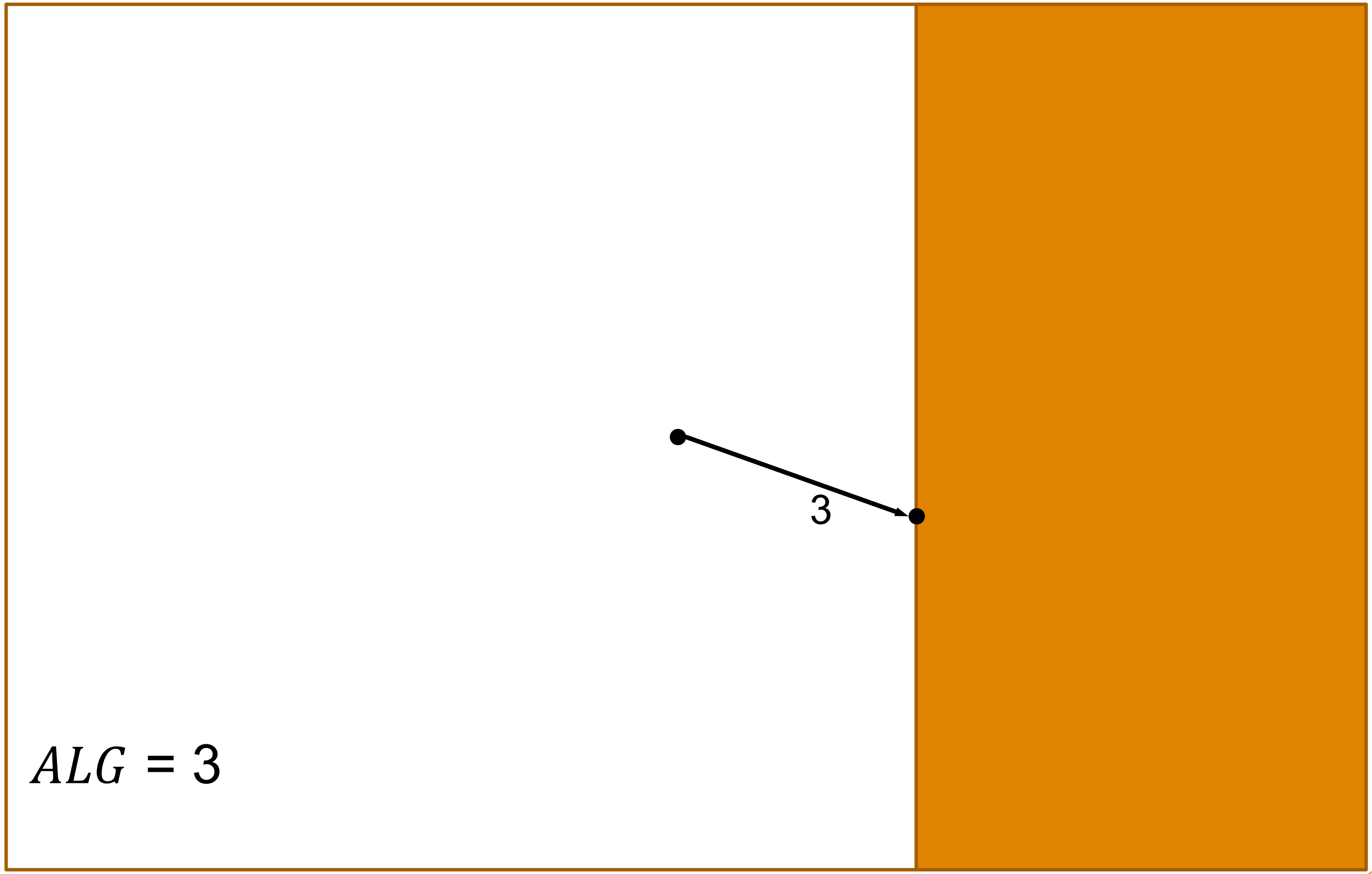
Convex Body Chasing

C.J. Argue

Joint with Anupam Gupta, Guru Guruganesh, Ziye Tang

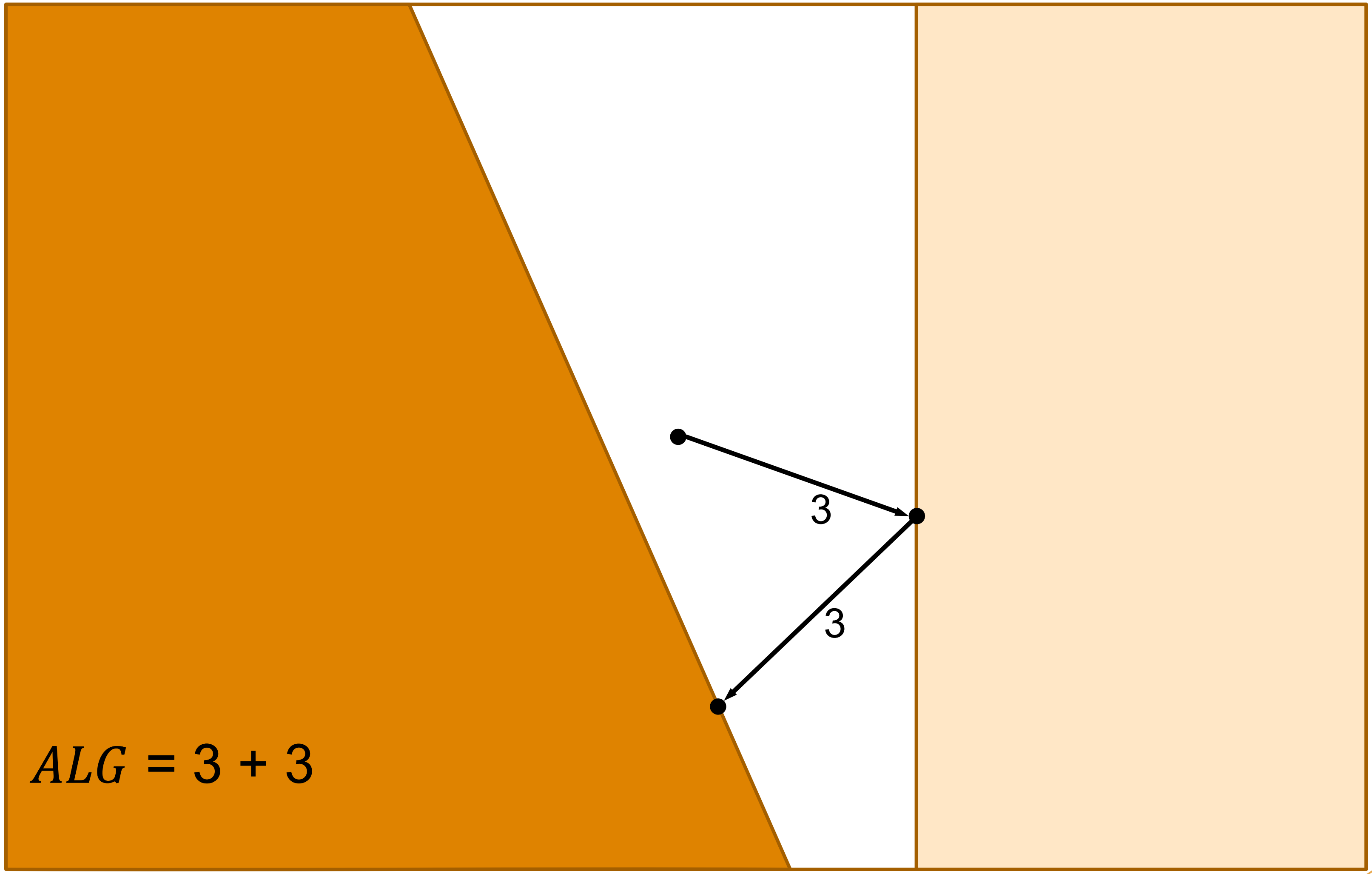
some slides courtesy of Guru Guruganesh

Convex Body Chasing – The Problem



$ALG = 3$

Convex Body Chasing – The Problem



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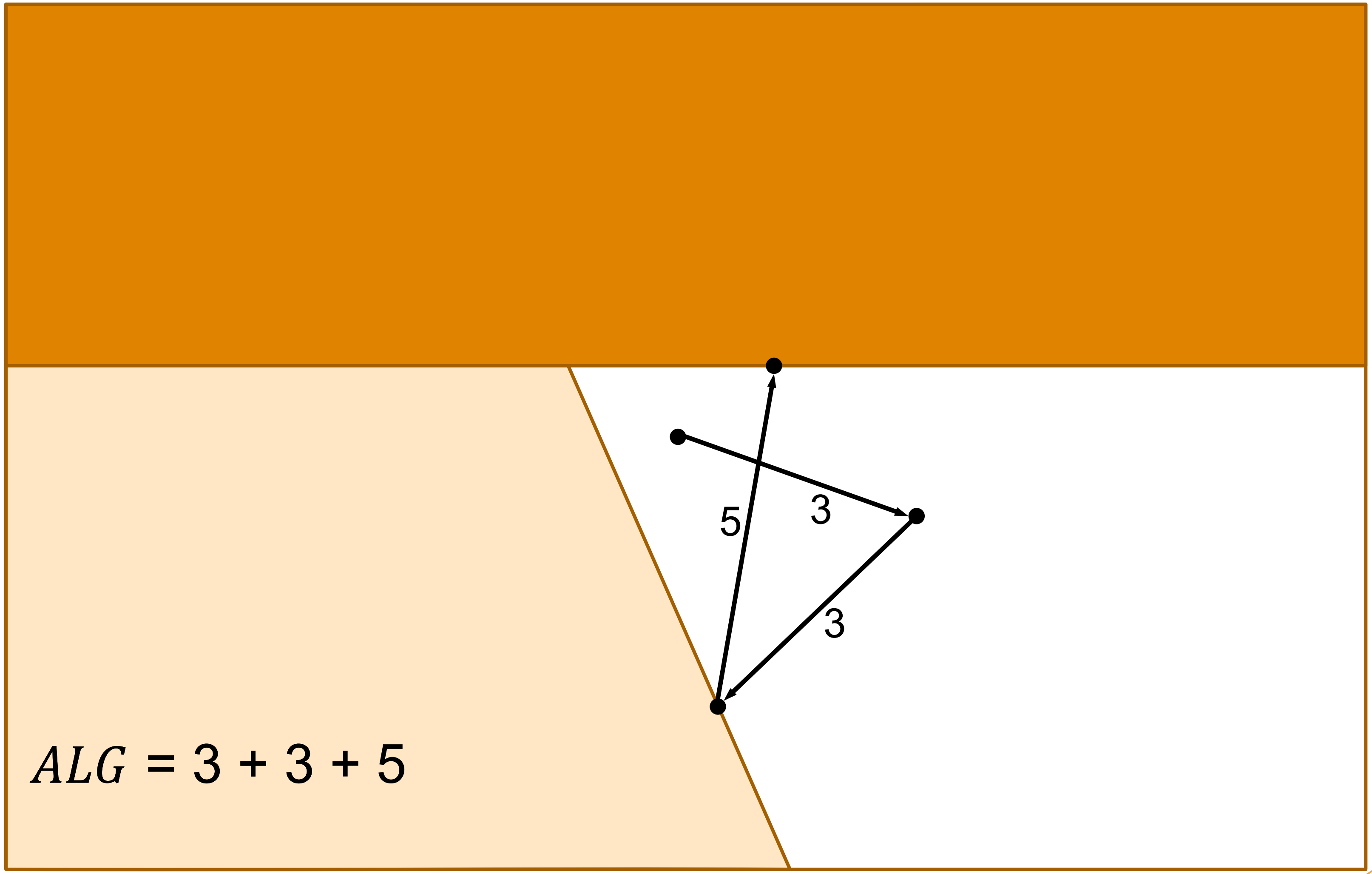
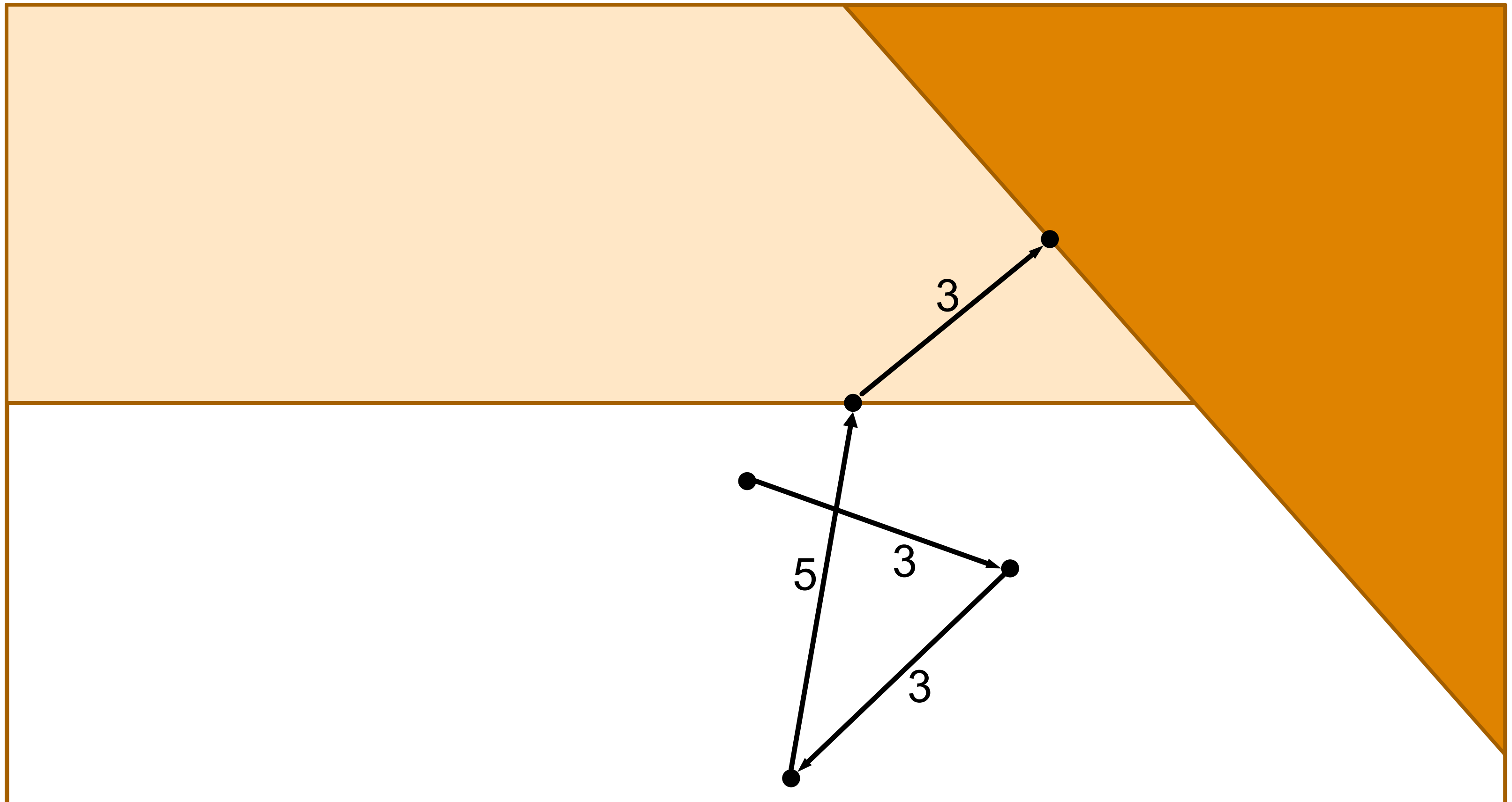


