** of A. In this case, when $R_A(f_1, \ldots, f_k)$ is written as a polynomial in the brackets $[i_1, \ldots, i_k]$, it is the same as the Chow form R_{X_A} ([GKZ94, Corollary 8.3.2]).

The A**-discriminant** Δ_A is the defining equation of the closure of the set

$$\nabla_0 := \left\{ f \in \mathbb{C}^A \;\middle|\; V\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{k-1}}\right) \neq \emptyset \text{ in } (\mathbb{C}^*)^{k-1} \right\}$$

when $\overline{\nabla}_0$ is a hypersurface in $\mathbb{C}^A$; otherwise Δ_A is set to be 1. $\overline{\nabla}_0$ is in fact conical (projective), as it is the affine cone of the projective dual variety of X_A.

Example 1.5. For $A = [0, 1, \ldots, d]$, the A-discriminant Δ_A is the classical discriminant of a univariate polynomial of degree d. For example, for $d = 2$ we get $a_1^2 - 4a_0a_2$.

The **principal** A**-determinant** E_A is defined as a particular type of the A-resultant defined above:

$$E_A := R_A\left(f, x_1\frac{\partial f}{\partial x_1}, \ldots, x_{k-1}\frac{\partial f}{\partial x_{k-1}}\right).$$

There is a nice prime factorization of E_A by [GKZ94, Theorem 10.1.2] as follows:

$$E_A = \pm \prod_{P \preceq Q} \Delta_{A\cap P}(f|_{A\cap P})^{\operatorname{mult}_{X_P} X_A}.$$

Note that when X_A is smooth, $\operatorname{mult}_{X_P} X_A = 1$. Hence, in this case, the Newton polytope $\operatorname{Newt}(E_A) \subset \mathbb{Z}^A$ is Minkowski sums of A-discriminants over all faces of Q.

Remark 1.6. If $v \in Q$ is a vertex, then $\Delta_{A\cap v}(f|_{A\cap v}) = a_i$ where ith column of A is v. Thus, when computing the Newton polytope of E_A, the contribution from the vertices of Q is just a translation, so we'll often ignore them when computing $\operatorname{Newt}(E_A)$. More generally, we can ignore the faces of Q that are standard simplices (i.e. simplices of induced volume 1) since k-standard simplices for $k > 1$ have A-discriminant 1. Moreover, just as in Example 1.1(2), we'll take some liberty in translation by a multiple of the all 1 vector and multiplication by -1 when identifying a polytope to another.

Example 1.7. In the Example 1.4, the A-discriminant Δ_A is $a_1^3 + 27a_2a_3a_4$, and the proper faces of Q are standard simplices. Hence, the Newton polytope of the principal A-determinant is $\operatorname{Newt}(E_A) \simeq \operatorname{Conv}((3,0,0,0),(0,1,1,1)) \simeq \operatorname{Conv}((0,3,3,3),(3,2,2,2)) = \Sigma(A)$ (subtract $(3,3,3,3)$ and take minus).

Example 1.8 $(\mathcal{A}_{n-1})$**.** As is shown in [GKZ94, Example 3.3.5(a)], the Chow polytope of a single point $(x_1 : \ldots : x_n) \in \mathbb{P}^{n-1}$ is the convex hull of $\{e_i \mid x_i \neq 0\}$. Hence the Chow polytope of $e_i - e_j$ for $i < j$ is just $[e_i, e_j]$. By definition, the Chow form of a cycle $\sum_i m_i X_i$ is $\prod_i R_{X_i}^{m_i}$ for an algebraic cycle; and we know by construction that the Newton polytope of a product of polynomials is the Minkowski sum of their individual Newton polytopes. Thus we will know that the Chow polytope of the finite set $\{e_i - e_j | 1 \leq i < j \leq n\}$ is the n-dimensional permutohedron, if we can show that it is the appropriate Minkowski sum of these $\binom{n}{2}$ intervals. To recall, the permutohedron is defined as the convex hull of $\{(\sigma(1), \sigma(2), \ldots, \sigma(n)) | \sigma \in S_n\} \subset \mathbb{R}^n$. The easiest way to prove the above claim is to use the fact that the permutohedron is a zonotope (this argument is from [Zieg95]): consider the $D = \binom{n}{2}$-dimensional cube I^D, and project it down to the Minkowski sum $[e_1, e_n] + [e_2, e_n] + \ldots + [e_{n-1}, e_n] + \ldots + [e_1, e_2]$. To show that this Minkowski sum is the desired permutohedron, it suffices to say that it is S_n-invariant, and that the point in it which maximizes $\alpha \in (\mathbb{C}^n)^*, \alpha = \sum_{i=1}^n \alpha_i e_i^*$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_n$, is simply the point $e_1 + 2e_2 + \ldots + ne_n$.

