MATH 54 FALL 2017: DISCUSSION 205/208 QUIZ#11

GSI: CHRISTOPHER EUR, DATE: 11/13/2017

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Problem 1. Let $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$.

(a) (4 points) Given that A^TA has eigenvalues 25, 9, and 0, find orthogonal diagonalization of A^TA.

(b) (3 points) Find the SVD of A.

(c) (extra point) Let $S = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\|^2 \leq 1\}$ be the unit ball in $\mathbb{R}^3$. Find the area of the image $A(S)$ using the fact that an ellipse with major and minor diameters $2a$ and $2b$ has area πab.

Problem 2. (3 points) Let B be a symmetric $n \times n$ matrix such that $B^2 = B$. Show that for any $y \in \mathbb{R}^n$, the vectors $y - By$ and By are orthogonal.

#1. (a) $A^TA = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$.

Eigenval:

$\lambda = 25 \rightsquigarrow \ker \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{normalize}} v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$\lambda = 9 \rightsquigarrow \ker \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ —— // —— $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \rightsquigarrow v_2 = \begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}$

$\lambda = 0 \rightsquigarrow \ker \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$ —— // —— $\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \rightsquigarrow v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}$

Let $P = \begin{bmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & 1/3 \end{bmatrix}$, then $P^TA^TAP = \begin{bmatrix} 25 & & \\ & 9 & \\ & & 0 \end{bmatrix}$.

P is already orthogonal since columns have been normalized and A^TA symmetric $\Rightarrow$ eigenvectors w/ distinct eigenval. are orthogonal.

(b) The V matrix is P above.

$u_1 = \frac{1}{5}Av_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$u_2 = \frac{1}{3}Av_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$.

So, $A = U \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} V^T$ is the SVD of A.

(c) $\pi \cdot \frac{5}{2} \cdot \frac{3}{2} = \frac{15\pi}{4}$, since U, V preserve the geometry.

#2 $(y - By) \cdot By = y \cdot By - By \cdot By = y^T By - y^T B^T B y = y^T By - y^T By = 0.$ ✓ (since $B^T = B$ and $B^2 = B$)