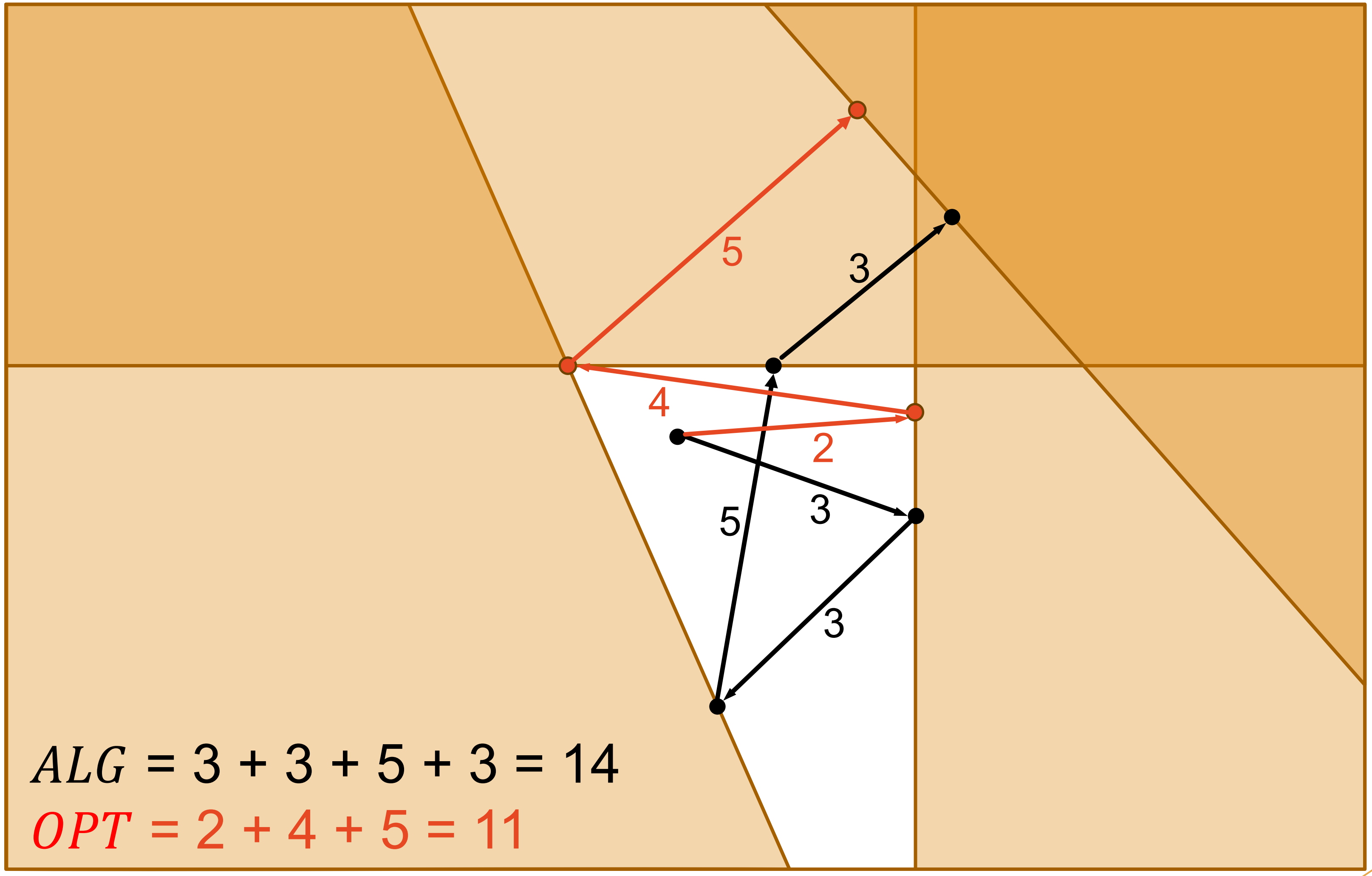
$ALG = 3 + 3 + 5$

Convex Body Chasing – The Problem

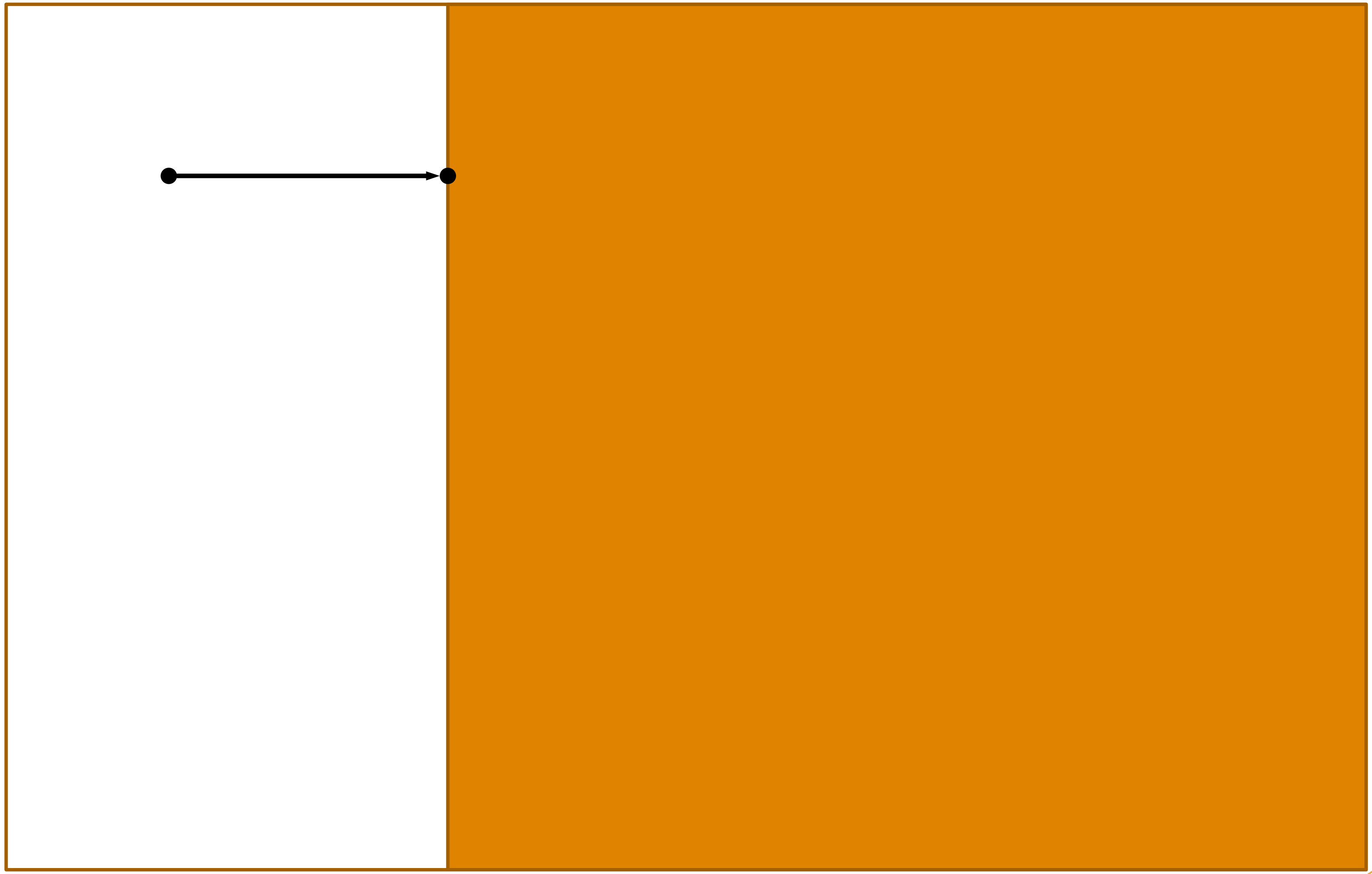


$ALG = 3 + 3 + 5 + 3$

Convex Body Chasing – The Problem



Nested Version



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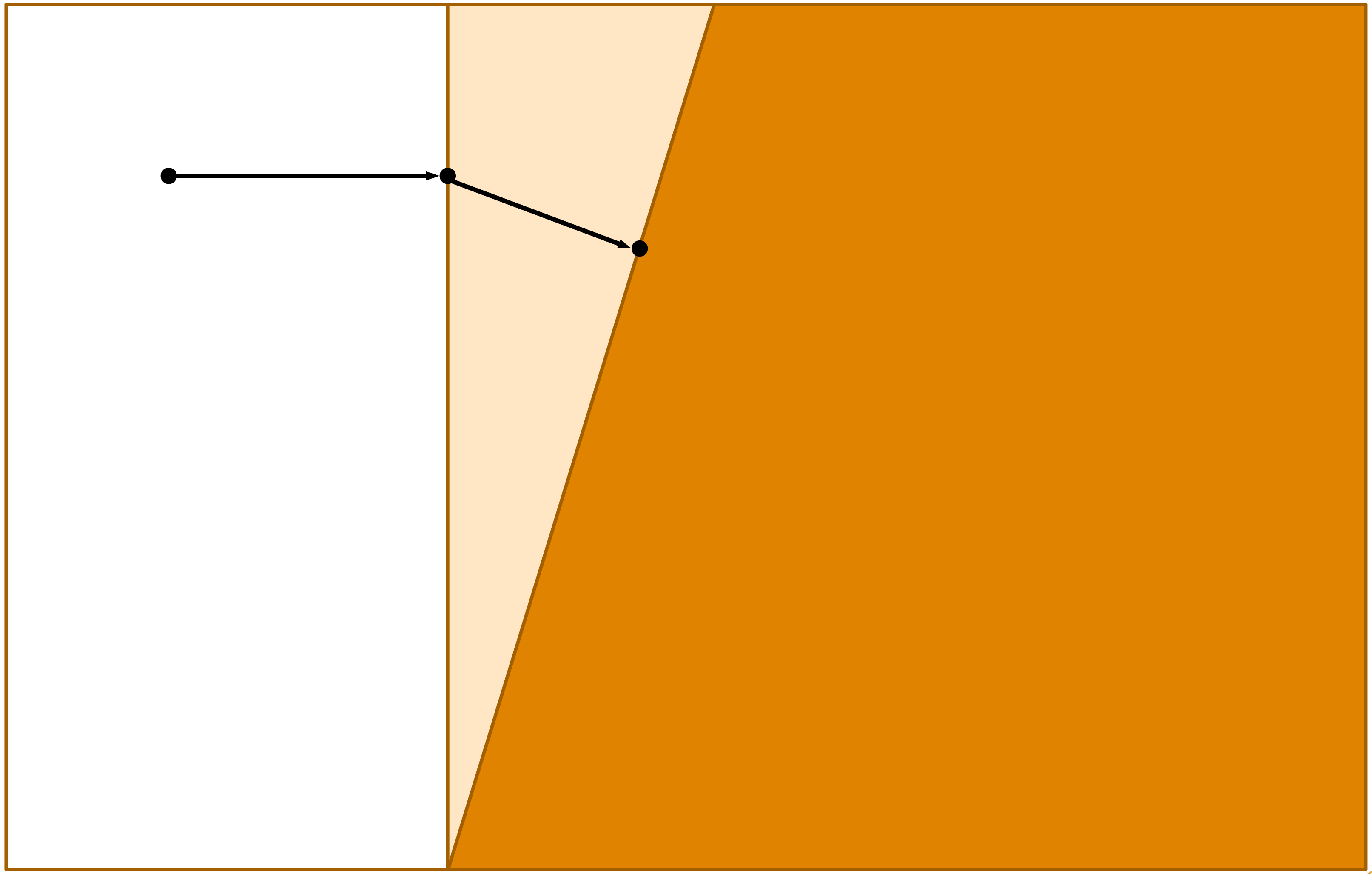

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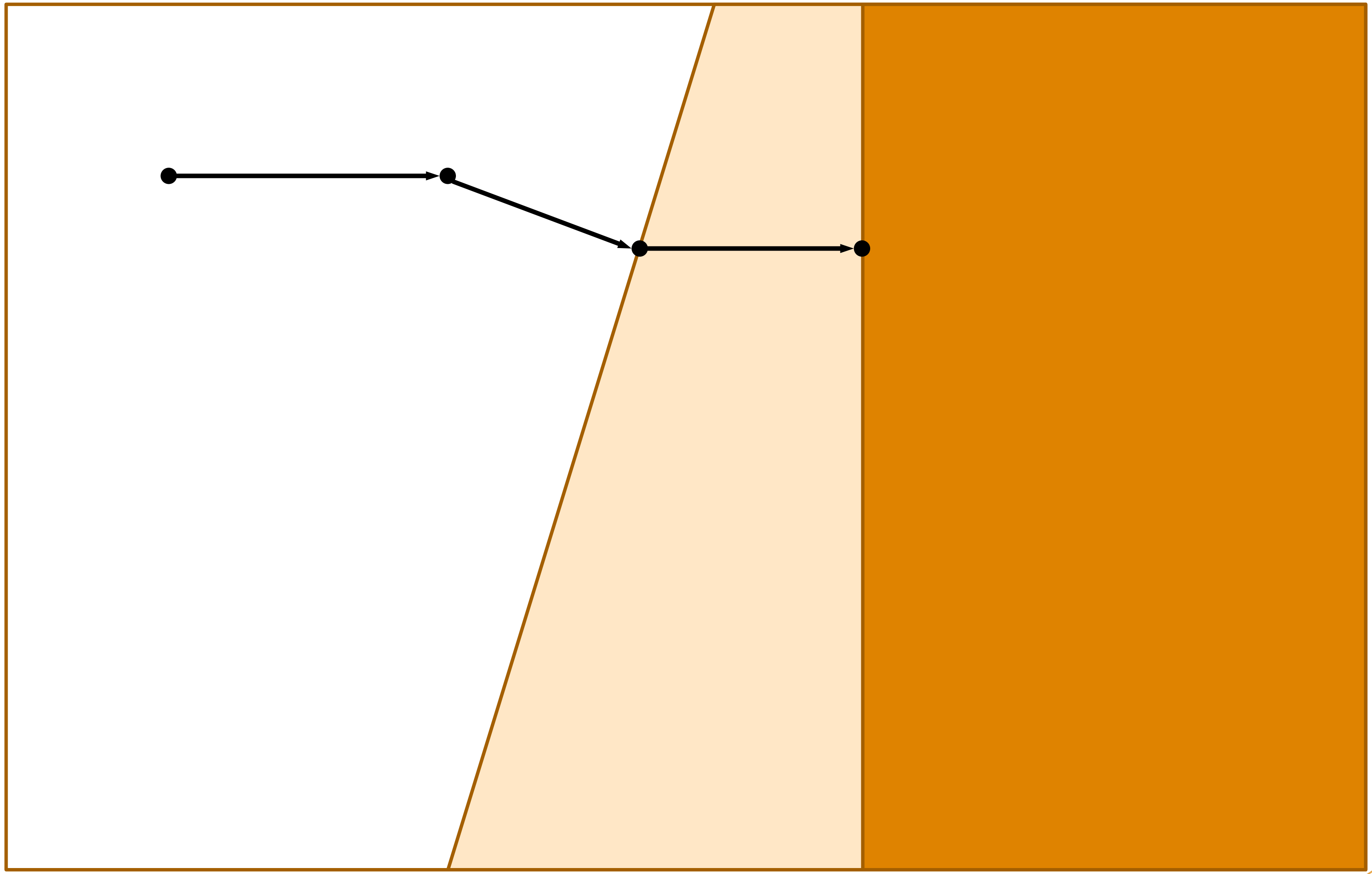

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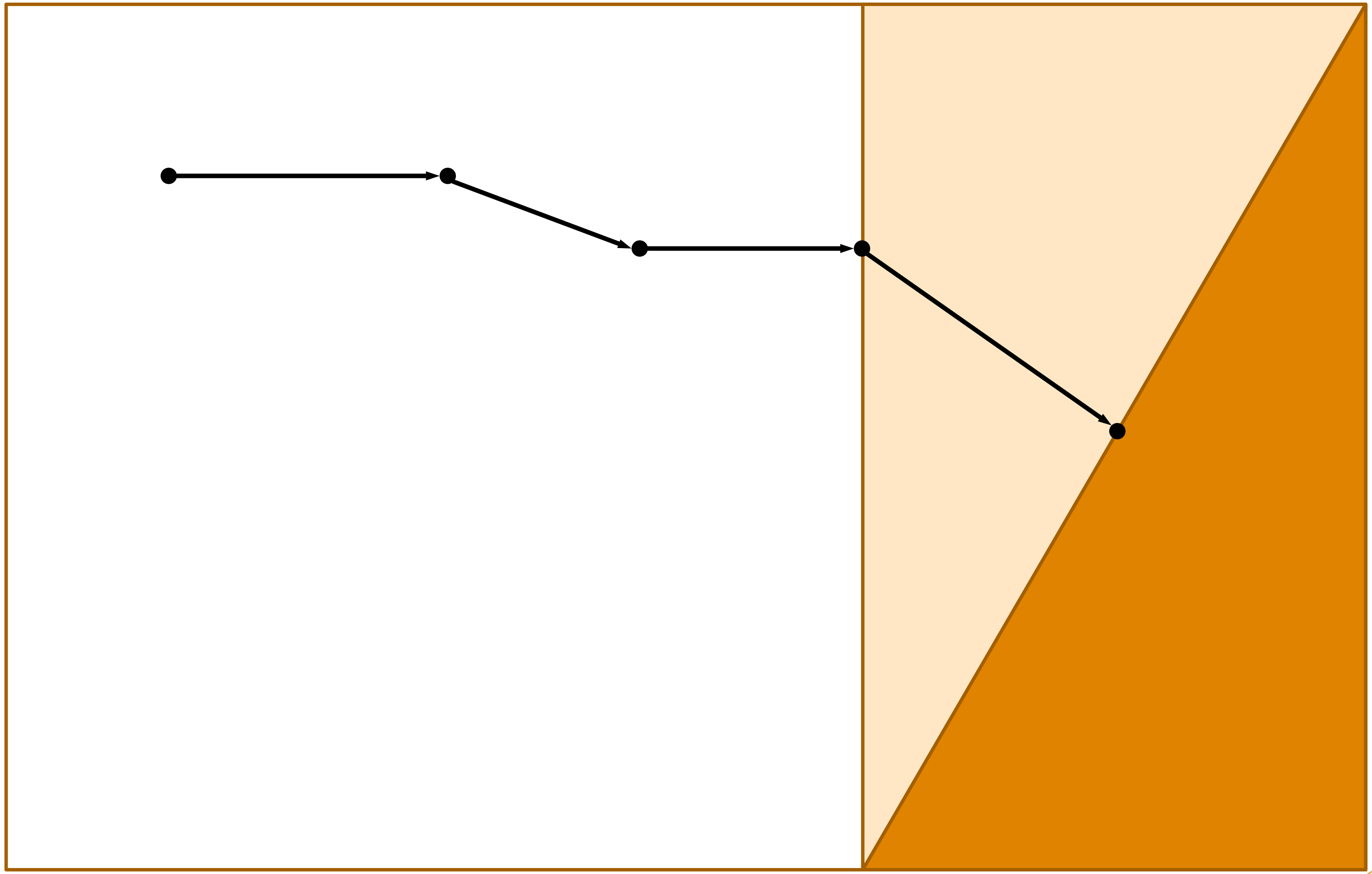

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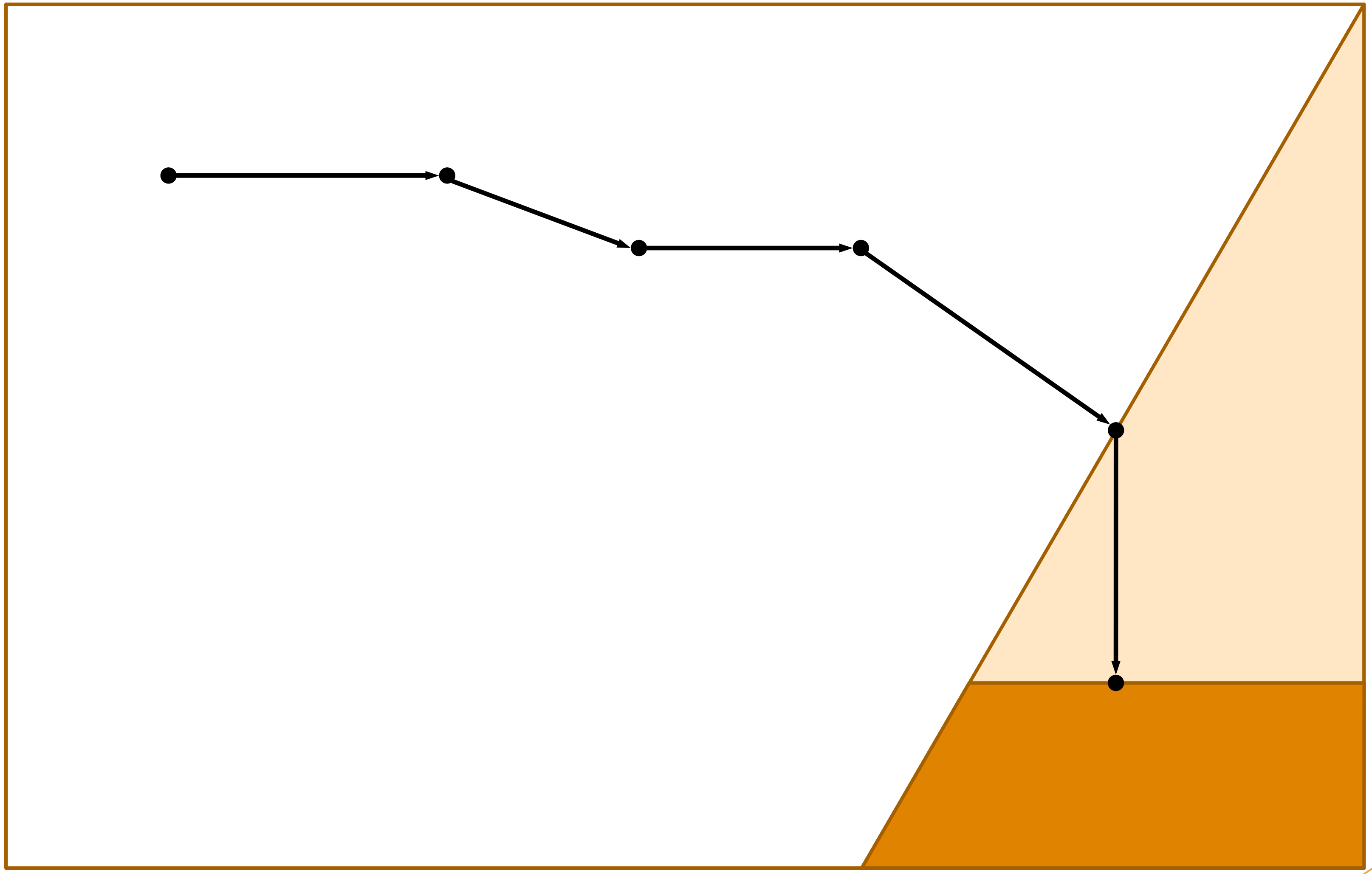

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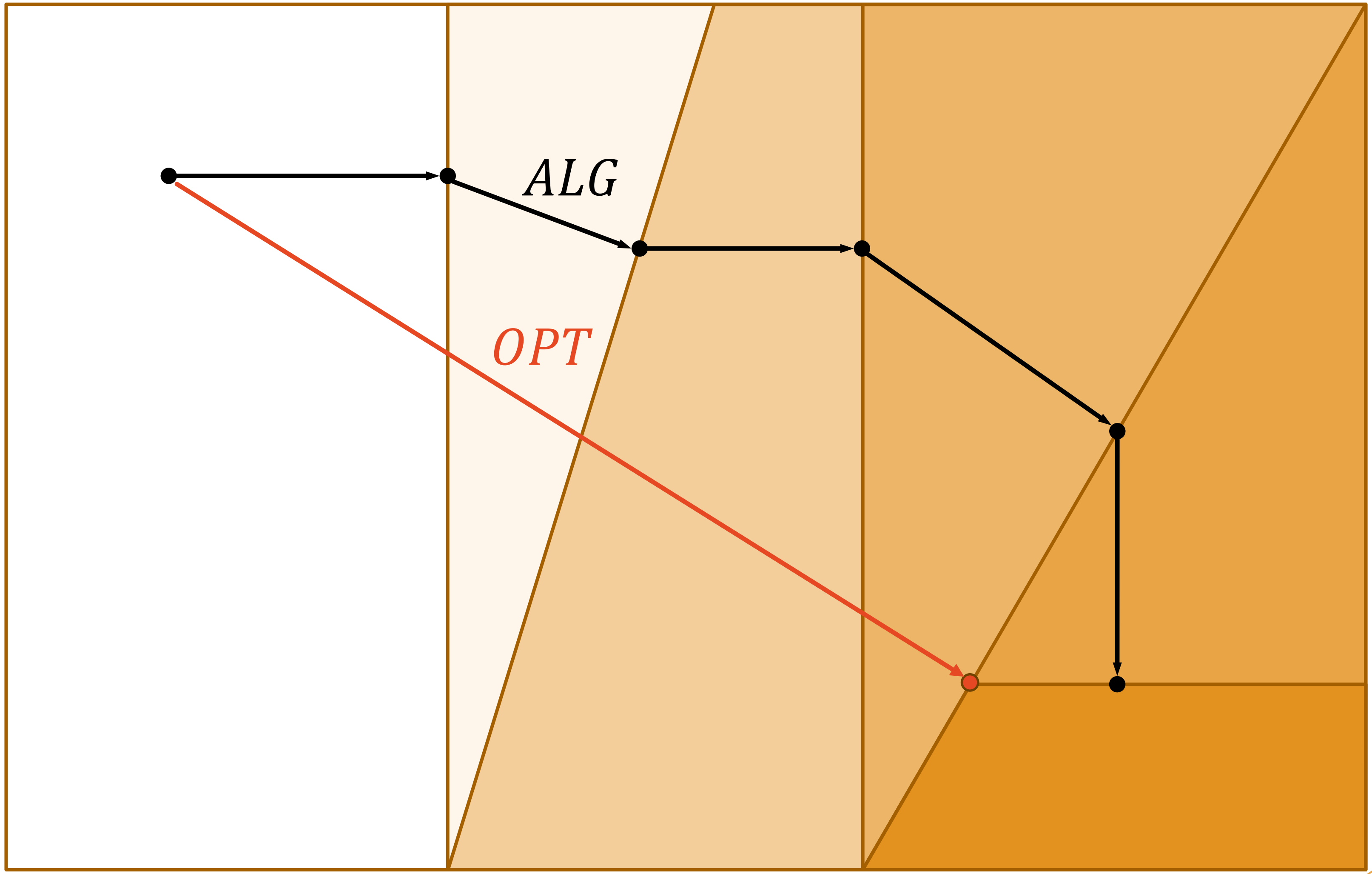