Example 1.9 (Twisted cubic)**.** Let $A = [0\ 1\ 2\ 3]$, so that $X_A \subset \mathbb{P}^3$ is the twisted cubic. Its Chow form defines a cubic hypersurface in $\mathrm{Gr}(2,4)$ given by

$$x_{1,2}^3 - x_{0,2}x_{1,2}x_{1,3} + x_{0,1}x_{1,3}^2 + x_{0,2}^2x_{2,3} - 2x_{0,1}x_{1,2}x_{2,3} - x_{0,1}x_{0,3}x_{2,3}.$$

The Chow polytope is thus the convex hull of $\begin{pmatrix} 3 & 2 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 3 & 0 & 1 & 2 \\ 3 & 2 & 1 & 2 & 2 & 1 \end{pmatrix}$, whose vertices are $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 2 & 3 \end{pmatrix}$.
The A-discriminant is the classical discriminant of a univariate degree 3 polynomial $a_1 + a_2x + a_3x^3 + a_4x^3$:

$$D_A = a_2^2a_3^2 - 4a_1a_3^3 - 4a_2^3a_4 + 18a_1a_2a_3a_4 - 27a_1^2a_4^2,$$

so that its principal A-determinant is $E_A = a_1a_4D_A$. The Newton polytope $\mathrm{Newt}(E_A)$ thus has (again) vertices $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 2 & 3 \end{pmatrix}$, as expected. The following figure also shows that the Chow polytope $Ch(X_A)$ (which coincides with $\mathrm{Newt}(E_A)$), is indeed the secondary polytope as well:

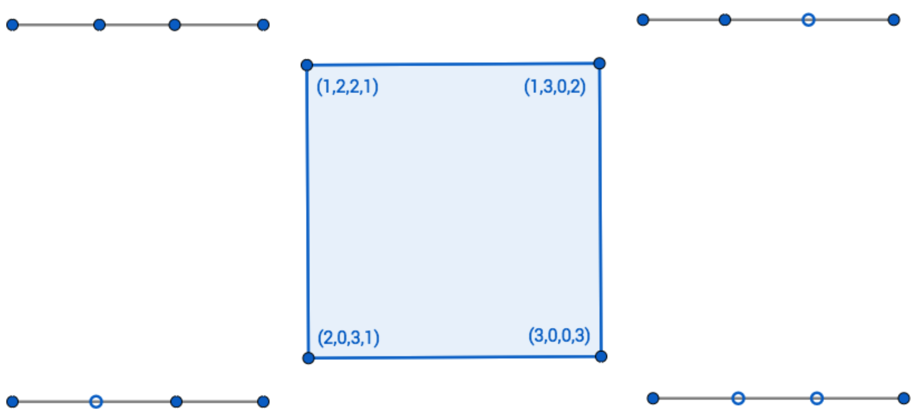

Figure 2: Secondary polytope of $A = [0\ 1\ 2\ 3]$, same as its Chow polytope and Newton polytope of E_A

Remark 1.10. The above example indeed generalizes to $A = [0\ 1\ \ldots\ d]$, the dth-dilate of Δ_1. Its secondary polytope is the $(d-1)$-dimensional cube, and its A-discriminant is the classical discriminant of a generic univariate polynomial of degree d.

2 The examples

2.1 $\Delta_2 \times \Delta_2$

In this section we study the example $A = \Delta_2 \times \Delta_2$. Following `Polymake`, we take the simplex to be the convex hull of the points $(1,0,0),(1,1,0),(1,0,1)$ in homogeneous coordinates. Then, (transpose of) A in our case is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The polytope $Q = \text{conv}(A)$ is 4 dimensional. It has 9 vertices, 18 edges, 16 two-dimensional faces, and 6 facets. Its Schlegel diagram is depicted below.

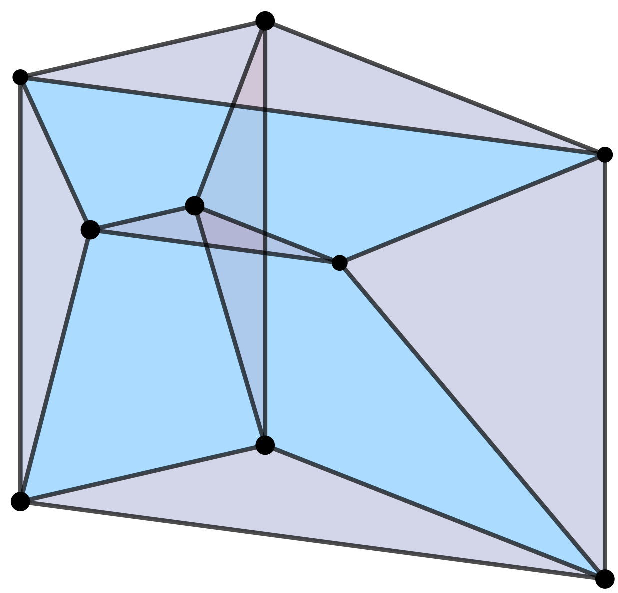

Figure 3: The Schlegel diagram of $\Delta_2 \times \Delta_2$

We now wish to study the secondary polytope $\Sigma(A)$ of A. The secondary fan of A will be the outer normal fan of $\Sigma(A)$, and we can compute the secondary fan in `Polymake`. From this, we know that the secondary polytope has dimension 4, with f-vector $(108, 222, 144, 30)$.

Next, we study $Ch(X_A)$. The variety X_A is the Segre variety; it is the embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ with

$$(x_0 : x_1 : x_2, y_0 : y_1 : y_2) \mapsto (x_0y_0, x_0y_1, x_0y_2, x_1y_0, x_1y_1, x_1y_2, x_2y_0, x_2y_1, x_2y_2).$$

This can be interpreted as the space of rank one matrices, because each rank one matrix is given by multiplying a vector and a covector, as we have done here. An attempt was made to compute its Chow polytope in `Macaulay2` using the code in the appendix, but this did not terminate.

The dual variety of X_A is therefore the set of 3×3 matrices which do not have full rank. This is the hypersurface whose defining polynomial is the 3×3 determinant.

