21-770 Introduction to Continuum Mechanics
Spring 2010

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April 28, 2010

Contents

0 Introduction

0.1 What is CM?

What is Continuum Mechanics? Essentially, it is a set of *axioms* that extend Newton's Laws for the motion of particles to the behavior of continua. Recall that Newton's Laws for particles are:

- Particles have mass m
- Positions are characterized by position $\underline{x}(t) \in \mathbb{R}^d$ (where $d = 3$, usually) and velocity $\underline{v}(t) = \dot{\underline{x}}(t)$, etc. This is *kinematics*.
- forces $\underline{f}(t)$ act on the particles and $m\ddot{\underline{x}}(t) = \underline{f}(t)$. This is *dynamics*.

0.2 Systems of Particles

Consider a collection of particles with masses m_i and positions $\underline{x}_i(t) \in \mathbb{R}^3$ with forces $\underline{f}_i(t)$ acting on them. We can now write down Newton's Laws for each particle, at least.

$$m_i\ddot{\underline{x}}_i(t) = \underline{f}_i(t) \Rightarrow \left(\sum_i m_i\underline{x}_i\right)^{\cdot\cdot} = \sum_i \underline{f}_i$$

This roughly corresponds to an "integral over a body" when we take the number of particles to infinity. We now define the *center of mass* to be

$$\underline{x}_C(t) = \frac{1}{M}\sum_i m_i\underline{x}_i(t) \qquad , \qquad M := \sum_i m_i$$

which is really a weighted average. Then,

$$M\ddot{\underline{x}}_C(t) = \sum_i \underline{f}_i(t)$$

We now decompose the forces into external and interparticle forces

$$\underline{f}_i = \underline{f}_i^e + \sum_{j\neq i} \underline{f}_{ij}$$

and so

$$M\ddot{\underline{x}}_C = \sum_i \left(\underline{f}_i^e + \sum_{j\neq i} \underline{f}_{ij} \right)$$

Next, we use Newton's Third Law, $\underline{f}_{ij} = -\underline{f}_{ji}$, to eliminate the second term in the sum, yielding

$$M\ddot{\underline{x}}_C = \sum_i \underline{f}_i^e$$

Example 0.1. We may have gravitational forces, where $\underline{f}_i^e = m_i \underline{g}$ so $M\underline{g} = \sum_i \underline{f}_i^e$.

0.2.1 Angular Momentum about a point

Consider a point $\underline{x}_0(t)$. Then

$$L_0(t) = \sum_i (\underline{x}_i(t) - \underline{x}_0(t)) \times m_i \dot{\underline{x}}_i(t)$$

and so

$$\begin{aligned} \dot{L}_0(t) &= -\dot{\underline{x}}_0 \times \left(\sum_i m_i \dot{\underline{x}}_i \right) + \sum_i (\underline{x}_i - \underline{x}_0) \times m_i \ddot{\underline{x}}_i \\ &= -M\dot{\underline{x}}_0 \times \dot{\underline{x}}_C + \sum_i (\underline{x}_i - \underline{x}_0) \times \underline{f}_i \end{aligned}$$

Recall that we write $\underline{f}_i = \underline{f}_i^e + \sum_{i\neq j} \underline{f}_{ij}$, so we can simplify the last term in the line above by utilizing the Newtonian assumptions $\underline{f}_{ij} = -\underline{f}_{ji}$ and $\underline{f}_{ij} \| \underline{x}_i - \underline{x}_j$. This allows us to cancel many terms and conclude

$$\dot{L}_0(t) = -M\dot{\underline{x}}_0 \times \dot{\underline{x}}_C + \sum_i (\underline{x}_i - \underline{x}_0) \times \underline{f}_i^e$$

We can now make convenient and natural choices for $\underline{x}_0$, namely either some fixed $\underline{x}_0 \in \mathbb{R}^3$ independent of t or $\underline{x}_0 = \underline{x}_C$. In the second case, we would just have the second term remaining, since $\dot{\underline{x}}_C \times \dot{\underline{x}}_C = \underline{0}$.