Formal Definition

- Instance σ: convex sets $K_1, K_2, K_3, \ldots, K_T \subseteq \mathbb{R}^d$
- Choose *online* $x_i \in K_i$
- Cost $ALG(\sigma) = \sum_{i=1}^{T} ||x_i - x_{i-1}||$
- Goal – minimize competitive ratio

$$\mathrm{cr}(ALG) \coloneqq \max_{\sigma} \frac{ALG(\sigma)}{OPT(\sigma)}$$

  - $OPT\ (\sigma)$ optimal *offline* cost

Motivation 1 – Function Chasing / SOCO

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Motivation 1 – Function Chasing / SOCO

functions $f_1, f_2, f_3, \dots, f_t : \mathbb{R}^d \to \mathbb{R}$

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- Cost $ALG(\sigma) = \sum_{i=1}^{T} ||x_i - x_{i-1}|| + f_i(x_i)$

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- Function chasing ≅ body chasing

  [Bubeck, Lee, Li, Sellke 18]

Motivation 2 – Metrical Task Systems

functions $f_1, f_2, f_3, \dots, f_t : \mathbb{R}^d \to \mathbb{R}$

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Motivation 2 – Metrical Task Systems

X ← Finite metric space

functions $f_1, f_2, f_3, \dots, f_t$: ~~$\mathbb{R}^d$~~ $\to \mathbb{R}$

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Motivation 2 – Metrical Task Systems

functions $f_1, f_2, f_3, \dots, f_t : \sout{\mathbb{R}^d}\ X \to \mathbb{R}$

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Finite metric space

Motivation 2 – Metrical Task Systems

X

functions $f_1, f_2, f_3, \dots, f_t$: ~~$\mathbb{R}^d$~~ $\to \mathbb{R}$

- Instance σ: convex ~~sets $K_1, K_2, K_3, \dots, K_T \subseteq \mathbb{R}^d$~~
- Choose *online* ~~$x_i \in K_i$~~ ~~$x_i \in \mathbb{R}^d$~~ $x_i \in X$
- Cost $ALG(\sigma) = \sum_{i=1}^{T} ||x_i - x_{i-1}|| + f_i(x_i)$

Finite metric space

Motivation 2 – Metrical Task Systems

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- Choose *online* ~~$x_i \in K_i$~~ ~~$x_i \in \mathbb{R}^d$~~ $x_i \in X$
- Cost $ALG(\sigma) = \sum_{i=1}^{T} ||x_i - x_{i-1}|| + f_i(x_i)$

Finite metric space

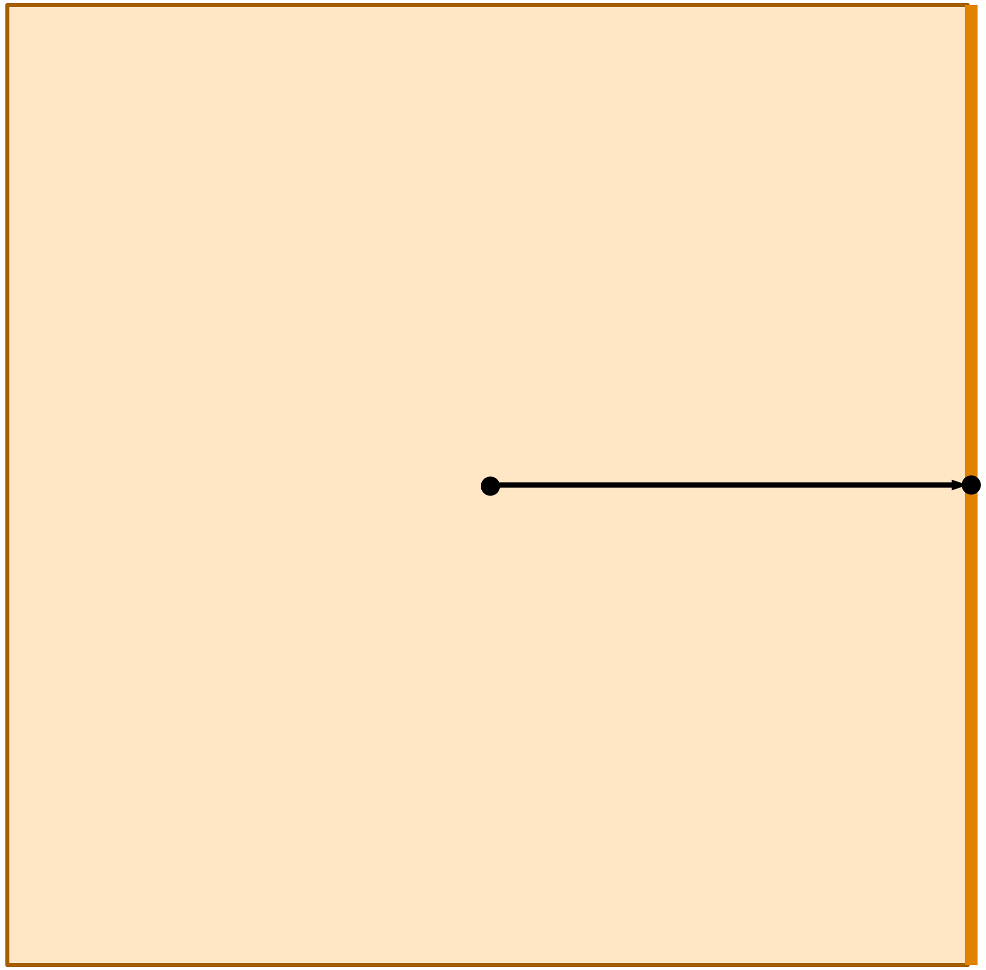
- Fundamental problem in online algorithms
- Body chasing ≈ role of geometry in MTS

Motivation 3 – Paging

- n total pages, cache holds $k < n$ pages
- Requested page r_t must be moved into cache
- Fractional paging ≅ body chasing
  - $K_t = \{x \in \mathbb{R}_+^n \mid x_{r_t} = 1, \sum_i x_i = k\}$
- k-Server ≅ paging generalized to arbitrary metric
  - Fundamental in online algorithms

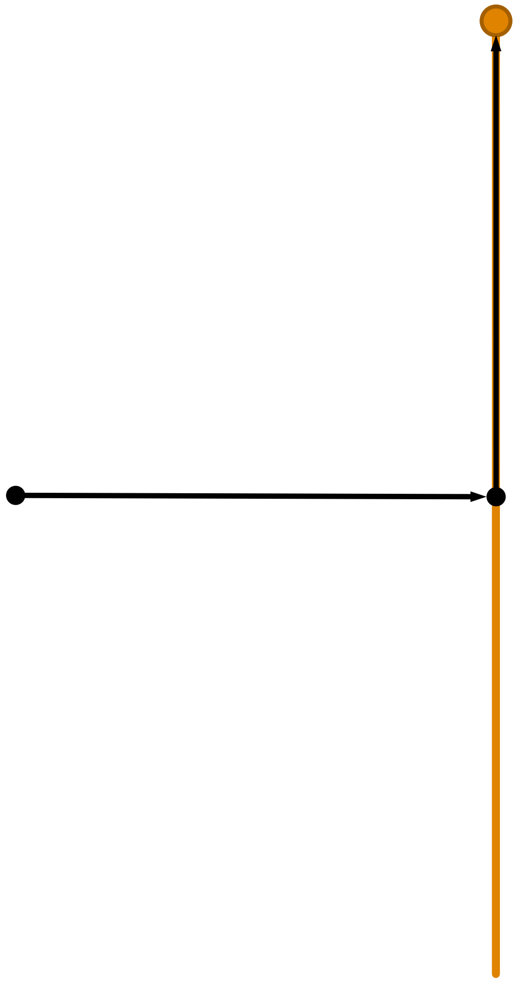
Lower Bound

[Friedman, Linial 93]



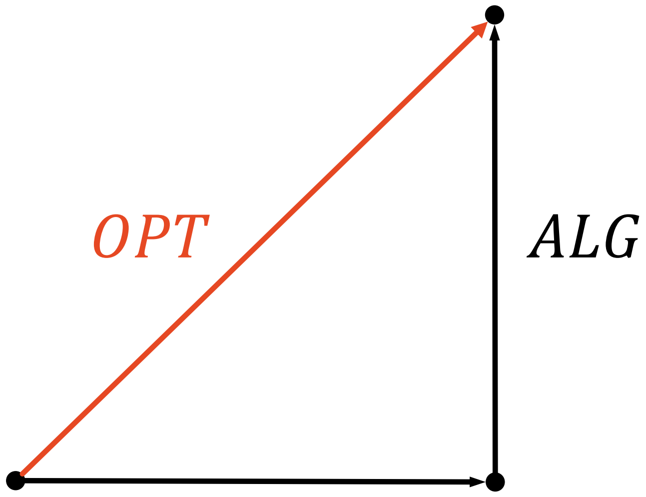
Lower Bound

[Friedman, Linial 93]



Lower Bound

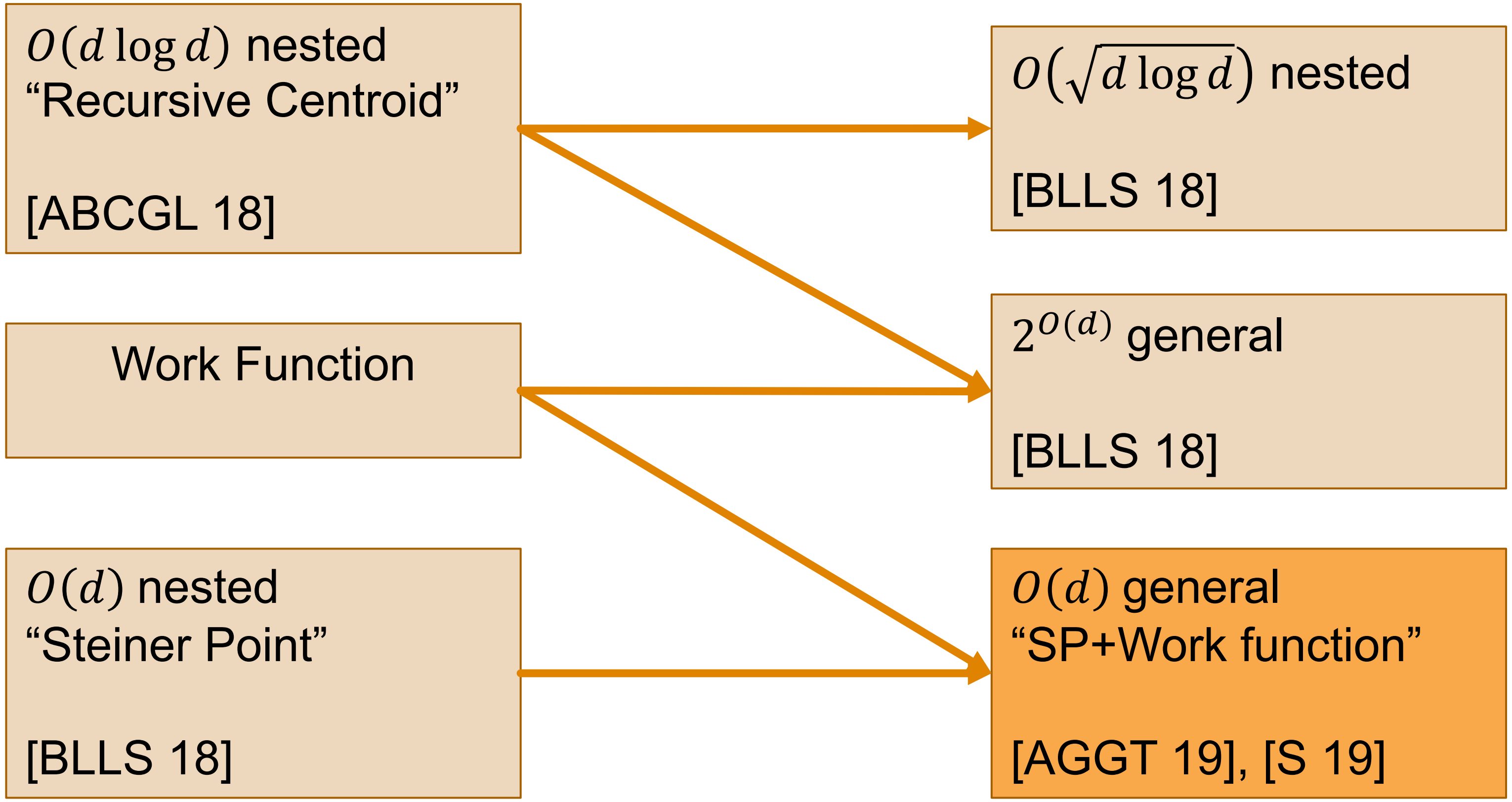
[Friedman, Linial 93]



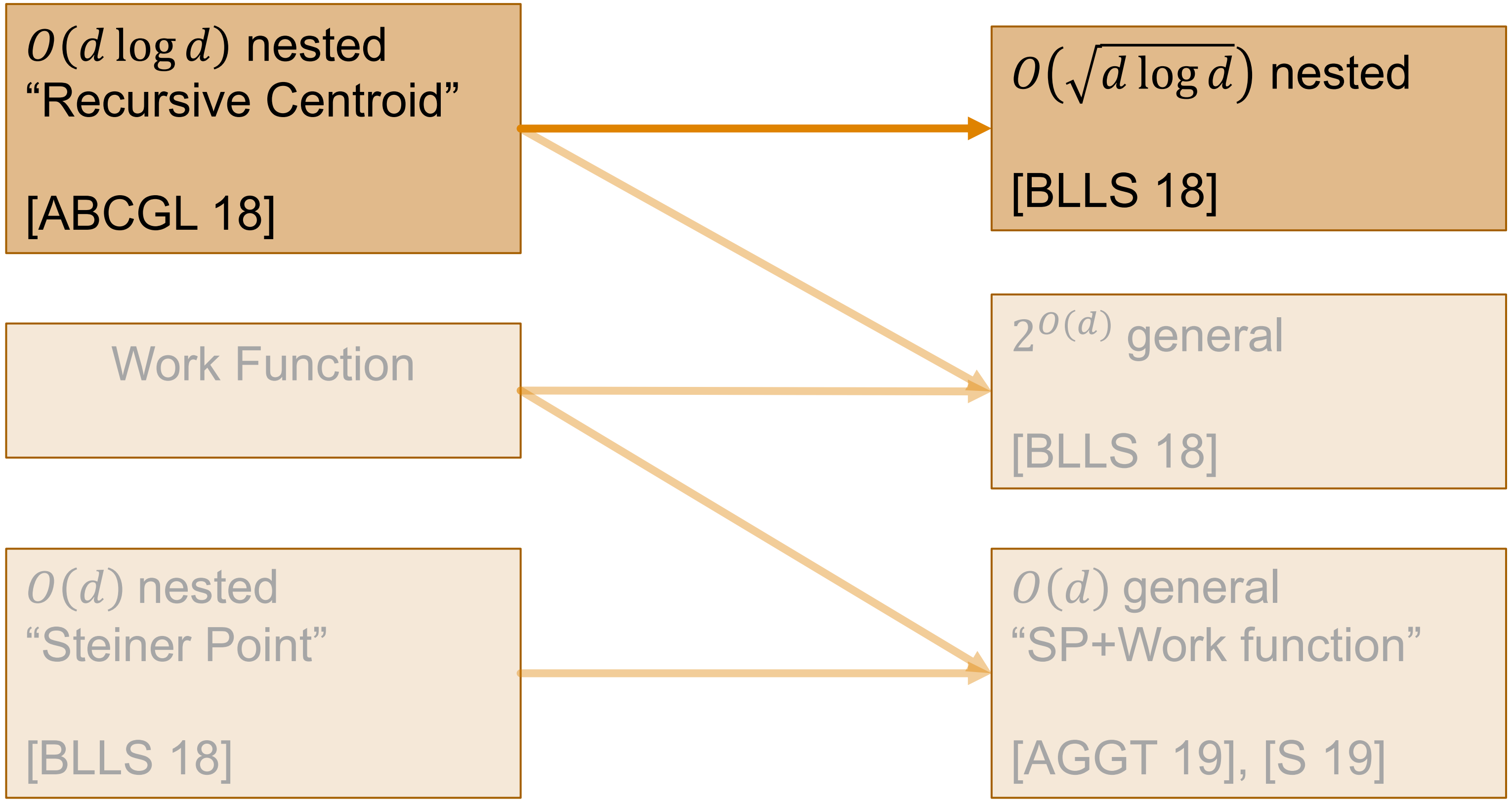
$$ALG \geq \sqrt{2} \cdot OPT$$

$$\boxed{ALG \geq \sqrt{d} \cdot OPT}$$

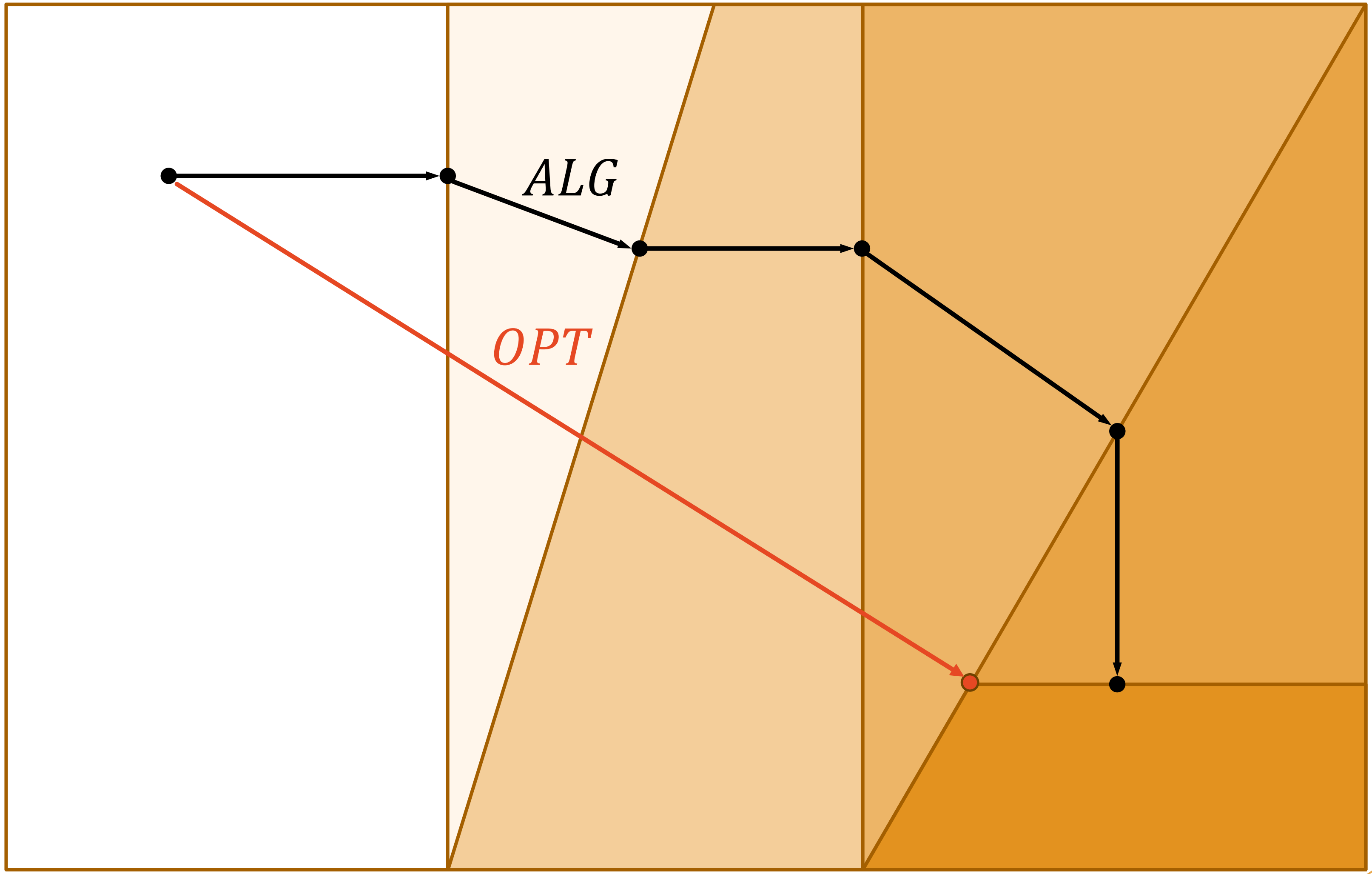
Progress


$O(d \log d)$ nested
"Recursive Centroid"
[ABCGL 18]
Work Function
$O(d)$ nested
"Steiner Point"
[BLLS 18]
$O(\sqrt{d \log d})$ nested
[BLLS 18]
$2^{O(d)}$ general
[BLLS 18]
$O(d)$ general
"SP+Work function"
[AGGT 19], [S 19]