Lastly, we consider $Newt(E_A)$. The faces of $\Delta_2 \times \Delta_2$ are products $\Gamma_1 \times \Gamma_2$ where Γ_1 is a face of the first simplex and Γ_2 is a face of the second. So, the faces correspond to pairs of nonempty subsets (I, J) of $\{0, 1, 2\}$. Denote by $\Gamma(I, J)$ the face corresponding to the pair (I, J). The $A \cap \Gamma(I, J)$ discriminant is 1 when I and J have different cardinalities, and when they are equal it is the minor $\Delta_{IJ}(a_{ij})$ of the matrix a_{ij}, on the rows from I and columns from J. Therefore, by Theorem 10.B.1.2, we have

$$E_A(f) = \prod_{I,J} \Delta_{IJ}(a_{ij}).$$

The above product has $9 + 9 + 1$ terms. In `Mathematica` we compute that this is a polynomial with 408 terms. We compute the Newton polytope of E_A in `Polymake`, and find that it is the polytope with vertex set given by

$\{(6,3,1,3,4,3,1,3,6), (6,3,1,3,2,5,1,5,4), (6,3,1,2,5,3,2,2,6), (6,3,1,2,2,6,2,5,3), (6,3,1,1,5,4,3,2,5),$
$(6,3,1,1,3,6,3,4,3), (6,2,2,3,5,2,1,3,6), (6,2,2,3,2,5,1,6,3), (6,2,2,2,6,2,2,2,6), (6,2,2,2,2,6,2,6,2),$
$(6,2,2,1,6,3,3,2,5), (6,2,2,1,3,6,3,5,2), (6,1,3,3,5,2,1,4,5), (6,1,3,3,3,4,1,6,3), (6,1,3,2,6,2,2,3,5),$
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$(5,3,2,1,6,3,4,1,5), (5,2,3,4,5,1,1,3,6), (5,2,3,3,6,1,2,2,6), (5,2,3,2,2,6,3,6,1), (5,2,3,1,3,6,4,5,1),$
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$(2,2,6,3,6,1,5,2,3), (2,2,6,2,6,2,6,2,2), (2,2,6,2,5,3,6,3,1), (1,6,3,6,2,2,3,2,5), (1,6,3,6,1,3,3,3,4),$
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$(1,5,4,5,4,1,4,1,5), (1,5,4,4,1,5,5,4,1), (1,5,4,3,2,5,6,3,1), (1,4,5,6,1,3,3,5,2), (1,4,5,5,1,4,4,5,1),$
$(1,4,5,4,5,1,5,1,4), (1,4,5,3,5,2,6,1,3), (1,3,6,6,3,1,3,4,3), (1,3,6,6,2,2,3,5,2), (1,3,6,5,2,3,4,5,1),$
$(1,3,6,4,5,1,5,2,3), (1,3,6,3,5,2,6,2,2), (1,3,6,3,4,3,6,3,1)\}.$

The volume of $\Delta_2 \times \Delta_2$ is 6, which means that it will be divided in to 6 simplices which each contain 5 vertices. So, each vector should sum to 30 (which they do). For instance, the first vertex in the list induces the regular subdivision with the following simplices:

$\{\{v_0, v_1, v_4, v_7, v_8\}, \{v_0, v_1, v_2, v_5, v_8\}, \{v_0, v_1, v_4, v_5, v_8\}, \{v_0, v_3, v_4, v_7, v_8\}, \{v_0, v_3, v_6, v_7, v_8\}, \{v_0, v_3, v_4, v_5, v_8\}\}.$

We can verify that this triangulation corresponds to the expected vertex. The vertex v_0 is contained in all 6 simplices, each with volume 1; vertex v_1 is contained in 3 simplices; vertex v_2 is contained in one simplex; vertex v_3 is contained in 3 simplices; vertex v_4 is contained in 4 simplices, and so on. This gives the expected vertex of the secondary polytope:

$$(6, 3, 1, 3, 4, 3, 1, 3, 6).$$

An attempt to draw this triangulation is displayed in the figure.

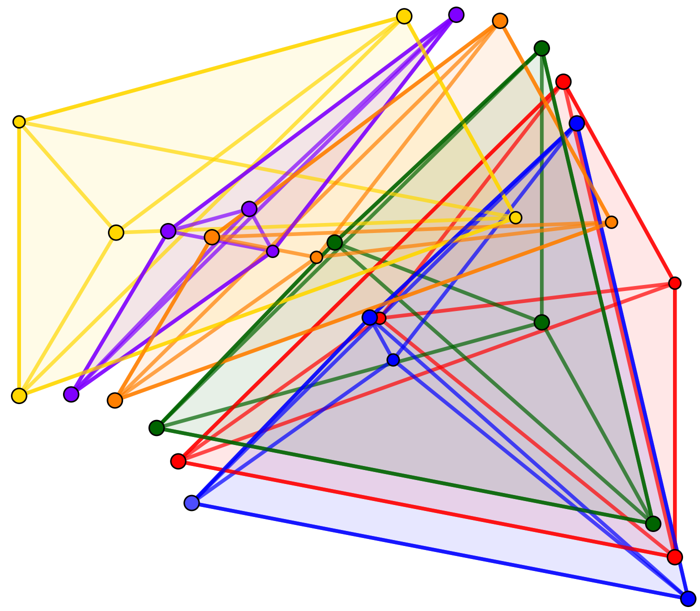


Figure 4: The sample triangulation $\Delta_2 \times \Delta_2$, in an exploded format. Vertex 0 and vertex 8 are contained in all simplices.