Example 0.2. Uf $\underline{f}_i^e = m_i \underline{g}$ (gravity), then

$$M(\underline{x}_i - \underline{x}_0) \times \underline{g} = \sum_i (\underline{x}_i - \underline{x}_0) \times \underline{f}_i^e$$

The formula $M\ddot{x}_C = F = \sum_i \underline{f}_i^e$ and $\dot{L}_0 = \sum_i (\underline{x}_i - \underline{x}_0) \times \underline{f}_i^e$ gives a system of 6 ODEs for certain averages of a system of particles.

0.2.2 Rigid Motions

Consider motions of particles where $|\underline{x}_i(t) - \underline{x}_j(t)| = |P_i - P_j|$, with $P_i = \underline{x}_i(0)$.

Theorem 0.3. *Assume $\mathcal{X} : \mathbb{R}^d : \mathbb{R}^d$ satisfies $|\mathcal{X}(\underline{p}) - \mathcal{X}(\underline{q})| = |\underline{p} - \underline{q}|$ for all $\underline{p}, \underline{q} \in \mathbb{R}^d$. Then $\exists \underline{x}_0 \in \mathbb{R}^d$ and $Q \in \mathbb{R}^{d\times d}$ an orthogonal matrix (i.e. $Q^T Q = I$) such that $\mathcal{X}(\underline{p}) = \underline{x}_0 + Q\underline{p}$.*

The matrix Q represents a rotation. If a system is undergoing a rigid motion, then we know

$$\underline{x}_i(t) = \underline{x}_0(t) + Q(t)\underline{p}_i$$

and notice, then, that

$$\underline{\dot{x}}_i = \underline{\dot{x}}_0 + \dot{Q}Q^T Q\underline{p}_i = \underline{\dot{x}}_0 + \dot{Q}Q^T\left(\underline{x}_i - \underline{x}_0\right)$$

Also, note that $\dot{Q}Q^T + Q\dot{Q}^T = 0$, so $\dot{Q}Q^T = -Q\dot{Q}^T = -(\dot{Q}Q^T)^T$, i.e. $\dot{Q}Q^T$ is *skew symmetric*. Let $W := \dot{Q}Q^T$. Then, $W\underline{a} = \underline{\omega} \times \underline{a}$, where

$$W = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Thus, $\underline{\dot{x}}_i(t) = \underline{\dot{x}}_0(t) + \underline{\omega}(t) \times (\underline{x}_i(t) - \underline{x}_0(t))$ and $\dot{Q} = W(\underline{\omega})Q$.

Note that the space of orthogonal matrices is a set of 2 3-manifolds in the 9-D space $\mathbb{R}^{3\times 3}$ (we have 2 since $\det W = \pm 1$).

If we selct the origin at $t = 0$ to be the origin, i.e. $\frac{1}{M}\sum_i m_i \varphi_i = \underline{0}$, then

$$\begin{aligned} \underline{x}_c(t) &= \frac{1}{M}\sum_i m_i \underline{x}_i(t) = \frac{1}{M}\sum_i m_i \left(\underline{x}_0(t) + Q\underline{p}_i\right) \\ &= \underline{x}_0(t) + Q\left(\frac{1}{M}\sum_i m_i \underline{p}_i\right) = \underline{x}_0(t) \end{aligned}$$

i.e. with this choice of origin at $t = 0$, we have

$$\underline{x}_i(t) = \underline{x}_C(t) + Q(t)\underline{p}_i$$

For angular momentum, we have

$$L_C = \sum_i \left(\underline{x}_i - \underline{x}_c\right) \times m_i \underline{\dot{x}}_i$$

We now use $\underline{x} = \underline{x}_C + Q\underline{p}$ and $\underline{\dot{x}} = \underline{x}_C + \omega \times \left(\underline{x}_i - \underline{x}_C\right)$ and the identity