Part 1 – Centroid



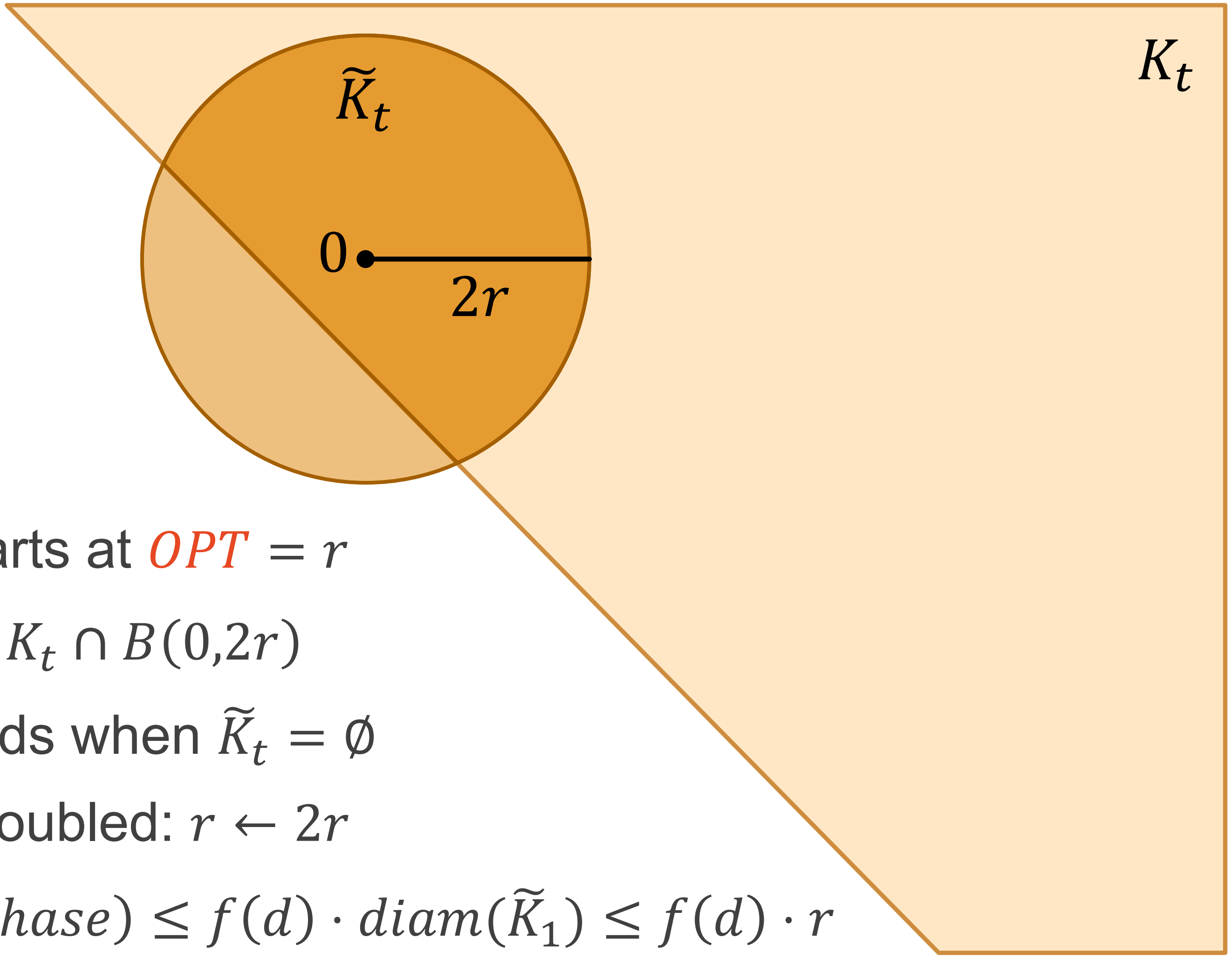
This Section – Nested



Idea – *Centroid*

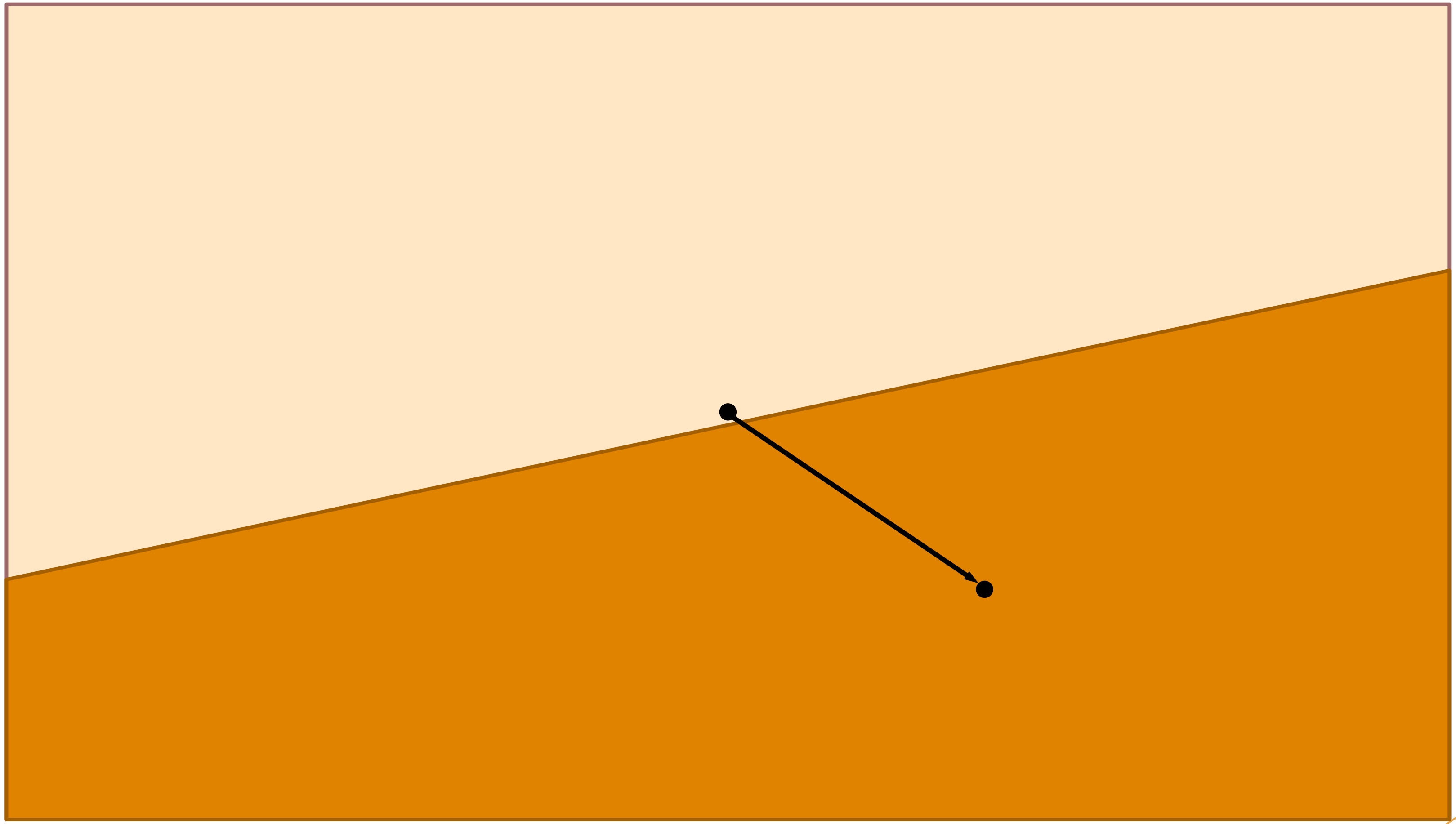
- Algorithm: $x_t = centroid(K_t)$
  - Assume K_t bounded
- Grunbaum's Theorem
  - Cut off centroid ⇒ volume decreases by constant

Unbounded Case – Guess and Double

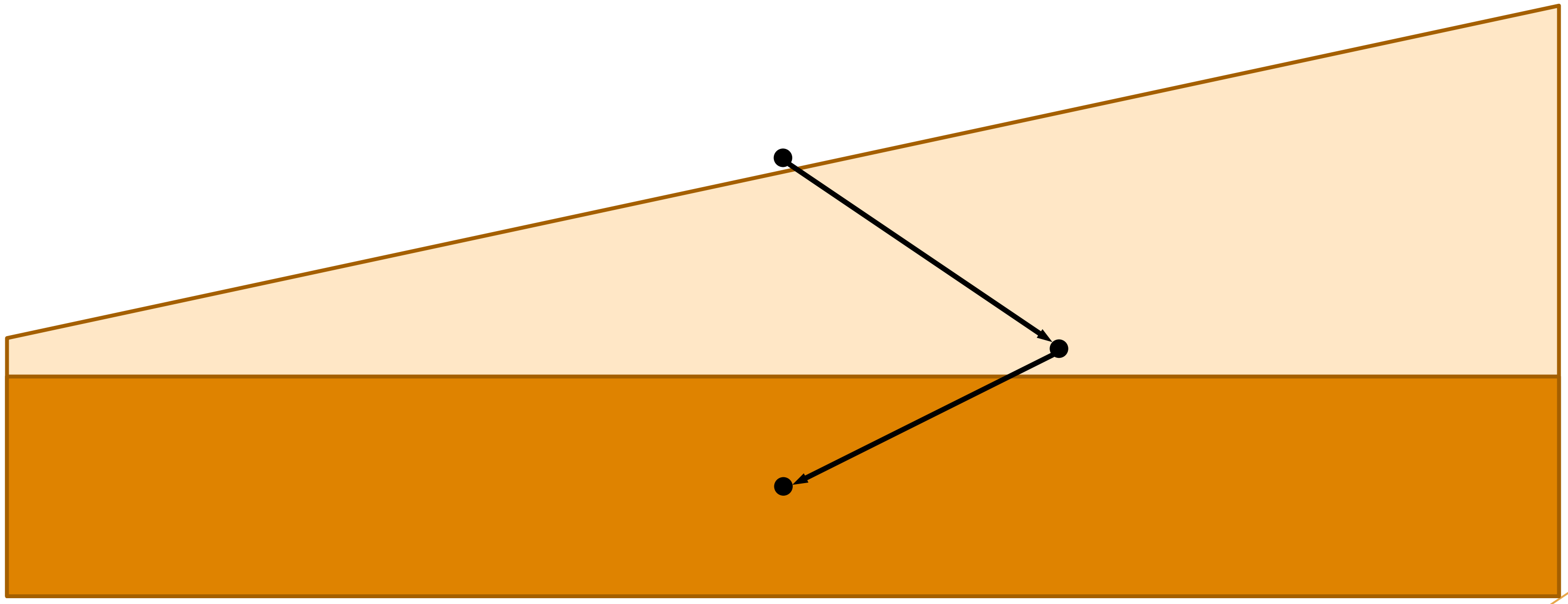


- Phase starts at $OPT = r$
- Use $\widetilde{K}_t = K_t \cap B(0,2r)$
- Phase ends when $\widetilde{K}_t = \emptyset$
  - OPT doubled: $r \leftarrow 2r$
  - $ALG(phase) \leq f(d) \cdot diam(\widetilde{K}_1) \leq f(d) \cdot r$

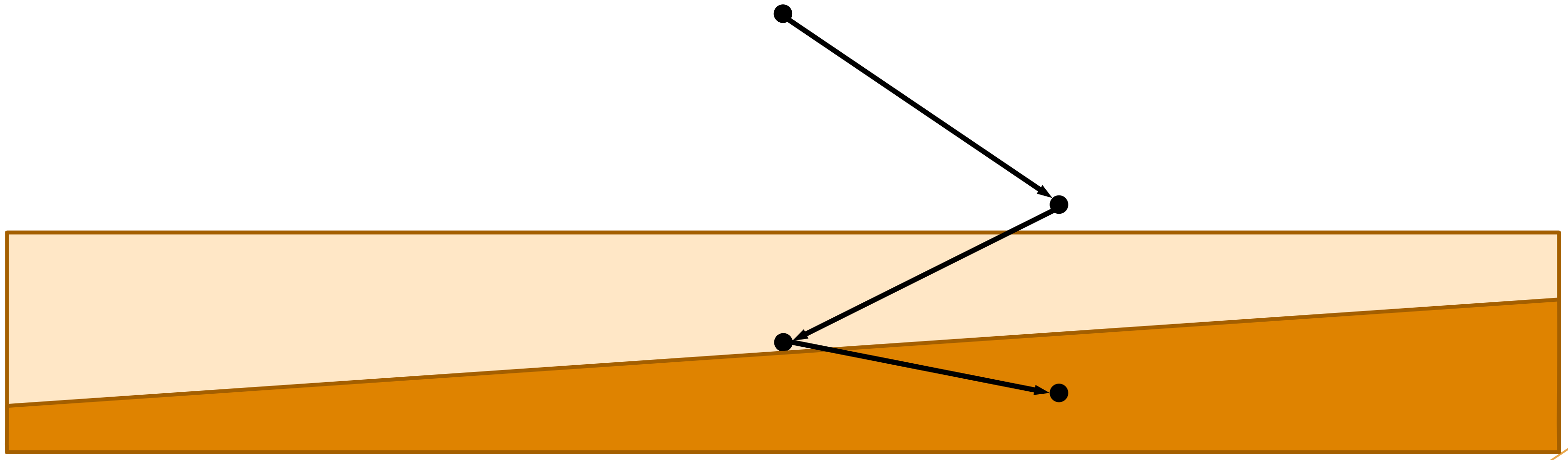
Problem with *Centroid*



Problem with *Centroid*

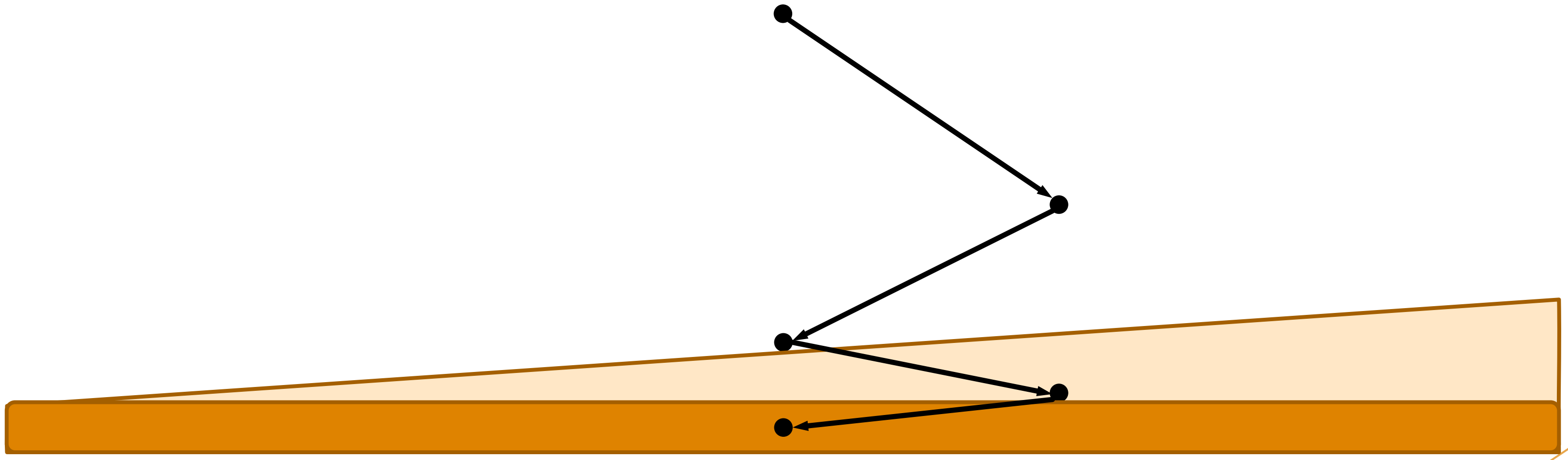


Problem with *Centroid*



Problem with *Centroid*

- ALG unbounded
- $OPT = O(1)$
- Not competitive



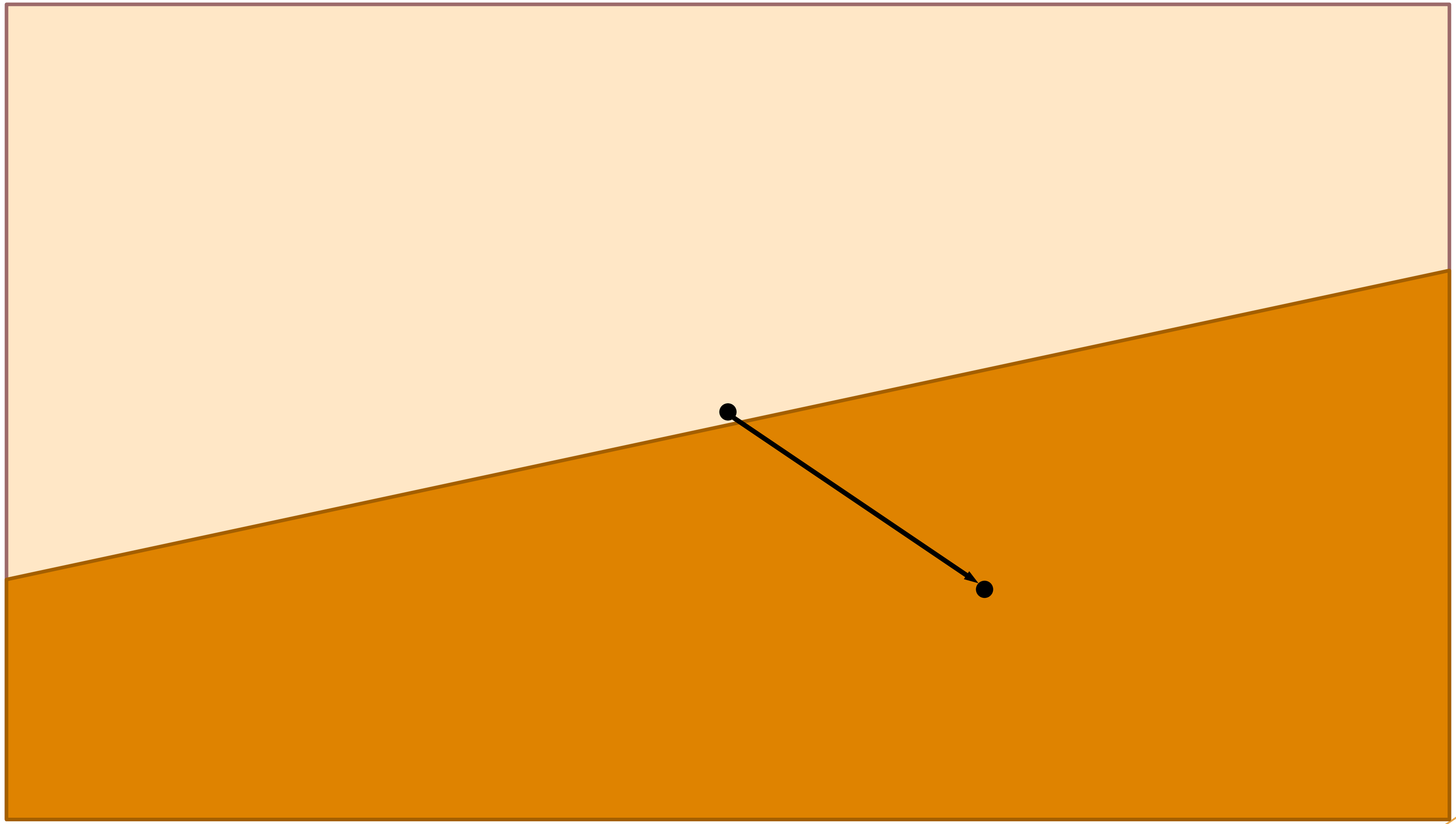
Recursive Centroid (Nested)

[Argue, Bubeck, Cohen, Gupta, Lee 18]

- Recursion on “skinny subspace”
  - Small steps
  - Hyperplane separation ⇒ cut parallel to skinny subspace
    - Shrink diameter

Recursive Centroid (Nested)

[Argue, Bubeck, Cohen, Gupta, Lee 18]