The discriminant Δ_A is the 3×3 determinant. Its monomials correspond to permutations of the set of 3 elements. We can verify this in `Macaulay2` as follows.

```
R = QQ[a_(0,0)..a_(2,2),x_0..x_2,y_0..y_2]
f = sum flatten for i from 0 to 2 list for j from 0 to 2 list a_(i,j)*x_i*y_j
XI= ideal(for i from 0 to 2 list diff(x_i,f))
YI = ideal(for i from 0 to 2 list diff(y_i,f))
F = ideal(f)
I = F + XI+YI
J =saturate(I,product flatten entries vars R)
eliminate(J,{x_0,x_1,x_2,y_0,y_1,y_2})
```

2.2 $\Delta^1 \times \Delta^{n-1}$

Let us compute the principal A-determinant $E_A, A = \Delta^1 \times \Delta^{n-1}$. Clearly $\mathbb{C}^A$ is identified with the space of biliear forms in $\{x_1, x_2\}, \{y_1, \dots, x_n\}$, which we identify with the space of matrices $\mathbb{C}^{2\times n}$. Following [GKZ94, p. 272], let $A' = \Delta^{m-1} \times \Delta^{n-1}$ for a moment, and define $f = \sum_{i,j} a_{ij} x_i y_j \in \mathbb{C}^A$ and the column vectors $x = [x_1, \dots, x_m]^t, y = [y_1, \dots, y_n]^t$. Then $\frac{\partial f}{\partial x_i} = \sum_j a_{ij} y_j$, and this is the ith entry of $[a_{ij}] \cdot y$; similarly for x. Thus for $f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial y_j}, 1 \le i \le m, 1 \le j \le n$ to have a common root is equivalent to the existence of x, y such that $x^t \cdot [a_{ij}] = 0, [a_{ij}] \cdot y = 0$, i.e. for there to be nontrivial kernels. This demonstrates that the A'-discriminant is $\Delta_{A'}(f) = \begin{cases} 1, & m \ne n \\ \det[a_{ij}], & m = n \end{cases}$.

Resetting back to $m = 2$ and working with A, and using the product formula [GKZ94, Theorem 10.1.2], we find that $E_A([a_{ij}]$ is the product of the entries in $[a_{ij}]$ and all 2×2 minors, of which there are $\binom{n}{2}$: $E_A = \left(\prod_{i,j=1}^{2,n} a_{ij} \right) \cdot \left(\prod_{1 \le k < \ell \le n} a_{1k} a_{2\ell} - a_{1\ell} a_{2k} \right)$. Expanding, Newton polytope of the second factor is a shifted permutohedron: each individual factor is an interval, and the product yields the permutohedron as a Minkowski sum as in 1.8. The first factor corresponds to a point in $\mathbb{C}^A$ and shifts the permutohedron, to agree with the secondary polytope as we will now compute.

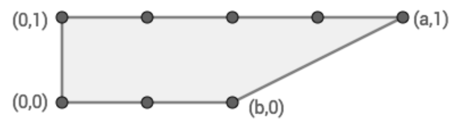
By contrast with the principal determinant, GKZ do not claim to have an effective way to enumerate the triangulations of $\Delta^{m-1} \times \Delta^{n-1}$. By the theorems cited above, i.e. the correspondence of $\Sigma(\Delta^{m-1} \times \Delta^{n-1})$ with coherent triangulations and $\Sigma(A) \cong \mathrm{Newt}(R_A)$, we know in principal the characteristic functions of these coherent triangulations; but the book claims to not be aware of whether all triangulations are in fact coherent.

This is the case, however, for $A = \Delta^1 \times \Delta^{n-1}$, as it does give an explicit enumeration of these triangulations, indeed of the entire face lattice of $\Sigma(A)$ as well, which we now review. Define $\omega_{ij} = e_i + f_j \in A, 1 \leq i \leq 2, 1 \leq j \leq n$, where e_1, e_2 are the vertices of Δ^1 and similarly for f_j. To two subsets $J_1, J_2 \subset \{1, 2, \ldots, n\}$ we associate the subset $\sigma(J_1, J_2) = \{\omega_{i,j_i} | i = 1, 2; j_i \in J_i\}$. Then by direct observation, $\dim \mathrm{Conv}\,(\sigma(J_1, J_2)) = \begin{cases} |J_1 \cup J_2|, & J_1 \cap J_2 \neq \emptyset \\ |J_1 \cup J_2| - 1, & J_1 \cap J_2 = \emptyset \end{cases}$. Any such set $\mathrm{Conv}(\sigma(J_1, J_2))$ which is a hyperplane section (e.g. of maximal dimension and not the whole $\mathrm{Conv}(A)$) and is not a facet of $\mathrm{Conv}(A)$ is then called a *diagonal*; we say two diagonals *cross* to mean they intersect at an interior point (e.g. not just because they share a vertex). The dimension formula then shows that $\mathrm{Conv}(\sigma(J_1, J_2))$ is a diagonal iff $J_2 = \{1, 2, \ldots, n\} - J_1$. Hence if two diagonals do not cross, their intersection is a common face of strictly smaller dimension, and so every set D of non-pairwise crossing diagonals yields a subdivision (not necessarily triangulation) S_D of (Q, A) (where as usual $Q := \mathrm{Conv}(A)$). It is obvious that this is a poset anti-homomorphism, i.e. S_D refines $S_{D'}$ iff $D' \subset D$. The book then shows that this is a bijection for all subdivisions: this is due to the fact that the diagonals are themselves simplices, hence cannot be subdivided. Given a subdivision $S = \{(Q_i, A_i)\}$, any interior facet of Q_i (i.e. not a facet of Q) belongs to some diagonal, and the previous sentence thus demonstrates that it must equal the entire diagonal. It is then possible to then explicitly construct a piecewise-convex functional for each such set of diagonals, i.e. showing that all subdivisions (triangulations) are coherent.