$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ to write

$$\begin{aligned}
L_C &= \sum_i Q\underline{p}_i \times m_i \left(\underline{\dot{x}}_C + \omega \times Q\underline{p}_i\right) \\
&= Q \sum_i m_i \underline{p}_i \times \underline{\dot{x}}_C + \sum_i m_i Q\underline{p}_i \times (\omega \times Q\underline{p}_i) \\
&= \sum_i m_i \left(|Q\underline{p}_i|^2 \omega - (Q\underline{p}_i \cdot \omega) Q\underline{p}_i\right) \\
&= \sum_i m_i \left(|\underline{p}_i|^2 QQ^T \omega - Q(\underline{p}_i \otimes \underline{p}_i) Q^T \omega\right) \\
&= Q \sum_i m_i \left(|\underline{p}_i|^2 I - \underline{p}_i \otimes \underline{p}_i\right) Q^T =: QJ_0Q^T
\end{aligned}$$

which is *independent* of t. We will use the notation J_0 to denote the sum term above. We can now write $L_C = J(t)\omega(t)$ where

$$J(t) = Q(t)J_0Q^T(t)$$

Let's summarize where we stand thus far:

$$\begin{aligned}
&M\underline{\ddot{x}}_C = F = \sum_i f_i^e \\
&(J\omega)^{\cdot} = N_C = \sum_i (\underline{x}_i - \underline{x}_C) \times f_i^e \\
&\dot{Q} = W(\omega)Q \qquad \text{where } W(\omega)a = \omega \times a
\end{aligned}$$

We can simplify the expression $(J\omega)^{\cdot}$ by using the relation $W = \dot{Q}Q^T = Q\dot{Q}^T$ and write

$$\begin{aligned}
(J\omega)^{\cdot} &= \left(Q^T J_0 Q\omega\right)^{\cdot} = J\dot{\omega} + \dot{Q}(Q^TQ)J_0Q^T\omega + QJ_0(Q^TQ)\dot{Q}^T\omega \\
&= J\dot{\omega} + WJ\omega + JW\omega = J\dot{\omega} + \omega \times J\omega + J\omega \times \omega \\
&= J\dot{\omega} + \omega \times J\omega = N_C
\end{aligned}$$

0.3 Rigid Bodies

We have some set $B_r \subset \mathbb{R}^3$ and a map $\underline{x} = \underline{x}_C(t) + Q(t)\underline{p}$ to a new set $B(t)$. We assume the following:

1. The continuum B_r can be "approximated" by a collection of particles $\{m_i\}_{i=1}^N$ with initial positions $\{\underline{p}_i\}_{i=1}^N$ undergoing a rigid motion.

2. If $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function, then

$$\lim_{N\to\infty} \sum_{i=1}^N \varphi(\underline{p}_i) m_i = \int_{B_r} \varphi(\underline{p}) \rho_r(\underline{p})\, d\underline{p}$$

   where $\rho_r : \mathbb{R}^3 \to [0, \infty)$ is the mass density of B_r.

Observe that

$$\underline{x}_C = \frac{1}{M}\sum_i m_i \underline{x}_i = \frac{1}{M}\sum_i m_i \left(\underline{x}_0(t) + Q(t)\underline{p}_i\right)$$
$$= \underline{x}_0(t) + \frac{1}{M}Q(t)\left(\sum_i m_i \underline{p}_i\right)$$

and so

$$\underline{x}_0(t) + Q(t)\frac{1}{M}\int_{B_r} \underline{p}\rho_r(\underline{p})\, d\underline{p}$$

We select the origin in the configuration so that $\int \underline{p}\rho(\underline{p})\, d\underline{p} = 0$. We have

$$M\ddot{\underline{x}}_C = F = \sum_i f_i = \sum_i \left(\frac{f_i}{m_i}\right) m_i$$

and we are thinking of $m_i \to 0$. If, for example, $\frac{f_i}{m_i} = \underline{g}(\underline{x}_C(t) + Q\underline{p}_i)$ with $\underline{g}$ the gravitational force at $\underline{x}_i$, then we have

$$M\ddot{\underline{x}}_C = \int_{B_r} f(\underline{x}_C + Q\underline{p})\rho_r(\underline{p})\, d\underline{p}$$

Note: under the change of variables $\underline{x} = \underline{x}_C + Q\underline{p}$, so $d\underline{x} = \det Q\, d\underline{p} = d\underline{p}$, then

$$M\ddot{\underline{x}}_C = \int_{B(t)} \underline{g}(\underline{x}, t)\rho_r\left(Q^T(\underline{x} - \underline{x}_C)\right)\, d\underline{x}$$

