Quiz #8; Wed, 3/14/2016

Math 53 with Prof. Stankova

Section 107; MWF10-11

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Problem. A cardboard box without a lid is to have a volume of 4 m^3. Find the dimensions that minimize the amount of cardboard used. (12 points for finding the dimensions that *locally* minimize the amount used; 3 points for justifying that it is indeed a global minimum).

Solution. If x, y, z are the width, length, height of the box, then its total surface area is $xy + 2yz + 2zx$. Now, since $xyz = 4$, we have that the total surface area of the box $(x, y) =$ (width, length) with volume 4 is:

$$f(x, y) = xy + (2)(4)/x + (2)(4)/y$$

Thus $\nabla f(x, y) = \langle y - 8/x^2, x - 8/y^2 \rangle$. Note that $x, y > 0$, and hence, the critical points are where $x^2y = xy^2 = 8$. That is, when $x = y$ and thus $x^3 = y^3 = 8$. So $(2, 2)$ is the unique critical point for the domain $x, y > 0$. For the second derivative test, we compute that

$$\det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \det \begin{bmatrix} 16/x^3 & 1 \\ 1 & 16/y^3 \end{bmatrix} = \frac{256}{x^3y^3} - 1$$

which is positive for $(2, 2)$ and $f_{xx} > 0$ as well for $(2, 2)$. Hence, $(2, 2)$ is a local minimum with $f(2, 2) = 12$. To see that this is a global minimum, note that if $x > 2$ and $y > 2$, then $y - 8/x^2 > 0$ and $x - 8/y^2 > 0$. So, outside the region $D = \{0 < x \leq 2,\ 0 < y \leq 2\}$, the function f is always increasing as x or y increases. But along the boundary $x = 2$ or $y = 2$, the minimum is achieved at $(2, 2)$.