This shows that the face lattice of $\Sigma(A)$ is anti-isomorphic to the lattice of non-crossing subdivisions; we thereby wish to determine the ordering on the latter, hence the maximal subdivisions, and hence the vertices of $\Sigma(A)$. Well, by definition of the convex hull, a point $(\alpha, \beta) \in Q \subset \mathbb{R}^2 \times \mathbb{R}^n$ is in the boundary ∂Q, iff one of its coordinates equals 0. Then, the diagonal d_J is given by $d_J = \{(\alpha, \beta) \in Q | \alpha_1 = \sum_{j \in J} \beta_j\}$, and hence: the diagonals $d_J, d_{J'}$ do not cross if and only if $J \subset J'$ or $J' \subset J$. At this point, defining a *flag in* $\{1, 2, \ldots, n\}$ to be a strictly increasing set of subsets, it follows that the faces of $\Sigma(A)$ are in bijection with flags, and that there are $n!$ maximal flags (choose a first subset, $F_1 = \{i\}, 1 \leq i \leq n$; choosing a following flag element is the same as choosing a distinct element of $\{1, 2, \ldots, n\}$). Since S_m acts on the vertices of Δ^{n-1} via the (unreduced) permutation representation, it acts on Q by affine automorphisms; and since it clearly acts transitively on the set of maximal flags in $\{1, 2, \ldots, n\}$, it acts transitively on the vertices of $\Sigma(A)$. Thus we only need to compute a single characteristic function (on whose coordinates S_n also acts), to now verify that $\Sigma(A)$ is a permutohedron. Pick the canonical flag, $\{1\} \subset \{1, 2\} \subset \ldots \subset \{1, 2, \ldots, n\}$, and see that the corresponding triangulation consists of the simplices $\sigma_p = \mathrm{Conv}(\omega_{11}, \omega_{12}, \ldots, \omega_{1p}, \omega_{2p}, \omega_{2,p+1}, \ldots, \omega_{2,n})$, where $1 \leq p \leq m$. Since simplices have volume normalized to 1, the characteristic function $\varphi(\omega_{ij})$ just counts the number $|\{p | \omega_{ij} \in \sigma_p\}|$. Clearly then $\varphi(\omega_{11}) = n, \varphi(\omega_{21}) = 1$, and similarly $\varphi(\omega_{1i}) = n - i + 1, \varphi(\omega_{2i}) = i$. It is here that we identify $\mathbb{C}^A$ with the space of $2 \times n$ matrices, as in the principal A-determinant computation, whereby we can identify the vector of the characteristic function of this particular triangulation, with the matrix $\begin{bmatrix} n & n-1 & \ldots & 2 & 1 \\ 1 & 2 & \ldots & n-1 & n \end{bmatrix}$. Finally, taking the S_n-action, taking the convex hull of these points, we find that this is precisely $\mathrm{Newt}(R_A)$ as claimed. Moreover, we see that the projection onto the first row is a an affine isomorphism, and the image is clearly, by definition, the standard embedding of the permutohedron in $\mathbb{R}^n$.

2.3 Hirzebruch surfaces

A Hirzebruch surface $\mathscr{H}_{a,b} (\simeq \mathscr{H}_r)$ is defined as the toric variety of $A = \begin{bmatrix} 0 & 1 & \cdots & a & 0 & 1 & \cdots & b \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$, where $a - b = r$, i.e. by a monomial parametrization $(x, y) \mapsto (y, xy, \ldots, x^a y, 1, x, \ldots, x^b)$. As schemes, Hirzebruch surfaces $\mathscr{H}_{a,b}$ with $a - b = r$ for a fixed r are isomorphic (although not as *embedded* projective schemes). Below is an illustration of the convex hull of A:



Hirzebruch surfaces in various different contexts. They are also called rational normal scrolls because they can be considered the join of Veronese embeddings of $\mathbb{P}^1$ of degree a and b. They can also be realized as a $\mathbb{P}^1$-fiber bundle over $\mathbb{P}^1$; for their properties as a ruled surface, see [Har77, V.2].

Write $\mathbb{C}^A = \{\alpha_0 y + \alpha_1 xy + \cdots + \alpha_a x^a y + \beta_0 + \beta_1 x + \cdots \beta_b x^b\}$. Denoting by p, q generic polynomials in x of degree a, b (respectively), we have that $\mathbb{C}^A = \{f \in \mathbb{C}^A : f = p(x)y + q(x)\}$. The Chow form of a Hirzebruch surface is generally hard to compute, but its A-discriminant and principal A-determinant are easier to understand.

Claim: The A-discriminant is the classical resultant of two polynomials of degree a and b. Moreover, the principal A-determinant is

$$E_A(f) = \alpha_0 \alpha_a \beta_0 \beta_b \Delta(p) \Delta(q) \operatorname{Res}(p, q)$$

where $\Delta(p), \Delta(q)$ denotes the classical discriminants of polynomials of degree a, b (respectively), and $\operatorname{Res}(p, q)$ is the classical resultant of two polynomials in degree a, b.