This shows a fundamental dichotomy of knowledge; when we integrate $\int f(\cdot\cdot)\rho(\cdot)$ versus $\int f(\cdot)\rho(\cdot\cdot)$, we know either the force or the point but not both. So as we let $m_i \to 0$, we have

$$J_0 = \sum_i m_i \left(|\underline{p}_i|^2 I - \underline{p}_i \otimes \underline{p}_i\right) \to \int_{B_r} \left(|\underline{p}|^2 I - \underline{p} \otimes \underline{p}\right)\rho_r(\underline{p})\, d\underline{p}$$

where we think of $\rho_r(\underline{p})\, d\underline{p}$ as a measure.

1 Balance Laws

1.1 Balance of Mass

Note that we don't discuss the "conservation" of mass. The content here is found in Chapter 3 of Gurtin's book.

Assumptions:

- We are given a **reference configuration** of a body $\mathcal{B}_r \subseteq \mathbb{R}^d$, where the set $\mathcal{B}_r$ is *measurable*.
- **Kinematics**. We have a *measurable* map $\mathcal{X} : \mathcal{B}_r \to \mathbb{R}^d$.

- **Mass density**. The *measurable* function $\rho_r : \mathcal{B}_r \to [0, \infty)$ represents the density in the reference configuration

In some sense, this set of assumptions is *minimal*. We don't require everything to be continuous, say, but *measurability* is mathematically essential.

Definition 1.1. *The mass density $\rho : \mathbb{R}^d \to [0, \infty)$ is the function characterized by*

$$\int_{\mathbb{R}^d} \varphi(x)\rho(x)\, dx = \int_{\mathcal{B}_r} \varphi\left(\mathcal{X}(\underline{p})\right) \rho_r\left(\underline{p}\right)\, d\underline{p} \quad \forall \varphi \in C_c(\mathbb{R}^d)$$

where

$$C_c(\mathbb{R}^d) = \{\varphi : \mathbb{R}^d \to \mathbb{R} \textit{ with compact support}\}$$

This is roughly akin to the "push-forward" of a measure.

Example 1.2. If $A_1, A_2 \subset \mathcal{B}_r$ both get mapped into a set $A \subset \mathcal{B}$, then we can take $\varphi(x) = \chi_A(x)$, the characteristic function of the set A, and find that

$$\int_{\mathbb{R}^d} \varphi\rho\, dx = \int_A \rho\, dx \Rightarrow \int_{\mathcal{B}_r} (\varphi \circ \mathcal{X})\rho_r\, d\underline{p} = \int_{A_1 \cup A_2} \rho_r\, d\underline{p}$$

so $\rho = 2$.

Remark 1.3. If $\mathcal{B} = \mathcal{X}(\mathcal{B}_r)$ then ρ vanishes outside $\mathcal{B}$; i.e. $\text{supp}(\rho) \subseteq \overline{\mathcal{B}}$.

Classical statements: Assume $\mathcal{X} : \mathcal{B}_r \to \mathbb{R}^d$ is a diffeomorphism onto its range $\mathcal{B} = \mathcal{X}(\mathcal{B}_r)$. Under the change of variables $x \in \mathcal{X}(\underline{p})$ we have

$$dx = \det\left[\frac{\partial x}{\partial p}\right] dp \qquad \text{where } \left[\frac{\partial x}{\partial p}\right]_{i\alpha} = \frac{\partial x_i}{\partial p_\alpha}$$

Standard notation: We write $F = \left[\frac{\partial x}{\partial p}\right]$ to be the Jacobian of the change of variables; it is also called the *deformation gradient*. Then the balance of mass says

$$\int_{\mathbb{R}^d} \varphi(x)\rho(x)\, dx = \int_{\mathcal{B}_r} (\varphi \circ \mathcal{X})(\rho \circ \mathcal{X}) \det(F)\, d\underline{p} = \int_{\mathcal{B}_r} (\varphi \circ \mathcal{X})\rho_r\, d\underline{p} \quad \forall \varphi \in C_c(\mathbb{R}^d)$$

Localization (as in the method of Calculus of Variations) yields

$$(\rho \circ \mathcal{X}) \det(F) = \rho_r \Rightarrow \rho(x) = \frac{\rho_r(\underline{p})}{\det(F(\underline{p}))}$$

where $x = \mathcal{X}(\underline{p})$. This is useful for solid mechanics when we want to compute $x = \mathcal{X}(\underline{p})$.