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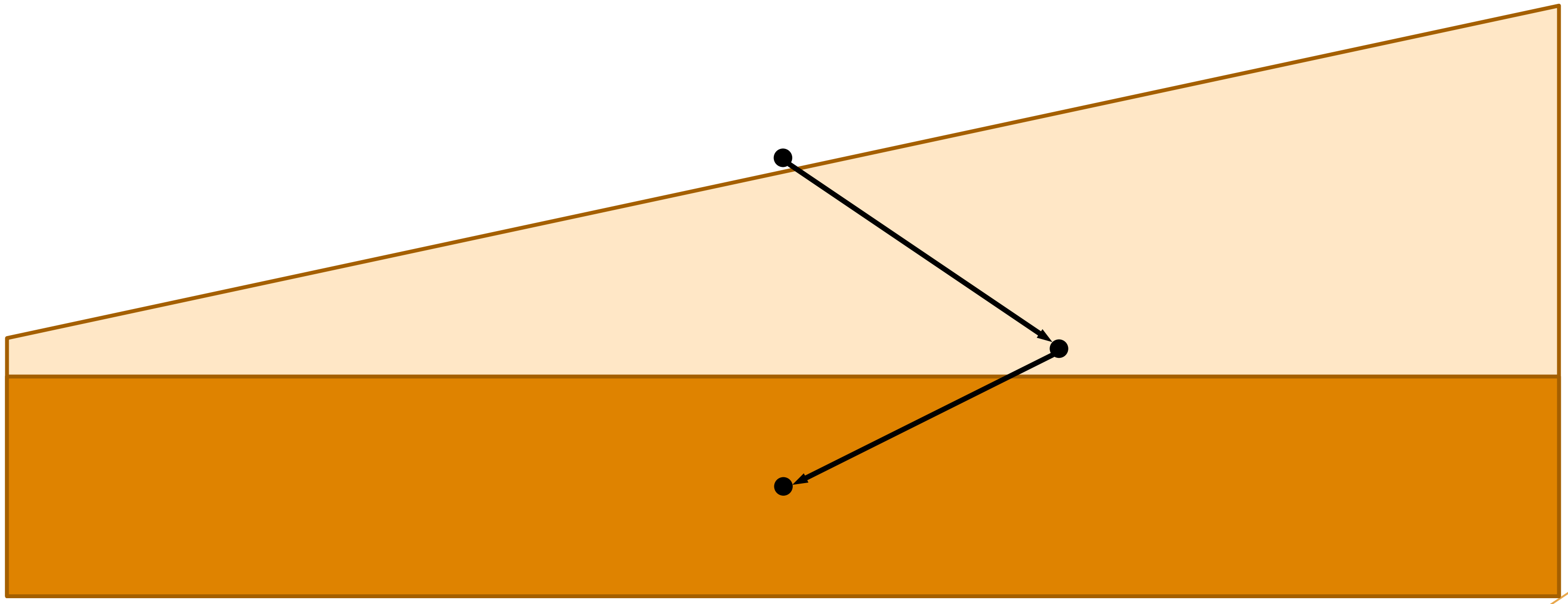

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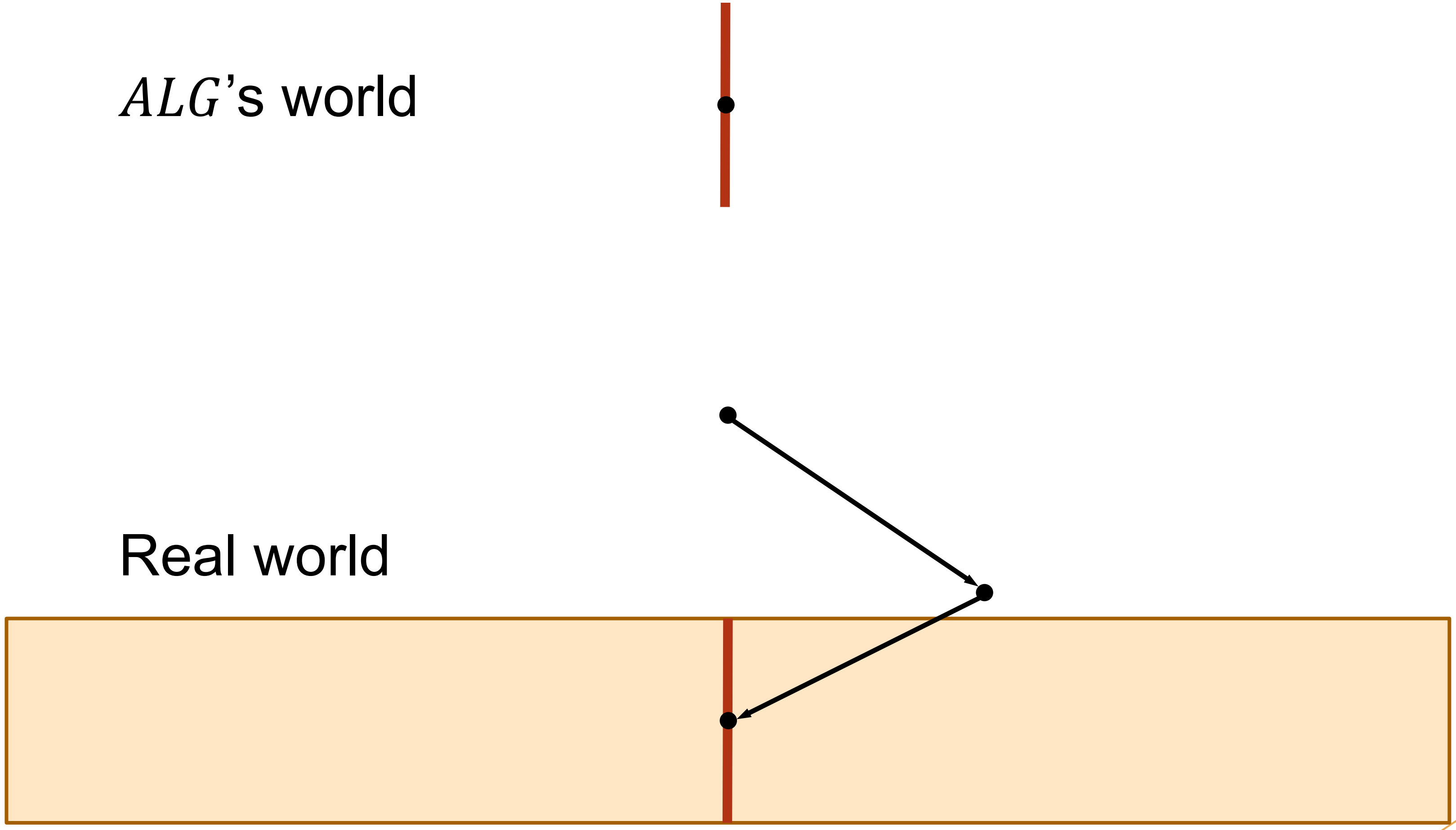

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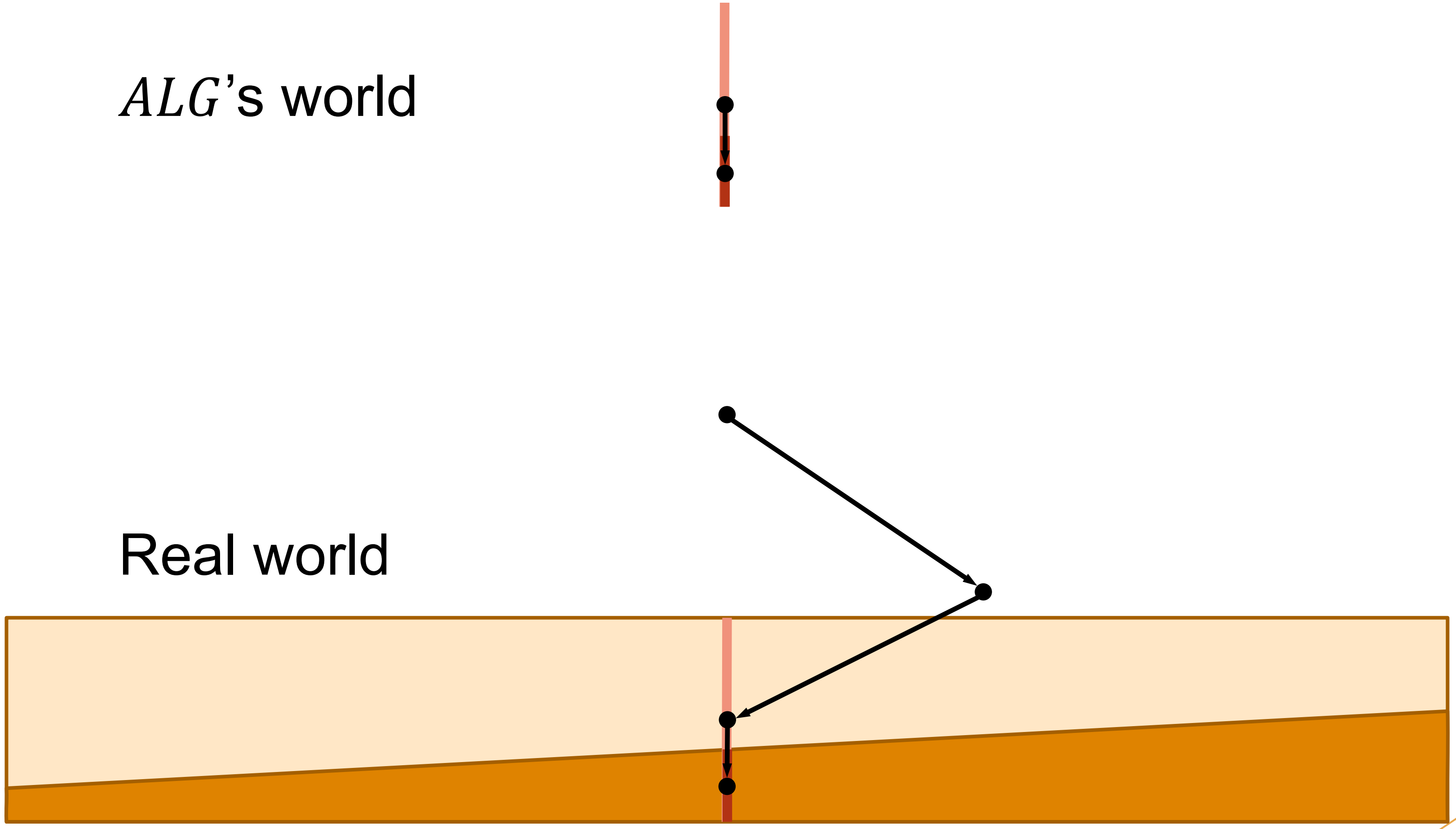

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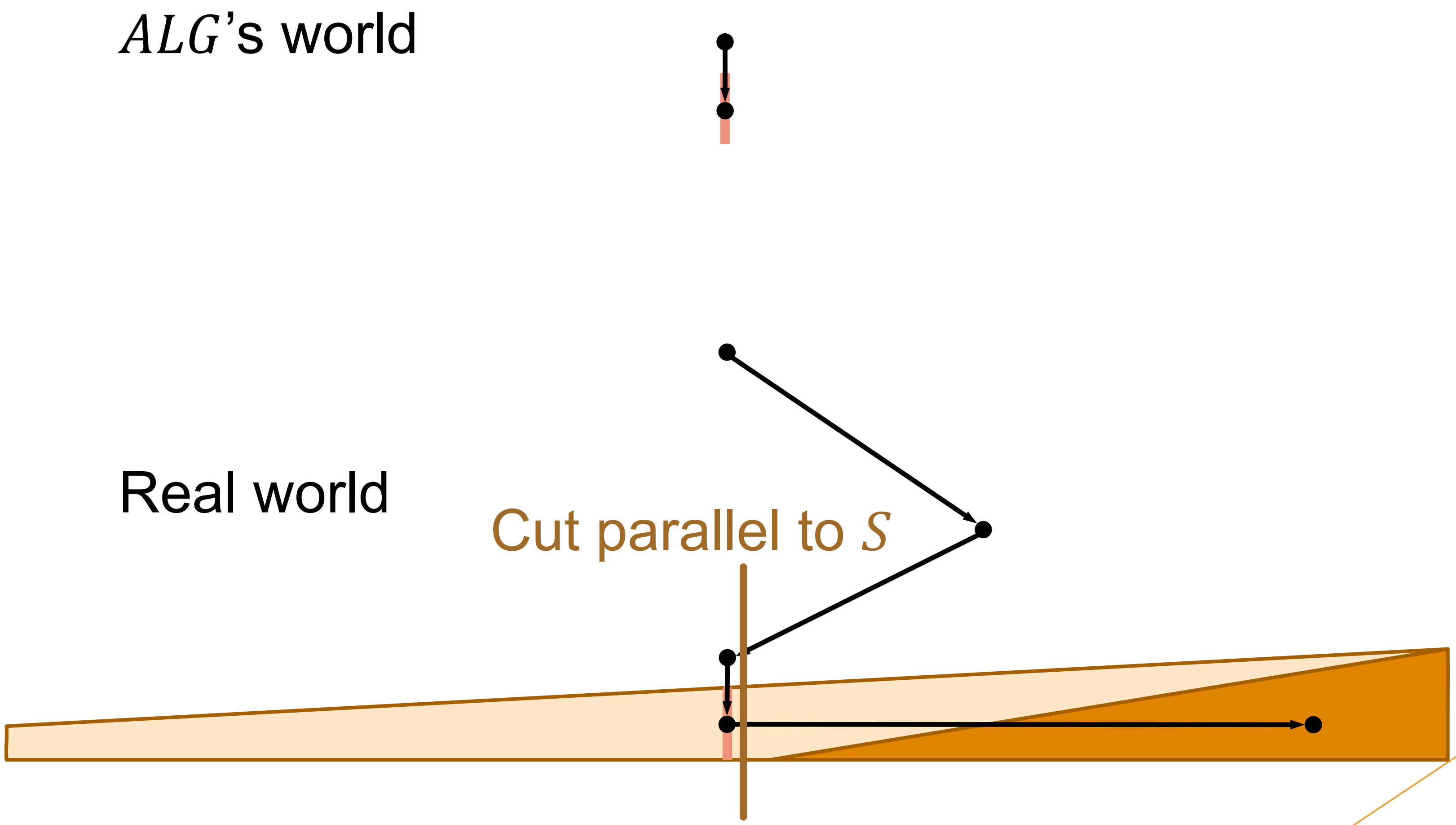


Recursive Centroid (Nested)

[Argue, Bubeck, Cohen, Gupta, Lee 18]

- $O(d \log d)$ competitive
  - Projected volume as potential function

Improvement to $O\left(\sqrt{d \log d}\right)$

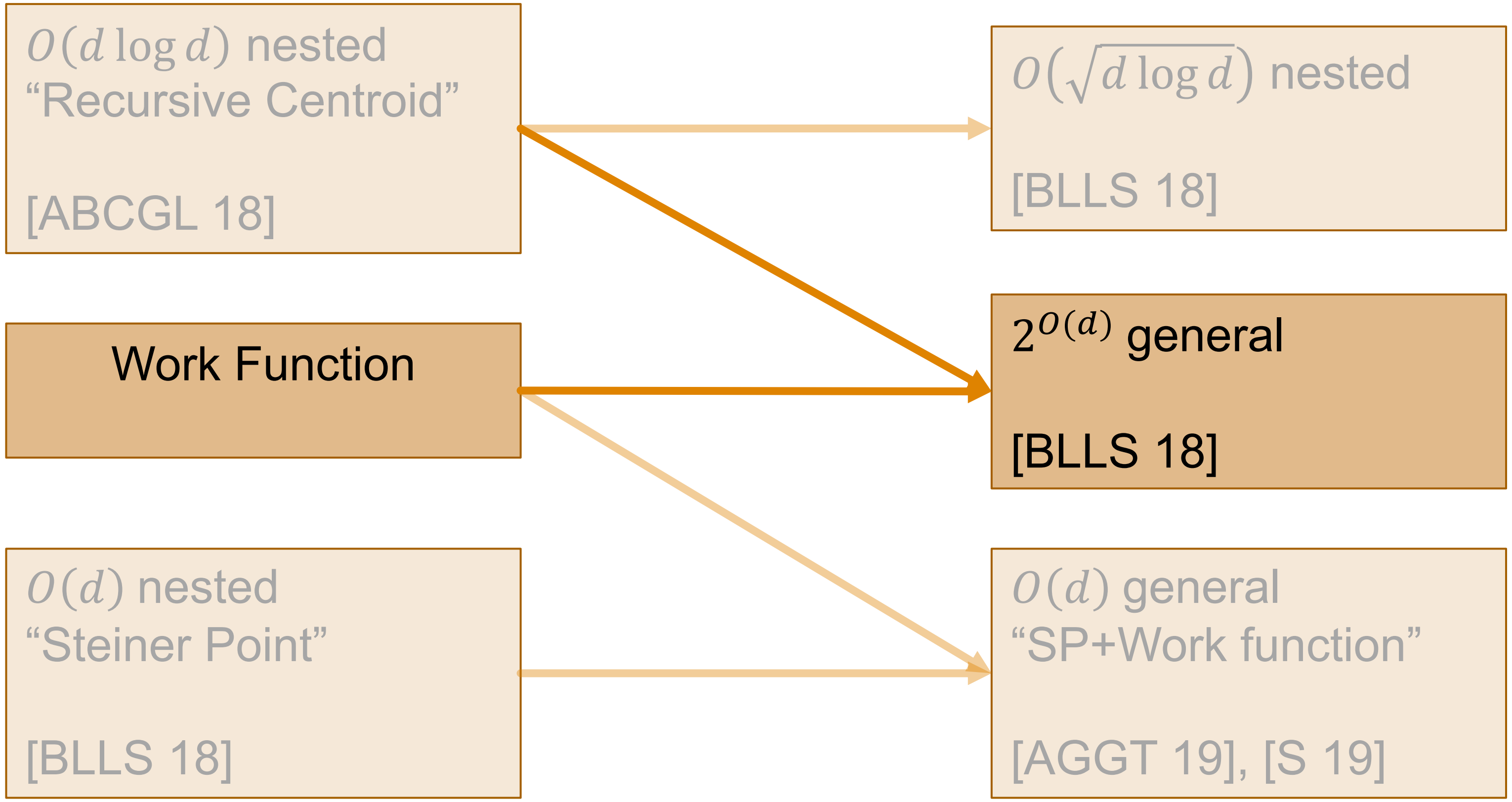
[Bubeck, Lee, Li, Sellke 18]

- Find centroid using Gaussian measure
  - Dampens movement
  - Retains volume-drop

Recap of Centroid

- Naïve centroid fails
- Recursive centroid
  - Move in ‘skinny’ directions
- Optimal for nested (up to log factor)

Part 2 – Work Function



Reduction Framework

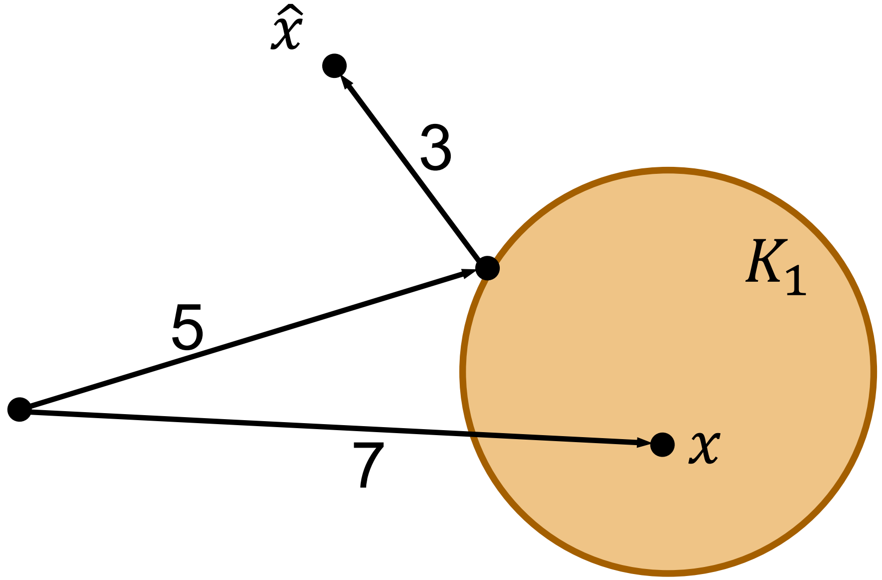
- Given:
  - General instance $K_1, \dots, K_T$
  - $f(d)$ competitive nested ALG
- Goal: Construct $\Omega_1, \dots, \Omega_T$ so that
  - Ω_t convex and $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$
  - $ALG(\Omega_1, \dots, \Omega_t) \leq O\big(f(d)\big) \cdot OPT(K_1, \dots, K_T)$
  - ALG outputs points $x_i \in K_i$