To see this, note that by writing $f \in \mathbb{C}^A$ as $p(x)y + q(x)$ we have that $V(f, f_x, f_y) = V(p(x), q(x), p'(x)y + q'(x)) \neq \emptyset$ in $(\mathbb{C}^*)^2$ when p, q share a simple nonzero root. Taking closure, the A-discriminant exactly measures when $p(x)$ and $q(x)$ share a factor; in other words, Δ_A should be the classical resultant $\operatorname{Res}(p, q)$ of two polynomials p, q in degree a, b (respectively). For the A-determinant, use the factorization of E_A ([GKZ94, Theorem 10.1.2]) along with observations made in Remark 1.6 and Remark 1.10.

Let's now carry out concretely the computations illustrating the main theorem regarding $Ch(X_A)$, $\Sigma(A)$, and $\operatorname{Newt}(E_A)$ for $\mathscr{H}_{3,2}$ and $\mathscr{H}_{4,2}$. All the computations are done via codes in `readingGKZ.m2`. For details see Section §3.1.

2.3.1 $\mathscr{H}_{3,2}$

Take $X = \mathscr{H}_{3,2}$, i.e. $A = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$. The Chow form is

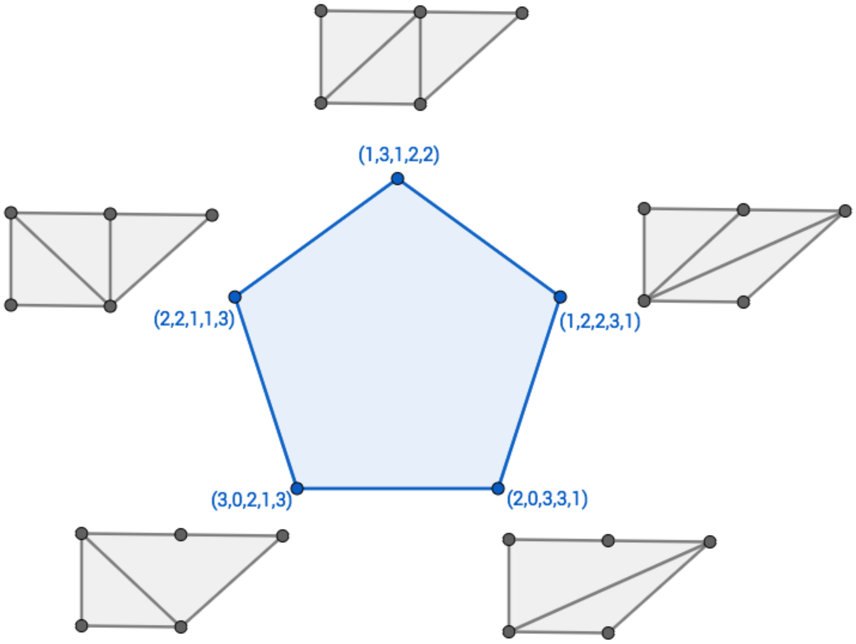
$$R_X = x_{1,2}x_{1,3}^2 - x_{0,2}x_{1,3}x_{2,3} + x_{0,1}x_{2,3}^2 - x_{0,2}x_{1,3}x_{1,4} + x_{0,1}x_{1,4}^2 + x_{0,2}x_{0,3}x_{2,4} - x_{0,1}x_{0,4}x_{2,4} - 2x_{0,1}x_{1,2}x_{3,4},$$

so that its Chow polytope is a convex hull of $\begin{bmatrix} 3 & 2 & 2 & 2 & 2 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 1 & 2 & 3 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \end{bmatrix}$, with vertices $\begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 0 & 3 & 2 & 0 \\ 2 & 3 & 1 & 1 & 2 \\ 3 & 3 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 & 3 \end{bmatrix}$.

Computing the principal A-determinant directly or using the *Claim* above gives

$$E_A = a_1 a_2^2 a_3^2 a_4^3 a_5 - 4a_1^2 a_3^3 a_4^3 a_5 - a_1 a_2^3 a_3 a_4^2 a_5^2 + 4a_1^2 a_2 a_3^2 a_4^2 a_5^2 + a_1^2 a_2^2 a_3 a_4 a_5^3 - 4a_1^3 a_3^2 a_4 a_5^3,$$

so that $\operatorname{Newt}(E_A)$ has vertices (again) $\begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 0 & 3 & 2 & 0 \\ 2 & 3 & 1 & 1 & 2 \\ 3 & 3 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 & 3 \end{bmatrix}$ as expected (the fourth term of E_A has weight that is half the sum of 2nd and 4th columns of the vertex matrix). Lastly, we can confirm that this polytope also arise as the secondary polytope $\Sigma(A)$:



2.3.2 $\mathscr{H}_{4,2}$

Take $X = \mathscr{H}_{4,2}$, i.e. $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$. The Chow form starts to become hard to compute (takes about 15 seconds on `M2`).