1.1.1 Calculus

Chain rule. Consider a time dependent motion $\mathcal{X}(t,\cdot) : \mathcal{B}_r \to \mathcal{B}(t) \subseteq \mathbb{R}^d$. Consider the change of variables $x = \mathcal{X}(t,\underline{p})$. Given $\varphi_r(t,\cdot) : \mathcal{B}_r \to \mathbb{R}$, define $\varphi(t,\cdot) : \mathcal{B} \to \mathbb{R}$ by $\varphi(t,x) = \varphi_r(t,\underline{p})$. Then,

$$\frac{\partial \varphi}{\partial t} \restriction_{\underline{p}} = \frac{\partial \varphi}{\partial t} + \sum_i \frac{\partial \varphi}{\partial x_i}\frac{\partial x_i}{\partial t} \restriction_{\underline{p}} = \varphi_t + \underline{v}\cdot\nabla\varphi$$

where $\restriction_{\underline{p}}$ indicates we are keeping $\underline{p}$ constant, and $\underline{v}(t,x) = \dot{\underline{x}}(t,\underline{p})$ and $\dot{\underline{x}} = \frac{\partial x}{\partial t} \restriction_{\underline{p}}$.

Definition 1.4. *The* convective derivative *of* $\varphi(t,\cdot) : \mathcal{B} \to \mathbb{R}$ *is* $\dot{\varphi} = \varphi_t + \underline{v}\cdot\nabla\varphi$.

Derivative of the Jacobian. We write

$$\begin{aligned}\dot{F}_{i\alpha} &= \frac{\partial}{\partial t} \restriction_{\underline{p}} F(t,\underline{p})_{i\alpha} = \frac{\partial}{\partial t}\frac{\partial x_i}{\partial p_\alpha}(t,\underline{p}) = \frac{\partial}{\partial p_\alpha}\frac{\partial x_i}{\partial t} = \frac{\partial \dot{x}_i}{\partial p_\alpha}\\ &= \sum_j \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial p_\alpha} = \sum_j \frac{\partial v_i}{\partial x_j} F_{j\alpha}\end{aligned}$$

and notice that the last expression above is a matrix product. We write

$$\dot{F} = (\nabla\underline{v})F \qquad \text{where } (\nabla\underline{v})_{ij} = \frac{\partial v_i}{\partial x_j}$$

If we write $F = F(t,x)$ then the equation becomes

$$(F_{i\alpha})_t + \underline{v}\cdot\nabla F_{i\alpha} = (\nabla v F)_{i\alpha}$$

and we write

$$F_t + (\underline{v}\cdot\nabla)F = (\nabla v)F$$

where the operation $(\underline{v}\cdot\nabla)$ is done *component-wise* on F.

Remark 1.5. Given $\rho_r : \mathcal{B}_r \to [0,\infty)$, we define ρ by $\rho\, dx = \rho_r\, d\varphi$ and think of it as the "push-forward" of a measure or the Radon-Nikodym derivative. That is, we require

$$\int_{\mathcal{B}_r} \varphi\cdot\mathcal{X}(\varphi)\rho_r(\varphi)\, d\varphi = \int_{\mathbb{R}^d} \varphi(x)\rho(x)\, dx \quad \forall\varphi\in C_c(\mathbb{R}^d)$$

Classical case: If $\mathcal{X} : \mathcal{B}_r \to \mathcal{B}$ is a diffeomorphism (smooth enough) with Jacobian $F = \left[\frac{\partial x_i}{\partial p_\alpha}\right]$, where $x = \mathcal{X}(\varphi)$, then

$$\rho(x) = \frac{\rho_r(\varphi)}{\det(F(\varphi))}$$

Note: this is the static (equilibirum) problem, with no time. Now, let's consider the same problem with time.

