MATH 54 FALL 2016: DISCUSSION 102/105 QUIZ#9

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STUDENT NAME: Strange things...

Problem 1. Define a linear operator $L : \mathcal{P}_2 \to \mathcal{P}_2$ by $L(p(x)) = p''(x) - 2xp'(x)$ (where $\mathcal{P}_2$ is the vector space of polynomials with real coefficients of degree ≤ 2).

(a) (2 points) Write down the matrix A that represents this linear operator L with respect to the basis $E = (1, x, x^2)$ on $\mathcal{P}_2$.

(b) (3 points) Find all eigenvalues of L and find the basis for each corresponding eigenspaces.

(c) (2 points) Use the previous part to find a matrix P such that $P^{-1}AP$ is diagonal, and check that $P^{-1}AP$ is indeed diagonal.

(d) (2 points) Compute $L^{50}(x^2)$ (You need not compute out powers of a single number such as 5^{23}).

(e) (1 point) Show that there is no nonzero polynomial $q(x)$ of degree ≤ 2 such that $L(q(x)) = 5q(x)$.

(a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ $\qquad L(1)=0, \quad L(x) = -2x, \quad L(x^2) = 2 - 4x^2$

(b) $\boxed{\begin{array}{lccc} & \lambda = 0, & -2, & -4 \\ & | & | & | \\ \text{Eigenspace basis:} & \{1\} & \{x\} & \{-1+2x^2\} \end{array}}$ $\longleftarrow$ $\text{Null}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) Let $B = (1, x, -1+2x^2)$.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "D"] \arrow[d, "B", "\wr"'] \arrow[dd, bend right=60, "P"'] & \mathbb{R}^3 \arrow[d, "\wr", "B"'] \arrow[dd, bend left=60] \\ \mathcal{P}_2 \arrow[r, "L"] & \mathcal{P}_2 \\ \mathbb{R}^3 \arrow[u, "E", "\wr"'] \arrow[r, "A"] & \mathbb{R}^3 \arrow[u, "\wr", "E"'] \end{tikzcd}$$

$D = P^{-1}AP$

$\boxed{P = \left(\underset{E \leftarrow B}{P}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \rightsquigarrow P^{-1} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ as expected ✓.

(d) $\begin{bmatrix} 0 & & \\ & -2 & \\ & & -4 \end{bmatrix}^{50} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}_B = \boxed{4^{50} \cdot \frac{1}{2}(-1+2x^2) = -2^{99} + 2^{100}x^2}$

$\hookrightarrow$ as $\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}_B$ corresp. to $x^2 = 1/2(1) + \frac{1}{2}(-1+2x^2)$

(e) $L(q) = 5q$ means that 5 is an eigenvalue (as $q \neq 0$).
But only eigenvalues of L are $0, -2, -4$. ※.