$$\begin{aligned} R_X = {} & x_{1,2,3}x_{1,2,4}^3 - x_{0,2,3}x_{1,2,4}^2x_{1,3,4} + x_{0,1,3}x_{1,2,4}x_{1,3,4}^2 - x_{0,1,2}x_{1,3,4}^3 + x_{0,2,3}x_{0,2,4}x_{1,2,4}x_{2,3,4} - 2x_{0,1,3}x_{1,2,4}^2x_{2,3,4} \\ & - x_{0,1,3}x_{0,2,4}x_{1,3,4}x_{2,3,4} + 3x_{0,1,2}x_{1,2,4}x_{1,3,4}x_{2,3,4} + x_{0,1,2}x_{0,3,4}x_{1,3,4}x_{2,3,4} + x_{0,1,3}x_{0,1,4}x_{2,3,4}^2 \\ & - x_{0,1,2}x_{0,2,4}x_{2,3,4}^2 - x_{0,2,3}x_{1,2,4}^2x_{1,2,5} + x_{0,1,3}x_{1,2,4}x_{1,2,5}^2 - x_{0,1,2}x_{1,2,5}^3 + x_{0,2,3}x_{0,2,4}x_{1,2,4}x_{1,3,5} \\ & - x_{0,1,3}x_{0,2,4}x_{1,3,4}x_{1,3,5} + x_{0,1,2}x_{0,3,4}x_{1,3,4}x_{1,3,5} - x_{0,1,3}x_{0,2,4}x_{1,2,5}x_{1,3,5} + x_{0,1,2}x_{0,2,5}x_{1,2,5}x_{1,3,5} \\ & + x_{0,1,3}x_{0,1,4}x_{1,3,5}^2 - x_{0,1,2}x_{0,1,5}x_{1,3,5}^2 - x_{0,2,3}x_{0,2,4}^2x_{2,3,5} + 2x_{0,1,3}x_{0,2,4}x_{1,2,4}x_{2,3,5} + x_{0,1,3}x_{0,2,4}x_{0,3,4}x_{2,3,5} \\ & - x_{0,1,2}x_{0,3,4}^2x_{2,3,5} - 2x_{0,1,2}x_{0,2,4}x_{1,3,4}x_{2,3,5} + 2x_{0,1,2}x_{0,1,4}x_{2,3,4}x_{2,3,5} + x_{0,1,3}x_{0,2,4}x_{0,2,5}x_{2,3,5} \\ & - x_{0,1,2}x_{0,2,5}^2x_{2,3,5} - 2x_{0,1,3}x_{0,1,4}x_{1,2,5}x_{2,3,5} + 2x_{0,1,2}x_{0,1,5}x_{1,2,5}x_{2,3,5} - x_{0,1,3}x_{0,1,4}x_{0,3,5}x_{2,3,5} \\ & + x_{0,1,2}x_{0,1,5}x_{0,3,5}x_{2,3,5} - 2x_{0,1,3}x_{1,2,3}x_{1,2,4}x_{1,4,5} + 3x_{0,1,2}x_{1,2,3}x_{1,3,4}x_{1,4,5} + 3x_{0,1,2}x_{1,2,3}x_{1,2,5}x_{1,4,5} \\ & - 2x_{0,1,2}x_{0,2,3}x_{1,3,5}x_{1,4,5} + x_{0,1,3}x_{0,2,3}x_{1,2,4}x_{2,4,5} - 3x_{0,1,2}x_{1,2,3}x_{1,2,4}x_{2,4,5} - 2x_{0,1,2}x_{0,2,3}x_{1,3,4}x_{2,4,5} \\ & + x_{0,1,2}x_{0,1,3}x_{2,3,4}x_{2,4,5} - x_{0,1,2}x_{0,2,3}x_{1,2,5}x_{2,4,5} + 2x_{0,1,2}x_{0,2,3}x_{0,3,5}x_{2,4,5} + x_{0,1,2}x_{0,1,3}x_{1,3,5}x_{2,4,5} \\ & + 3x_{0,1,2}^2x_{2,3,5}x_{2,4,5} - 2x_{0,1,3}^2x_{1,2,4}x_{3,4,5} + 3x_{0,1,2}x_{0,2,3}x_{1,2,4}x_{3,4,5} + 2x_{0,1,2}x_{0,1,3}x_{1,3,4}x_{3,4,5} \\ & - 3x_{0,1,2}^2x_{2,3,4}x_{3,4,5} - 2x_{0,1,2}x_{0,2,3}x_{0,2,5}x_{3,4,5} + 2x_{0,1,2}x_{0,1,3}x_{1,2,5}x_{3,4,5} - 3x_{0,1,2}^2x_{1,3,5}x_{3,4,5} \end{aligned}$$

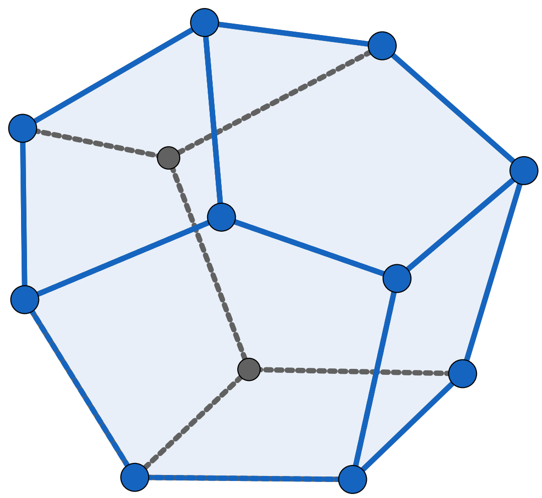
The Chow polytope $P = Ch(R_X)$ has the f-vector $(12, 18, 8, 1)$. Its 12 vertices are

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 & 1 & 1 & 2 & 3 & 2 & 4 \\ 2 & 0 & 3 & 0 & 2 & 0 & 3 & 4 & 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 0 & 3 & 4 & 2 & 0 & 2 & 3 & 0 & 0 \\ 2 & 2 & 3 & 4 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 \end{bmatrix}$$