1.1.2 Evolutionary form

Let $\mathcal{X} : (0,T) \times \mathcal{B}_r \to \mathbb{R}^d$. Then

$$\int_{\mathcal{B}_r} \varphi \cdot \mathcal{X}(t,\varphi)\rho_r(\varphi)\,d\varphi = \int_{\mathbb{R}^d} \varphi(t,x)\rho(t,x)\,dx \quad \forall C_c\left((0,T)\times\mathbb{R}^d\right)$$

We still get

$$\rho(t,x) = \frac{\rho_r(\varphi)}{\det\left(F(t,\varphi)\right)}$$

where $x = \mathcal{X}(t,\varphi)$.

Chain rule: $\varphi(t,x) = \varphi_r(t,\varphi)$ under $x = \mathcal{X}(t,\varphi)$, a family of (smooth enough) diffeomorphisms. Given $\varphi_r : (0,T)\times\mathcal{B}_r \to \mathbb{R}$, then

$$\varphi(t,x) = \varphi_r\left(t, \mathcal{X}^{-1}(t,x)\right)$$

Alternatively, given $\varphi : (0,T)\times\mathcal{B}(t) \to \mathbb{R}$, define

$$\varphi_r(t,\varphi) = \varphi\left(t, \mathcal{X}(t,\varphi)\right)$$

So,

$$\dot{\varphi}_r(t,\varphi) = \frac{\partial}{\partial t} \restriction_\varphi \varphi_r(t,\varphi) = \varphi_t(t,x) + (\underline{v}\cdot\nabla)\varphi(t,x)$$

Derivative of Jacobian. Recall $F = \left[\frac{\partial x_i}{\partial p_\alpha}\right]$. Then, using $\underline{v}(t,x) = \dot{\underline{x}}(t,\varphi)$, we have

$$\dot{F}_{i\alpha} = \frac{\partial}{\partial t}\restriction_\varphi \frac{\partial x_i}{\partial p_\alpha} = \frac{\partial \dot{x}_i}{\partial p_\alpha} = \sum_j \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial p_\alpha}$$

and thus $\dot{F} = (\nabla\underline{v})F$. Note that $\nabla\underline{v}$ is a *matrix* with entries $(\nabla\underline{v})_{ij} = \frac{\partial v_i}{\partial x_j}$. Gurtin uses the notation $L = \nabla\underline{v}$. Also, we point out that Roman indices (like i) are used for "real world" variables, and Greek indices (like α) are used for "reference" variables.

Derivative of determinant. Observe that

$$\begin{aligned}\det\left(A+\delta A\right) &= \det\left(A\left(I + A^{-1}\delta A\right)\right) = \det(A)\det\left(I + A^{-1}\delta A\right)\\ &= \det(A)\left(1 + \operatorname{tr}\left(A^{-1}\delta A\right) + O(\delta A^2)\right)\end{aligned}$$

and so

$$\det(A+\delta A) - \det(A) = \det(A)\operatorname{tr}\left(A^{-1}\delta A\right) + O(\delta A^2)$$

Recall the Frobenius inner product on matrices $A, B \in \mathbb{R}^{d\times d}$, defined by

$$A : B = \sum_{i,j} A_{ij}B_{ij} \equiv \sum_i\sum_j A_{ij}(B^T)_{ji} = \sum_i (AB^T)_{ii} = \operatorname{tr}(AB^T)$$

Similarly, one can show

$$A : B = \operatorname{tr}(AB^T) = \operatorname{tr}(A^TB) = \operatorname{tr}(B^TA) = \operatorname{tr}(BA^T)$$

and so $|A|^2 = A : A = \mathrm{tr}(A^T A)$. This implies

$$\det(A + \delta A) - \det(A) = \det(A)\left(A^{-T} : \delta A\right) + O(\delta A^2)$$