Work Function

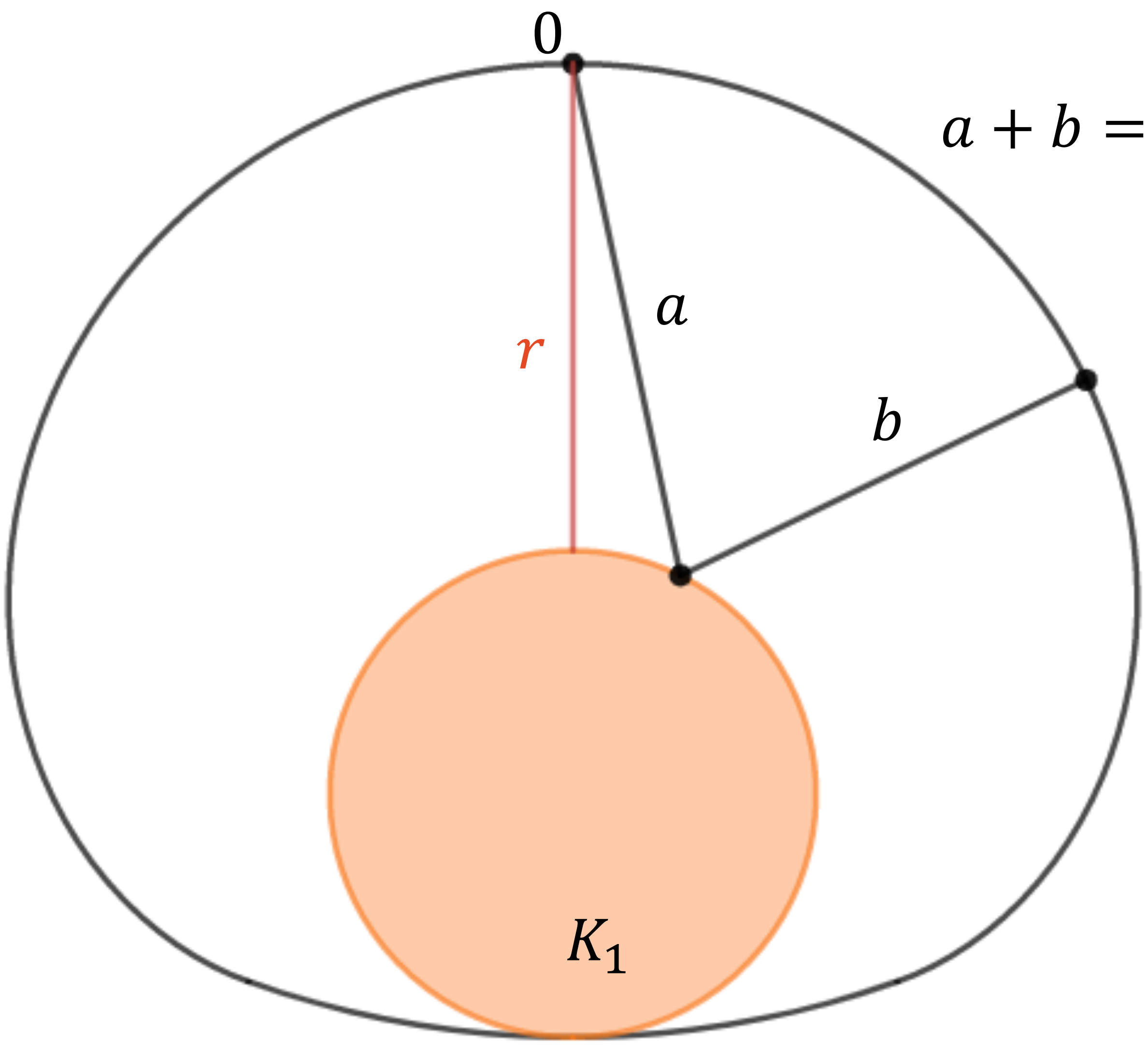
- Central to k-Server, MTS
- $w_t(x) :=$ min cost to satisfy requests $1, \dots, t$ and end at x

$w_1(x) = 7$

$w_1(\hat{x}) = 5 + 3$



Work Function Sublevel Set



$a + b = 2r$

$\Omega_1 = \{x \mid w_1(x) \leq 2r\}$

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  - ALG outputs points $x_i \in K_i$

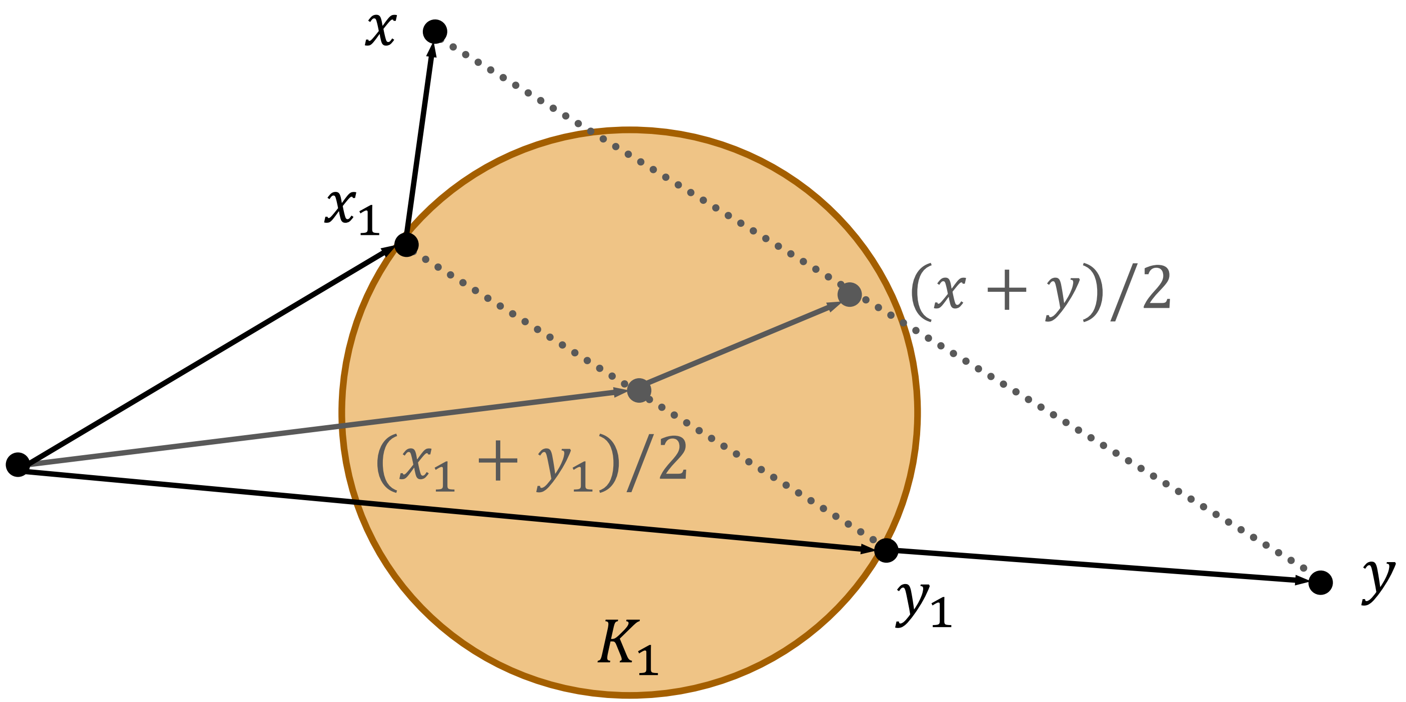
$r \le OPT \le 2r$

Candidate

$\Omega_t = \{x \mid w_t(x) \le 2r\}$

Convexity

w_t is convex $\quad\Longrightarrow\quad$ $\Omega_t = \{x \mid w_t(x) \leq 2r\}$ is convex



Nested

$$w_t(x) \leq w_{t+1}(x) \qquad \Longrightarrow \qquad \{x \mid w_t(x) \leq 2r\} \supseteq \{x \mid w_{t+1}(x) \leq 2r\}$$

$$\Omega_t \supseteq \Omega_{t+1}$$

Cost to satisfy requests $1, \ldots, t+1$ and end at x

Cost to satisfy requests $1, \ldots, t$ and end at x

Reduction Framework

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  - ALG outputs points $x_i \in K_i$

Candidate

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

✓

Cost

- $\{x \mid w_t(x) \leq 2r\} \subseteq B(0, 2r)$
- $\Rightarrow diam\{x \mid w_t(x) \leq 2r\} \leq 4r \leq 4 \cdot OPT$
- $\Rightarrow ALG \leq f(d) \cdot diam\{x \mid w_t(x) \leq 2r\} \leq O\big(f(d)\big) \cdot OPT$

Reduction Framework

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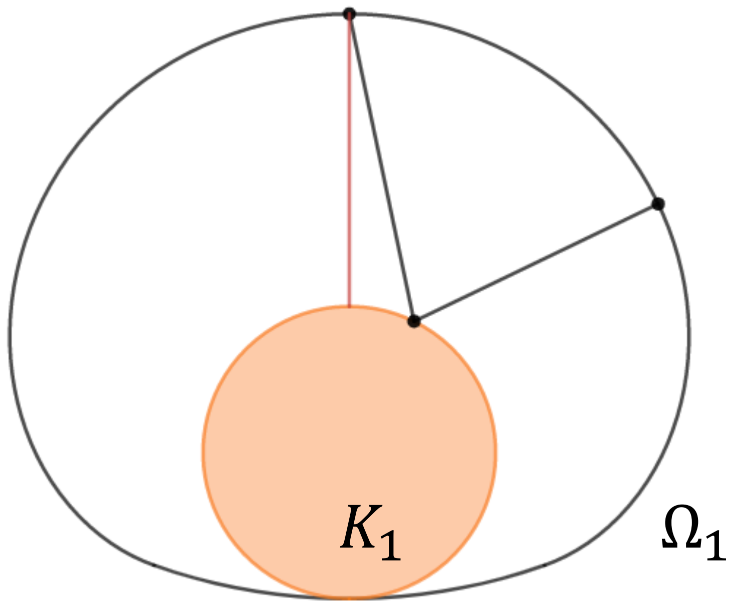
  - ALG outputs points $x_i \in K_i$

Candidate

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

(In)feasibility

- $\Omega_t \nsubseteq K_t$
  - May play infeasible point
  - No black-box reduction
- Fix: project onto K_t each step?
  - Hard to control extra cost
- Challenge: how to reason about Ω_t?



Reduction Framework

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- Goal: Construct $\Omega_1, \dots, \Omega_T$ so that
  - Ω_t convex and $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$ ✓
  - $ALG(\Omega_1, \dots, \Omega_t) \leq O\big(f(d)\big) \cdot OPT(K_1, \dots, K_T)$ ✓
  - ALG outputs points $x_i \in K_i$ ✗

Candidate

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

Recursive Centroid (General)

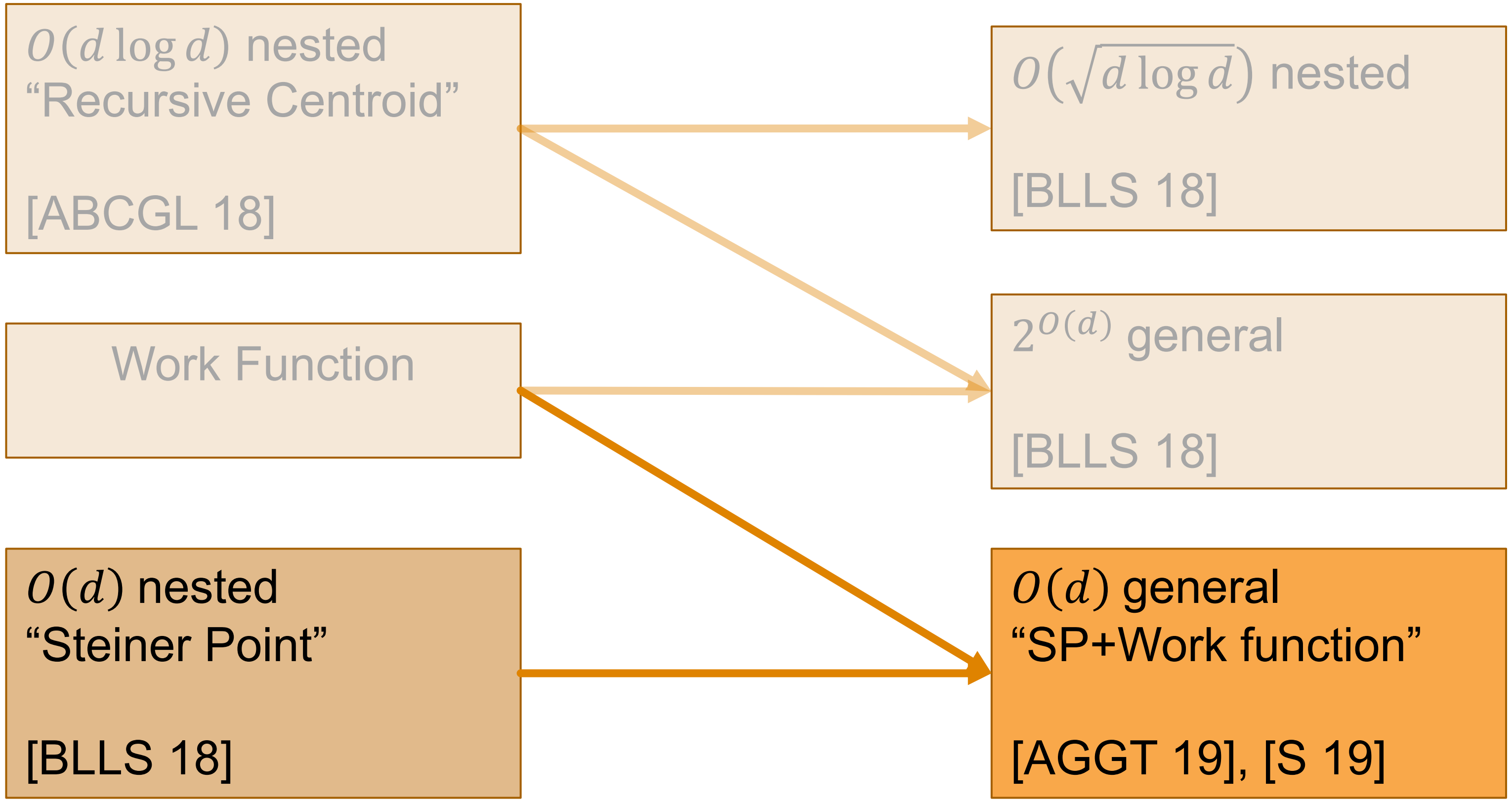
[Bubeck, Lee, Li, Sellke, 18]

- Roughly uses reduction framework
- Many sophisticated work-arounds
- $2^{O(d)}$ competitive

Recap of Work Function Reduction

- Construct nested instance
  - Asymptotically same cost
- May play infeasible point

Part 3 – Steiner Point

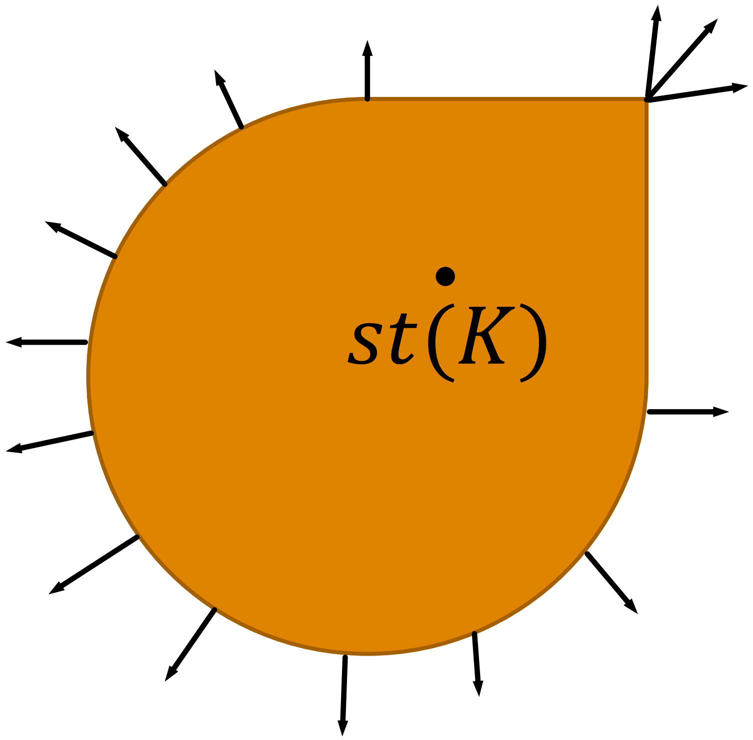


Steiner Point

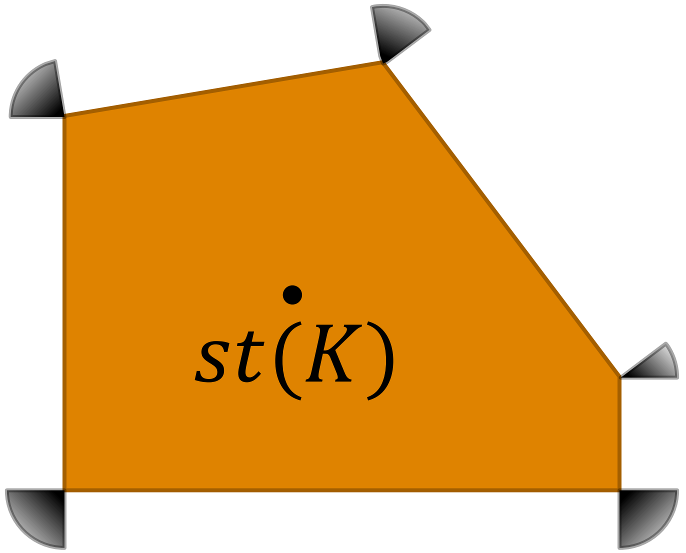
- Alternate “center” of convex body
- Introduced by Steiner in 1826
- Long history in convex geometry
- Lipschitz w.r.t. Hausdorff Distance
  - Natural metric on sets

Steiner Point

- Average of extreme points in all directions



- Average of extreme points weighted by size of normal cone



Steiner Point Definitions

Visually intuitive

$$st(K) = \int_{\|\theta\|=1} \nabla s_K(\theta)\, d\theta$$

$$\nabla s_K(\theta) := \underset{x \in K}{\operatorname{argmax}} \langle \theta, x \rangle$$

$$= d \cdot \int_{\|\theta\|=1} s_K(\theta) \cdot \theta \, d\theta$$

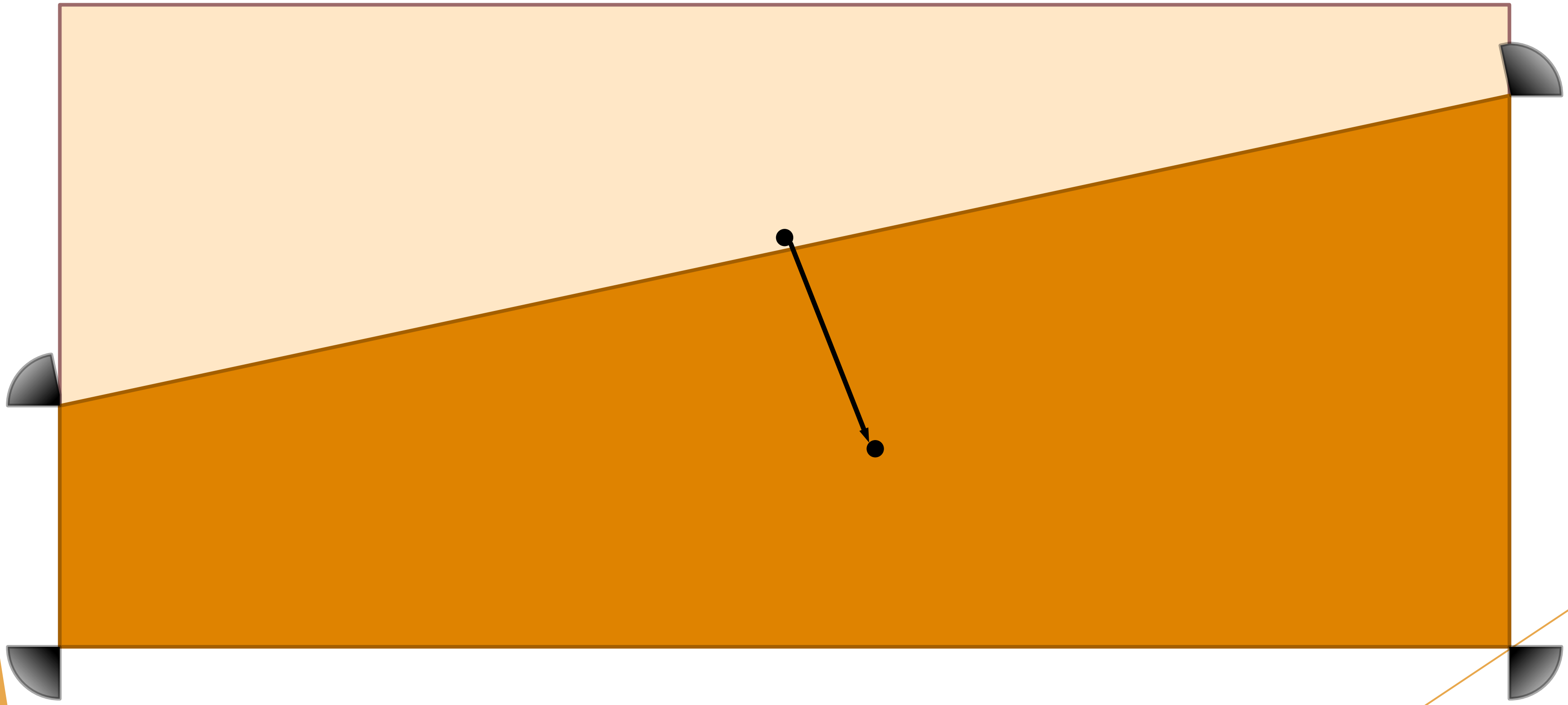
$$s_K(\theta) := \max_{x \in K} \langle \theta, x \rangle$$