The principal A-determinant is a lot easier to compute:

$$
\begin{aligned}
E_A = {} & a_1a_2^2a_3^2a_4^2a_5^4a_6 - 4a_1^2a_3^3a_4^2a_5^4a_6 - 4a_1a_2^3a_4^3a_5^4a_6 + 18a_1^2a_2a_3a_4^3a_5^4a_6 - 27a_1^3a_4^4a_5^4a_6 \\
& - a_1a_2^2a_3^3a_4a_5^3a_6^2 + 4a_1^2a_3^4a_4a_5^3a_6^2 + 4a_1a_2^3a_3a_4^2a_5^3a_6^2 - 18a_1^2a_2a_3^2a_4^2a_5^3a_6^2 + 27a_1^3a_3a_4^3a_5^3a_6^2 \\
& + a_1a_2^3a_3^2a_4a_5^2a_6^3 - 4a_1^2a_2a_3^3a_4a_5^2a_6^3 - 4a_1a_2^4a_4^2a_5^2a_6^3 + 18a_1^2a_2^2a_3a_4^2a_5^2a_6^3 - 27a_1^3a_2a_4^3a_5^2a_6^3 \\
& - a_1^2a_2^2a_3^2a_4a_5a_6^4 + 4a_1^3a_3^3a_4a_5a_6^4 + 4a_1^2a_2^3a_4^2a_5a_6^4 - 18a_1^3a_2a_3a_4^2a_5a_6^4 + 27a_1^4a_4^3a_5a_6^4
\end{aligned}
$$

whose convex hull is the same as the Chow polytope. The polytope P can be visualized as follows:



Remark 2.4 (Missing polytope!)**.** The above polytope is a Minkowski sum of $\Delta_1, \Delta_1, \Delta_3$. We weren't able to find the name of this polytope however; if you know what its name is, please let us know!

3 Macaulay2: "readingGKZ.m2"

The file `readingGKZ.m2` is available for download at `https://math.berkeley.edu/~ceur/notes`. Loading `readingGKZ.m2` requires packages `Polyhedra.m2` and `Resultants.m2`. The package `Resultants.m2` is not a standard package for Macaulay2 distribution 1.10; one can download the package at `https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2-1.10/share/doc/Macaulay2/Resultants/html/`.

The core functions in `readingGKZ.m2` are as follows.

`AtoIdeal`
Input: a matrix A of columns in $\mathbb{Z}^{k-1}$
Output: the homogeneous ideal defining X_A

`ChowForm`
Input: a homoegeneous ideal I defining $X \subset \mathbb{P}^n$
Output: the Chow form R_X in Plücker coordinates
Comments: – imported from the package `Resultants.m2`
– implemented using Gauss maps

`ChowForm2`
Input: a homoegeneous ideal I defining $X \subset \mathbb{P}^n$
Output: the Chow form R_X in Plücker coordinates
Comments: – an implementation of Chow form using incidence variety and elimination.
– (unfortunately) about 10 times slower than `ChowForm` in the current state

`ChowPolytope` Option: `Normalize=>true/false`
Input: a homogeneous ideal I or a matrix A
Output: the Chow polytope of $V(I)$ or X_A
Comments: – implemented using weights as set in the *Note* below Example 1.1.
– has an option to normalize (shift by an appropriate all 1 vector) so that it will match the secondary polytope. Default is not to normalize when the input is an ideal, and to normalize when the input is a matrix.

`pDual`
Input: a homogeneous ideal defining $X \subset \mathbb{P}^n$
Output: ideal of the projective dual $X^\vee$
Comments: – when $X = X_A$, this is the same as the A-discriminant

`Adiscriminant`
Input: a matrix A
Output: the A-discriminant Δ_A.
Comments: – computed by forming the incidence variety and elimination.

`principalAdet`
Input: a matrix A for which the toric variety X_A is smooth
Output: the principal A-determinant E_A of X_A
Comments: – computed using [GKZ94, Theorem 10.1.2].
– usually much faster than computing the Chow form.
– when X_A not smooth, the Newton polytope of the output is at least combinatorially equivalent to the secondary polytope $\Sigma(A)$.

The computations for the examples in Section §2.3 are done as follows:

```
--Section 2.3: Hirzebruch H_{3,2}
A = matrix{{0,1,2,0,1},{1,1,1,0,0}}
RX = ChowForm AtoIdeal A
P = ChowPolytope A, << ", vertices" << vertices P
DA = Adiscriminant A, pDual AtoIdeal A --should be the same
EA = principalAdet A
Q = newtonPolytope EA, << "vertices" << vertices Q
vertices P == vertices Q


--Section 2.3: Hirzebruch H_{4,2}
A = matrix{{0,1,2,3,0,1},{1,1,1,1,0,0}}
time RX = ChowForm AtoIdeal A; -- 15 seconds
P = ChowPolytope A, << ", vertices" << vertices P -- 15 seconds also
DA = Adiscriminant A, pDual AtoIdeal A --should be the same
EA = principalAdet A
Q = newtonPolytope EA, << "vertices" << vertices Q
vertices P == vertices Q


--Section 2.3: Hirzebruch H_{5,2} --optional
A = matrix{{0,1,2,3,4,0,1},{1,1,1,1,1,0,0}}
time EA = principalAdet A; -- 0.22 seconds!
Q = newtonPolytope EA
fVector Q
```

References

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