Algebraically useful

Steiner Point (Nested)

[Bubeck, Klartag, Lee, Li, Sellke 18]

- $x_t = st(K_t)$



Steiner Point (Nested)

[Bubeck, Klartag, Lee, Li, Sellke 18]

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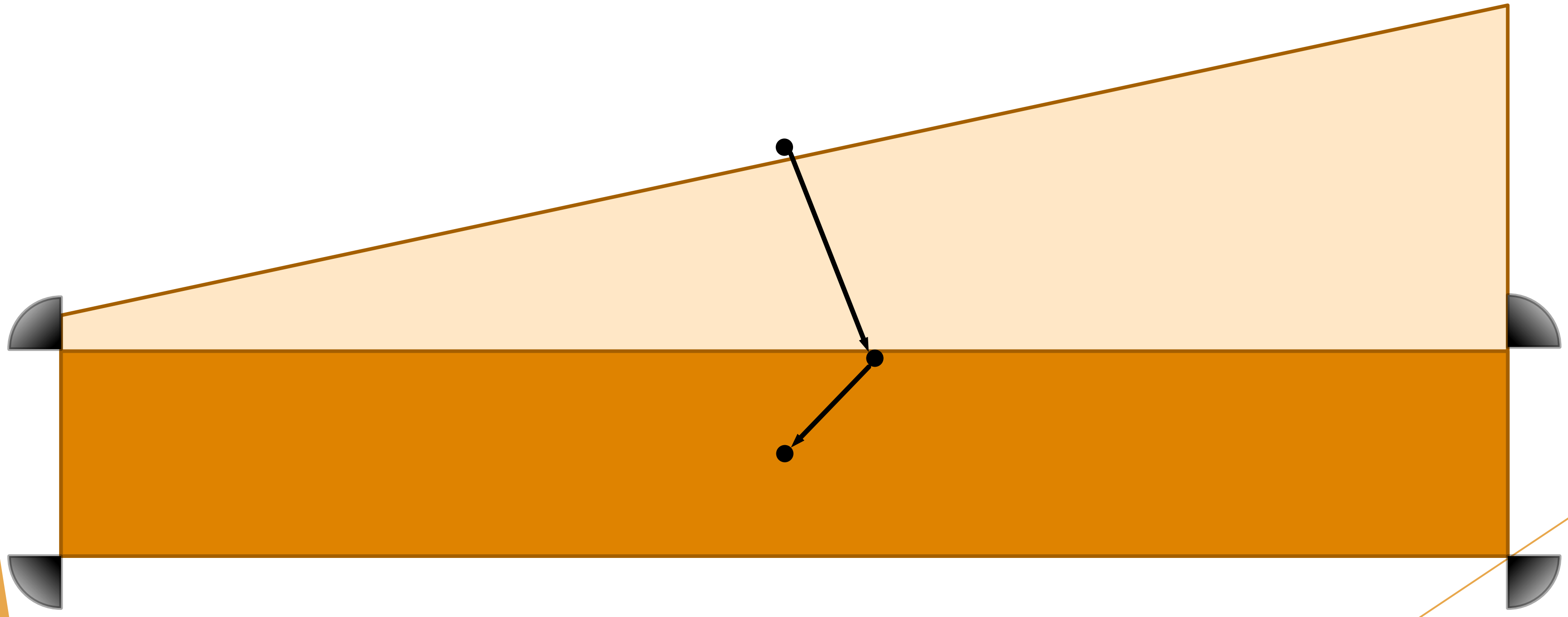

Steiner Point (Nested)

[Bubeck, Klartag, Lee, Li, Sellke 18]

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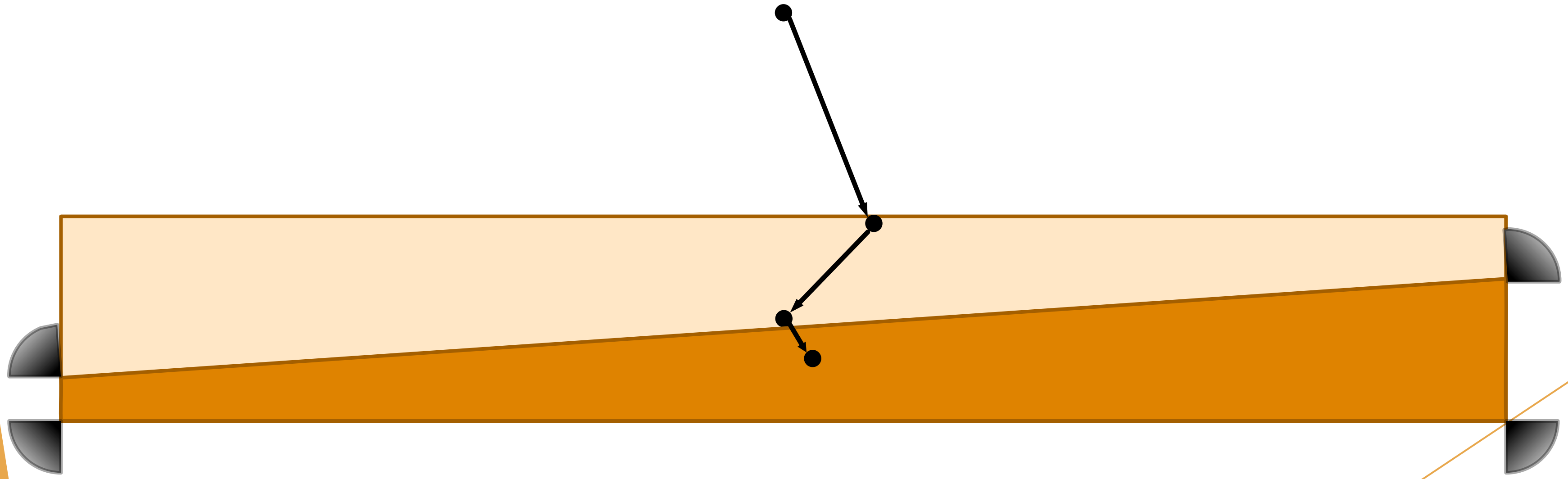
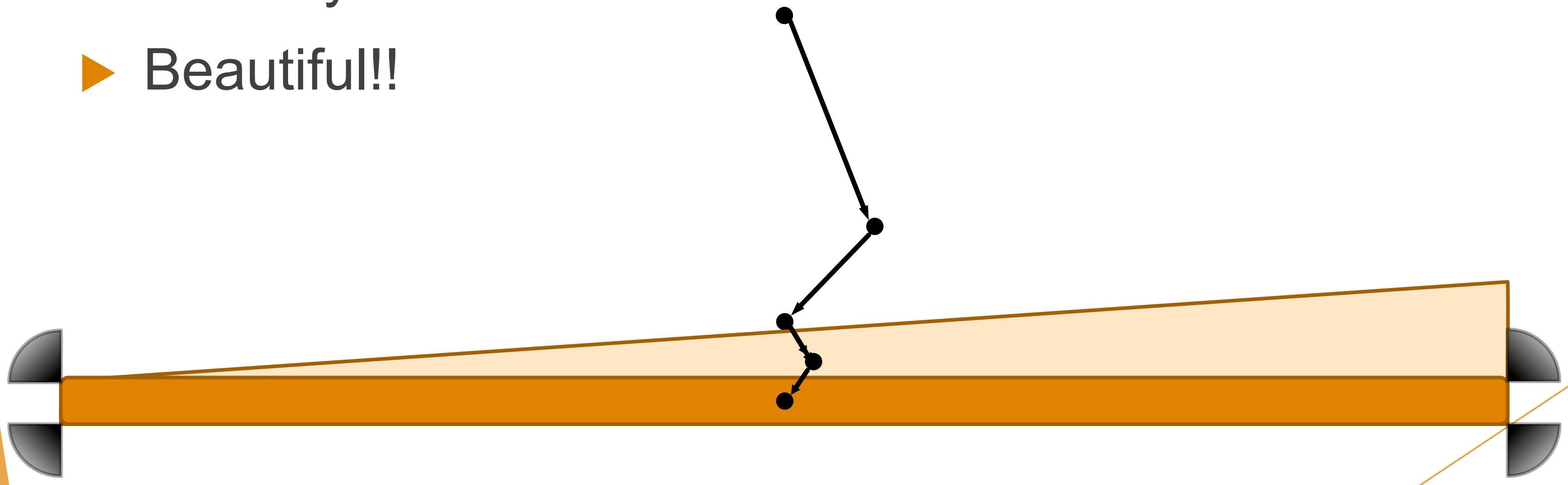


Steiner Point (Nested)

[Bubeck, Klartag, Lee, Li, Sellke 18]

- $x_t = st(K_t)$
- “Smoother version of recursive centroid”
- $O(d)$ competitive
- Memoryless
- Beautiful!!



Analysis

$$ALG = \sum_{i=1}^{T-1} \|st(K_i) - st(K_{i+1})\|$$

$$= \sum_{i=1}^{T-1} \left\| d \cdot \int_{\|\theta\|=1} \left(s_{K_i}(\theta) - s_{K_{i+1}}(\theta)\right) \theta \, d\theta \right\|$$

(Jensen) $$\leq d \cdot \int_{\|\theta\|=1} \left(\sum_{i=1}^{T-1} \left| s_{K_i}(\theta) - s_{K_{i+1}}(\theta) \right| \right) \overset{1}{\cancel{\|\theta\|}} \, d\theta$$

(Nested) $$= d \cdot \int_{\|\theta\|=1} \left(\sum_{i=1}^{T-1} s_{K_i}(\theta) - s_{K_{i+1}}(\theta) \right) d\theta$$

$$= d \cdot \int_{\|\theta\|=1} \left(s_{K_1}(\theta) - s_{K_T}(\theta) \right) d\theta$$

(Bounded) $$\leq d \cdot diam(K_1) \leq O(d) \cdot OPT$$

Steiner Point + Work Function

[Argue, Gupta, Guruganesh, Tang 19]

- Given:
  - General instance $K_1, \dots, K_T$
  - $\boldsymbol{ALG}$ = **Steiner Point**
- Goal: Construct $\Omega_1, \dots, \Omega_T$ so that
  - Ω_t convex and $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$
  - $ALG(\Omega_1, \dots, \Omega_t) \leq O(d) \cdot OPT(K_1, \dots, K_T)$
  - ALG outputs points $x_i \in K_i$

Candidate

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

✓

✓

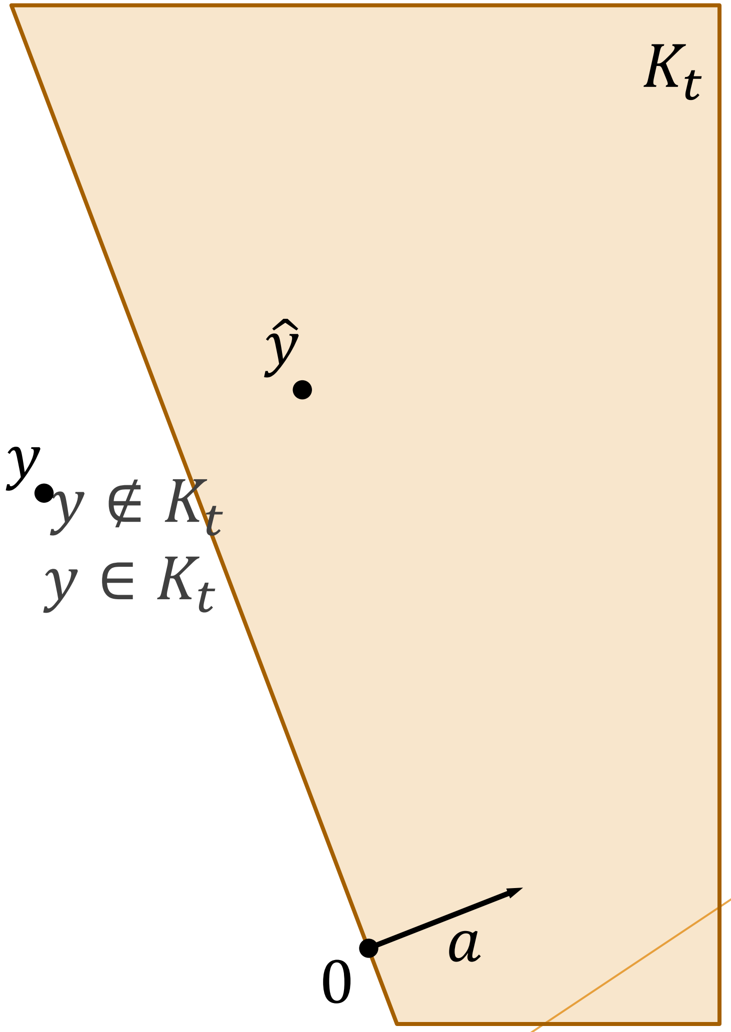
~~✗~~ ✓

Main Theorem: $O(d)$ competitive general algorithm

Proof of Feasibility Lemma

- $K_t = \{\, x \mid \langle a, x \rangle \geq b \,\}$ (w.l.o.g.)
  - Translate so $b = 0$
- Define

$$\hat{y} = \begin{cases} reflect(y) & y \notin K_t \\ y & y \in K_t \end{cases}$$



Goal: $st(\Omega_t) \in K_t$

$\Omega_t = \{x \mid w_t(x) \leq 2r\}$

$st(\Omega_t) = d \cdot \int_{\|\theta\|=1} s_t(\theta) \cdot \theta \, d\theta$

$s_t(\theta) = \max_{x \in \Omega_t} \langle \theta, x \rangle$

Proof of Feasibility Lemma

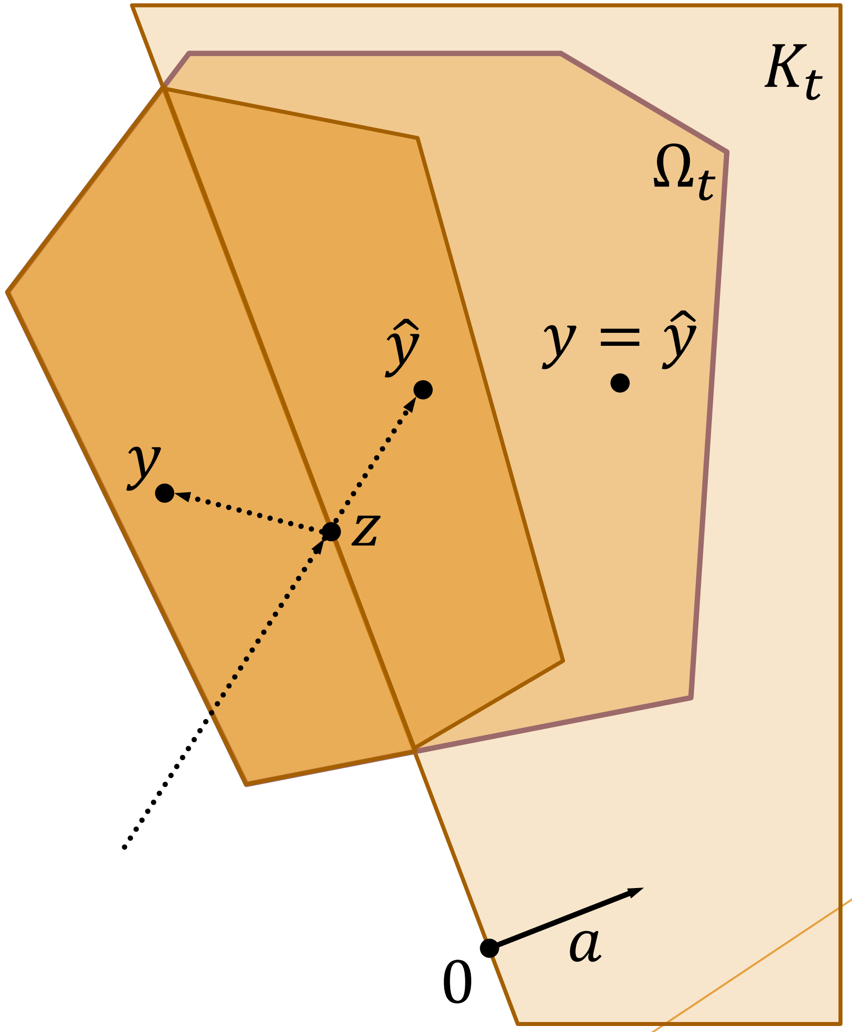
Claim: If $\boldsymbol{y} \in \boldsymbol{\Omega_t}$ then $\hat{\boldsymbol{y}} \in \boldsymbol{\Omega_t}$

If $\langle a, y \rangle \geq 0$ then $\hat{y} = y$

Else, $\langle a, y \rangle < 0$

$$w_t(y) = \min_{z \in K_t} \|y - z\| + w_{t-1}(z)$$

$$\Rightarrow \; w_t(\hat{y}) \leq w_t(y) \leq 2r$$ ☺



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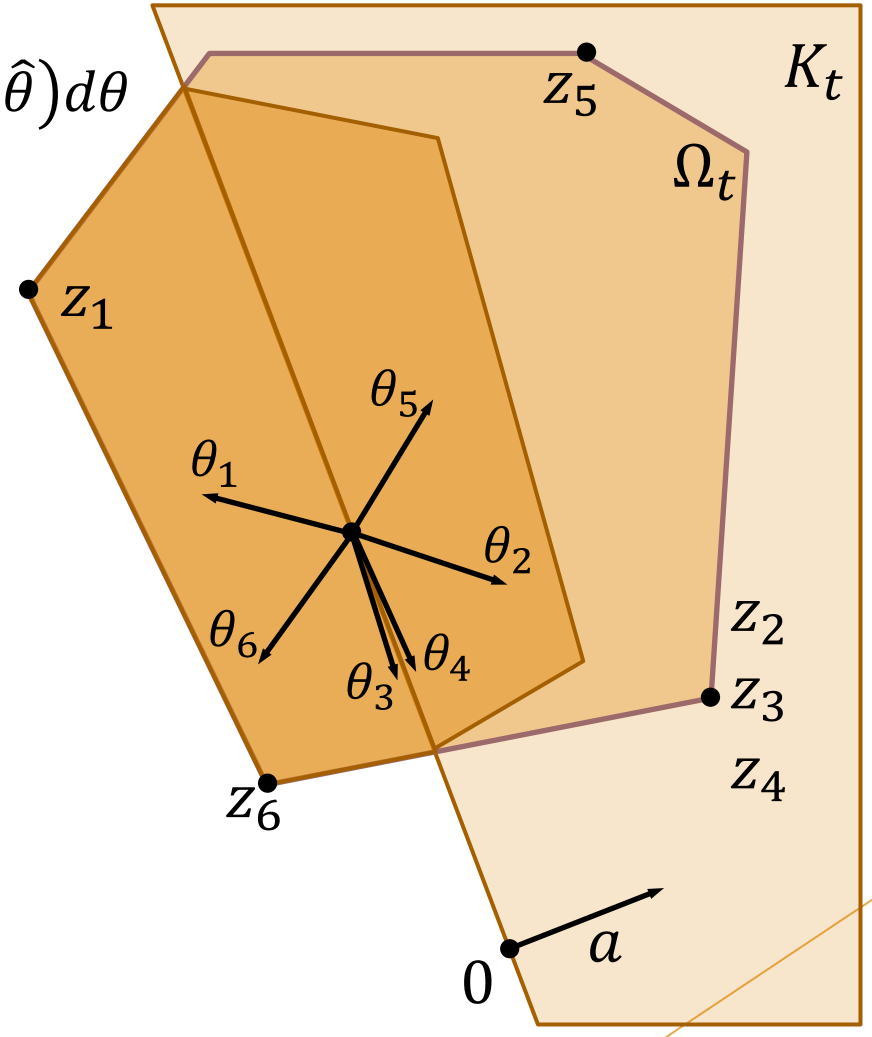
Proof of Feasibility Lemma

$$st(\Omega_t) = d \cdot \iint_{\substack{\|\theta\|=1 \\ \langle a,\theta\rangle < 0}} \big(s_t(\theta) \cdot \theta + s_t(\hat{\theta}) \cdot \hat{\theta}\big) d\theta$$



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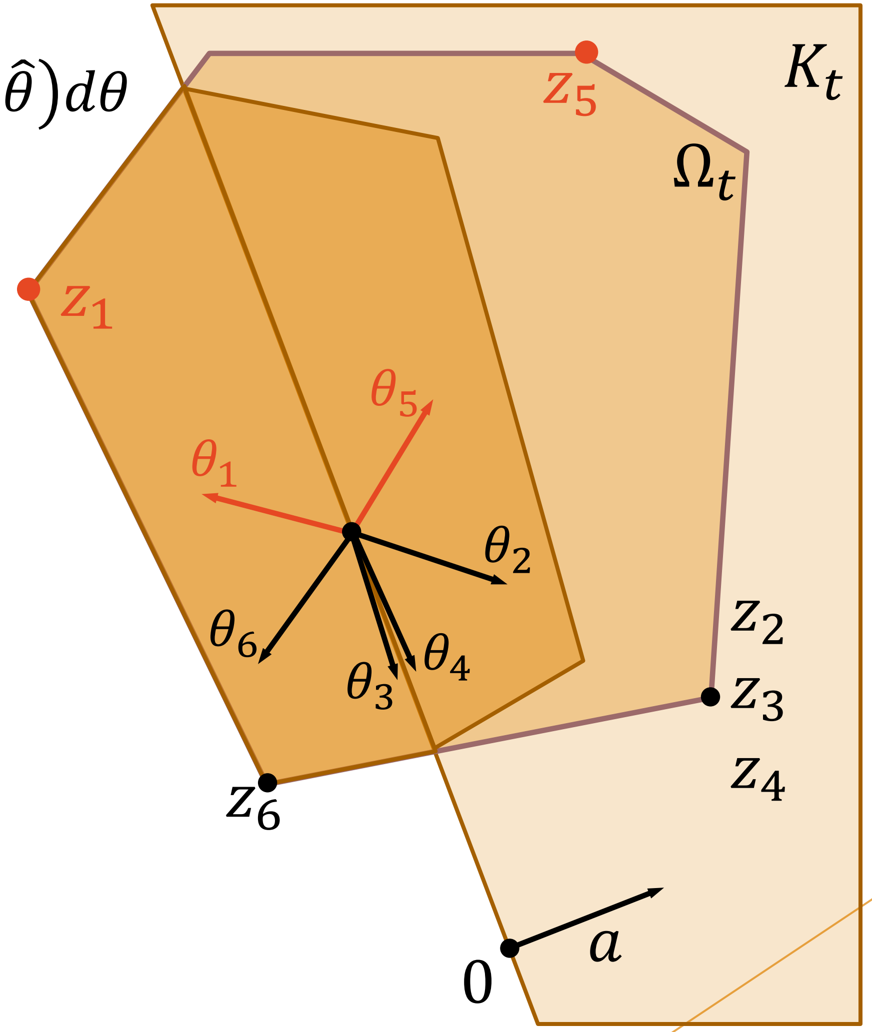
Proof of Feasibility Lemma

$$st(\Omega_t) = d \cdot \int_{\substack{\|\theta\|=1 \\ \langle a,\theta\rangle < 0}} \left(s_t(\theta)\cdot\theta + s_t(\hat{\theta})\cdot\hat{\theta}\right) d\theta$$



Goal: $st(\Omega_t) \in K_t$

$\Omega_t = \{x \mid w_t(x) \le 2r\}$

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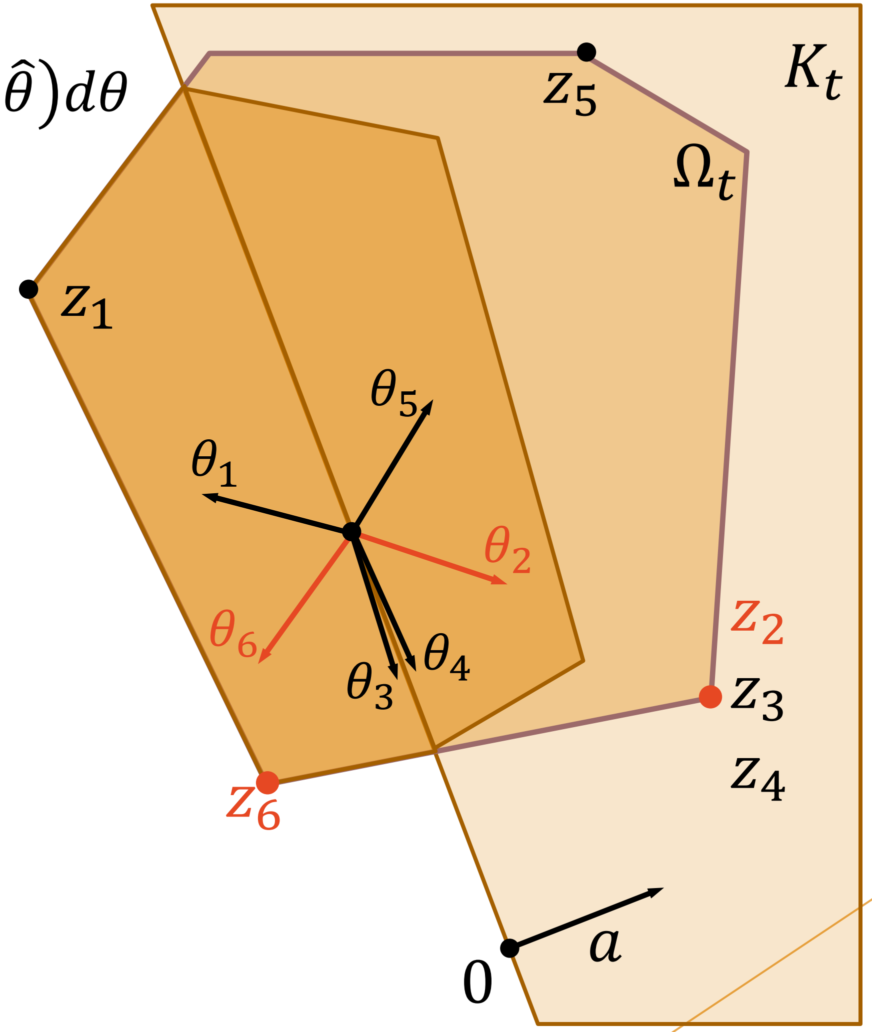

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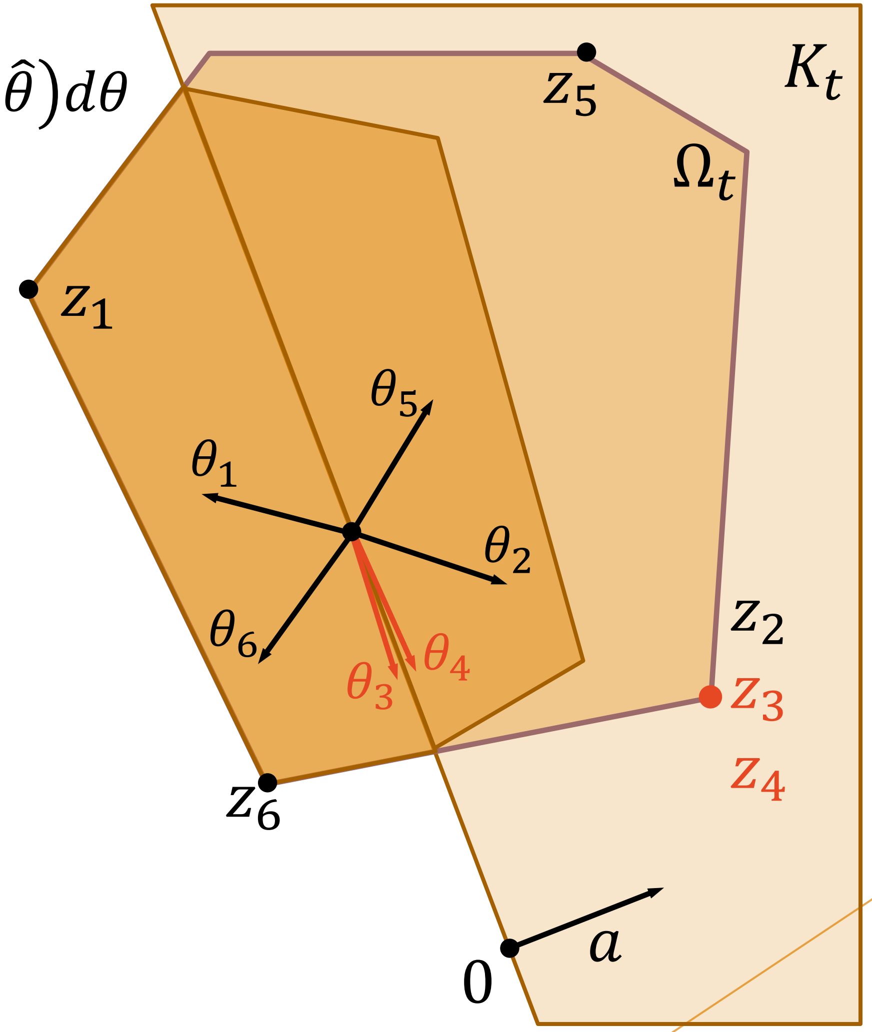

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Proof of Feasibility Lemma

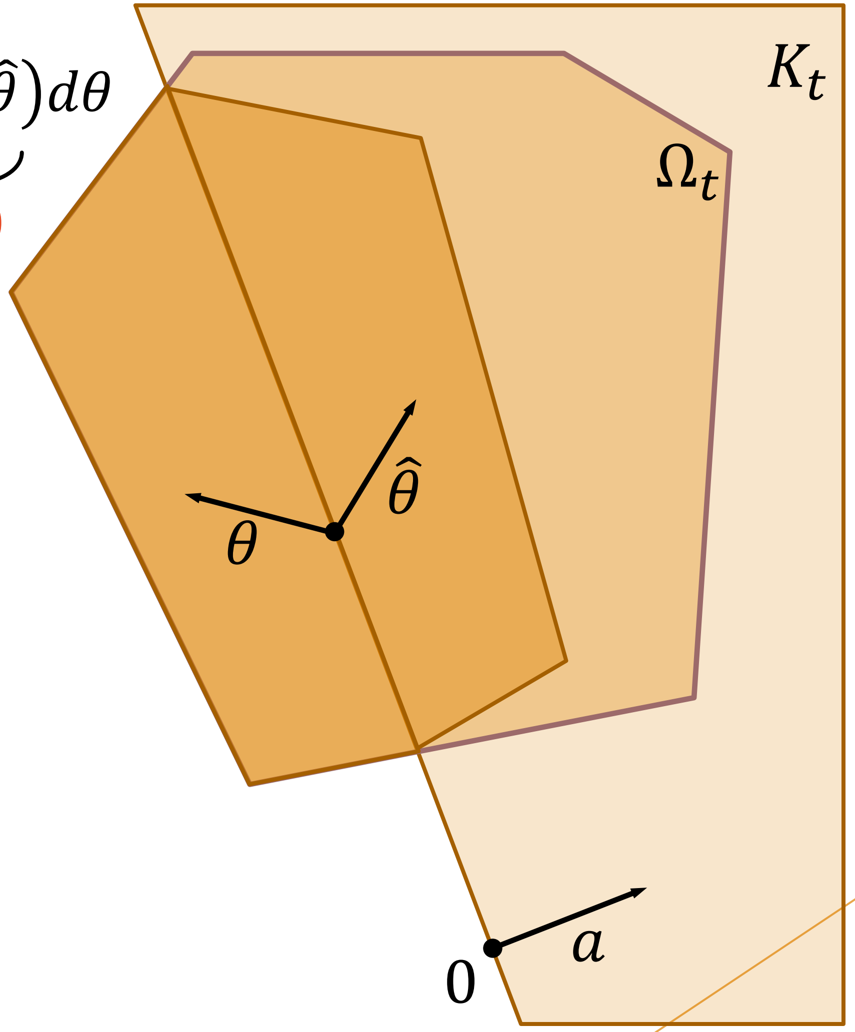
$$st(\Omega_t) = d \cdot \int_{\substack{\|\theta\|=1 \\ \langle a,\theta\rangle < 0}} \underbrace{\left(s_t(\theta)\cdot\theta + s_t(\hat\theta)\cdot\hat\theta\right)}_{\text{To show: } \langle\ \ ,a\rangle \geq 0} d\theta$$

$$0 \leq \langle a, s_t(\theta)\cdot\theta + s_t(\hat\theta)\cdot\hat\theta\rangle$$

$$0 \leq s_t(\theta) \cdot \langle a,\theta\rangle + s_t(\hat\theta)\cdot\langle a,\hat\theta\rangle$$

$$-\langle a,\hat\theta\rangle$$

$$s_t(\theta) \leq s_t(\hat\theta)$$



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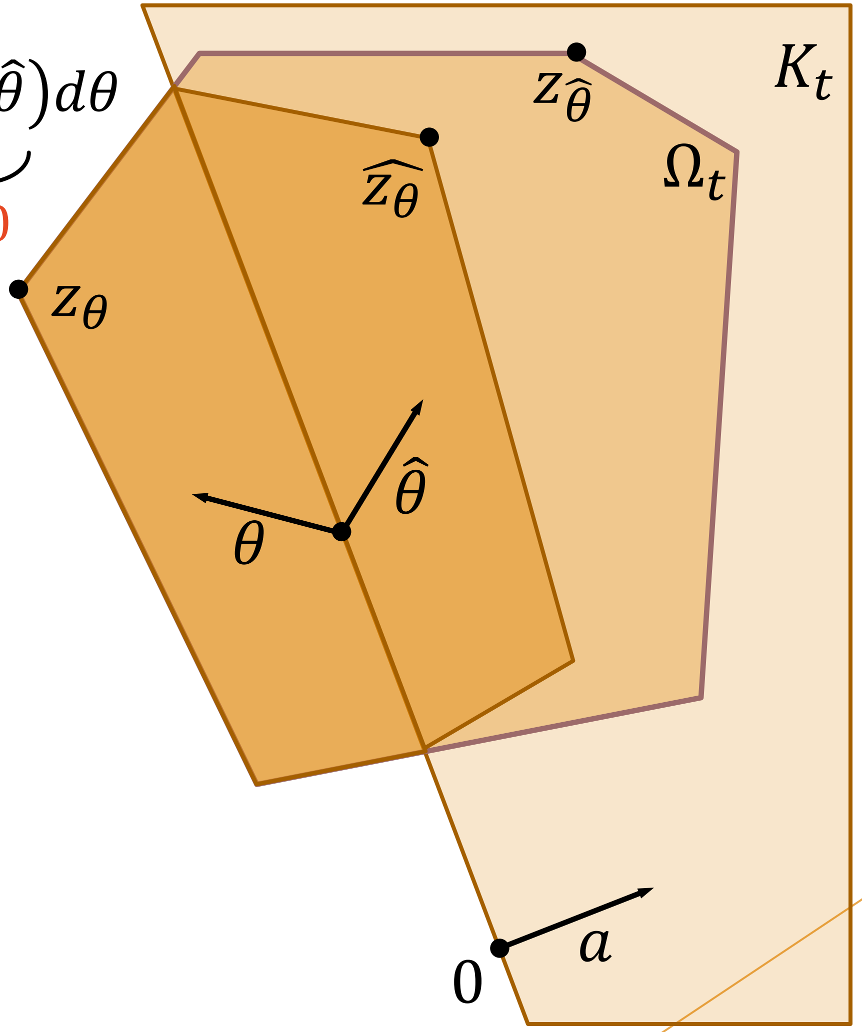
Proof of Feasibility Lemma

$$st(\Omega_t) = d \cdot \int_{\substack{\|\theta\|=1 \\ \langle a,\theta\rangle < 0}} \underbrace{\left(s_t(\theta)\cdot\theta + s_t(\hat{\theta})\cdot\hat{\theta}\right)}_{\text{To show: } \langle\ \ , a\rangle \geq 0} d\theta$$

$$\begin{aligned} s_t(\theta) &= \langle z_\theta, \theta\rangle \\ &= \langle \widehat{z_\theta}, \hat{\theta}\rangle \\ &\leq \langle z_{\hat{\theta}}, \hat{\theta}\rangle \\ &= s_t(\hat{\theta}) \end{aligned}$$

$s_t(\theta) \leq s_t(\hat{\theta})$



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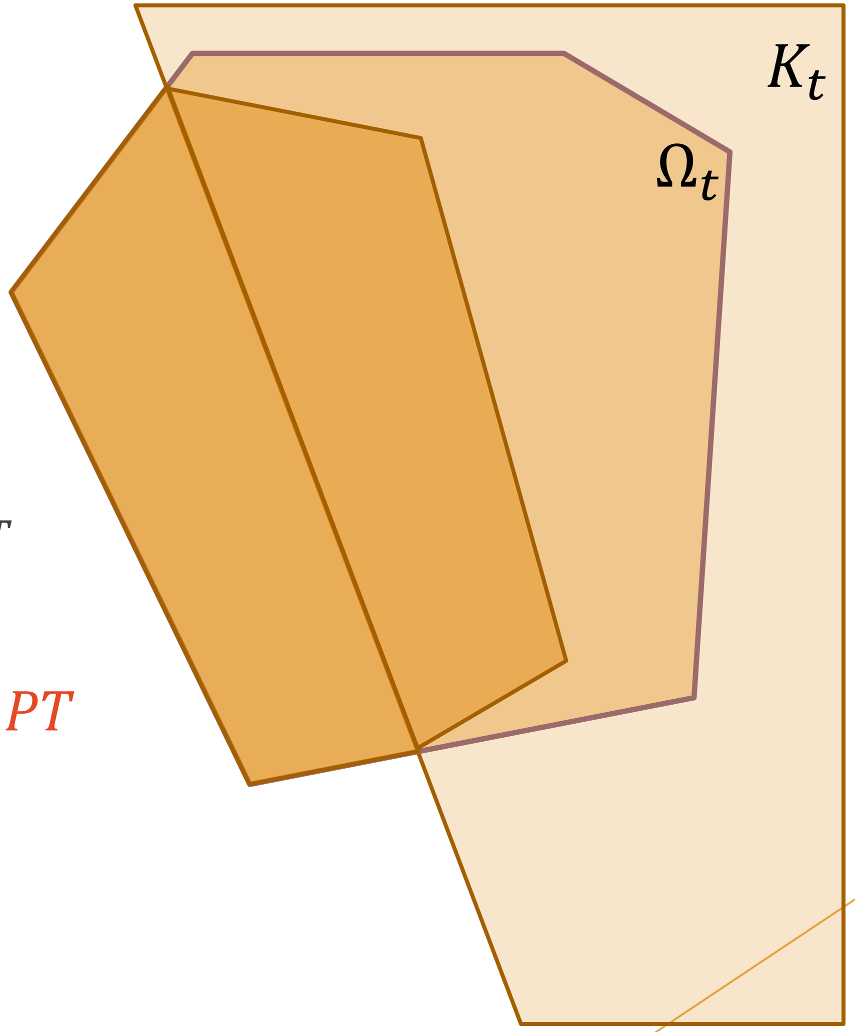
$s_t(\theta) = \max_{x\in\Omega_t} \langle \theta, x\rangle$

Recap of Main Theorem

- Algo: $x_t = st(\Omega_t)$
  - $\Omega_t = \{x \mid w_t(x) \leq 2r\}$
- Proof
  - Ω_t convex, $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_T$
  - Feasibility: $x_t \in K_t$
  - $ALG \leq O(d) \cdot r_{final} \leq O(d) \cdot OPT$



Steiner Point of Work Function

[Sellke 19]

- $st(f) := \int_{\theta \in B^*} \nabla f^*(\theta) d\theta$
  - $f^*(\theta) =$ Fenchel dual
  - $B^* :=$ dual space unit ball
- Algorithm: $x_t = st(w_t)$
  - Analysis similar to nested Steiner point
  - Arbitrary norm

Open questions

- $O(\sqrt{d})$-competitive general chasing
- Applications to related problems
  - Paging
  - MTS
  - k-server

Thank you!

Questions?

References

- "Chasing Convex Bodies with Linear Competitive Ratio"
  Argue, Gupta, Guruganesh, Tang, *SODA '20* *[This talk]*
- "A Nearly-Linear Bound for Chasing Nested Convex Bodies"
  Argue, Bubeck, Cohen, Gupta, Lee, *SODA '19*
- "Chasing Nested Convex Bodies Nearly Optimally,"
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- "Competitively Chasing Convex Bodies"
  Bubeck, Lee, Li, Sellke, *STOC '19*
- "Chasing Convex Bodies and Functions"
  Friedman, Linial, *Discrete and Computational Geometry '93*
- "Chasing Convex Bodies Optimally"
  Sellke, *SODA '20* *[Similar results to this talk]*