That is,

$$\frac{\partial \det(A)}{\partial A_{ij}} = \det(A)\left(A^{-T}\right)_{ij}$$

which is sometimes written as $D\det(A) = \det(A)A^{-T}$. Thus,

$$\begin{aligned}
\dot{\rho}(t,x) &= \frac{-\rho_r(\varphi)}{\det\left(F(t,\varphi)\right)^2}\det(F)^{\cdot} \\
&= -\frac{\rho_r}{\det(F)^2}\det(F)\left(F^{-T} : \dot{F}\right) \\
&= -\frac{\rho_r}{\det(F)}\left(F^{-T} : (\nabla \underline{v})F\right) \\
&= -\rho\,(I : \nabla \underline{v}) = -\rho\,\mathrm{div}(\underline{v})
\end{aligned}$$

where we have used the facts that $A : BC = B^T A : C = AC^T : B$ and $I : A = \mathrm{tr}(A)$. Finally, we can write

$$\rho_t + \underline{v}\cdot\nabla\rho + \rho\,\mathrm{div}(\underline{v}) = 0 \Rightarrow \rho_t + \mathrm{div}(\rho\underline{v}) = 0$$

Reynolds transport formula. Suppose $\mathcal{X} : (0,T)\times\mathcal{B}_r \to \mathcal{B}(t) \subseteq \mathbb{R}^d$ is a family of diffeomorphisms and $\varphi : (0,T)\times\mathbb{R}^d \to \mathbb{R}$ is smooth. Then,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{B}(t)} \varphi(t,x)\rho(t,x)\,dx &= \frac{d}{dt}\int_{\mathcal{B}_r} \varphi\circ\mathcal{X}(t,\varphi)\rho_r(\varphi)\,d\varphi \\
&= \int_{\mathcal{B}_r} \dot{\varphi}\circ\mathcal{X}(t,\varphi)\rho_r(\varphi)\,d\varphi
\end{aligned} \tag{1}$$

so then

$$\frac{d}{dt}\int_{\mathcal{B}(t)} \varphi\rho\,dx = \int_{\mathcal{B}(t)} \dot{\varphi}\rho\,dx$$

where $\mathcal{B}(t)$ is the image of $\mathcal{B}_r$ under $\mathcal{X}$ and ρ is the density on $\mathcal{B}(t)$ under $\mathcal{X}$.

Leibniz's Formula. In 1-D,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\,dx = \int_{a(t)}^{b(t)} f_t(t,x)\,dx + f\left(t,b(t)\right)b'(t) - f\left(t,a(t)\right)a'(t)$$

and in many-D

$$\frac{d}{dt}\int_{\mathcal{B}(t)} f(t,x)\,dx = \int_{\mathcal{B}(t)} f_t(t,x)\,dx + \int_{\partial\mathcal{B}(t)} f\underline{v}_n\,da$$

where $\underline{v}_n(t,s)$ is the *normal velocity* of the point $s \in \underline{\partial\mathcal{B}(t)}$. We often write $\underline{v}_n = \underline{v}\cdot\underline{n}$ where $\underline{v}(t,\underline{x})$ is the velocity of points $x \in \overline{\partial\mathcal{B}(t)}$ and $\underline{n}(t,s)$ is the normal at $s \in \partial\mathcal{B}(t)$.

Note: if $\mathcal{R} \subseteq \mathbb{R}^d$ is a *fixed* region, then the divergence theorem implies

$$0 = \int_{\mathcal{R}} \rho_t + \operatorname{div}(\rho \underline{v})\, dx = \int_{\mathcal{R}} \rho_t + \int_{\partial \mathcal{R}} \rho \underline{v} \cdot \underline{n}$$

and therefore

$$\frac{d}{dt} \int_{\mathcal{R}} \rho = - \int_{\partial \mathcal{R}} \rho \underline{v} \cdot \underline{n}$$

1.2 Balance of Momentum

Q: How can we generalize Newton's Laws?

Kinematics: Suppose $\mathcal{X} : (0,T) \times \mathcal{B}_r \to \mathcal{B}(t) \subseteq \mathbb{R}^d$ is a smooth family of diffeomorphisms. Given a "part" $\mathcal{P}_r$ of the body $\mathcal{B}_r$ (i.e. $\mathcal{P}_r \subseteq \mathcal{B}_r$) at the current location $\mathcal{P}(t) \subseteq \mathcal{B}(t)$, then

1. The *linear momentum* of $\mathcal{P}_r$ is

$$\mathbf{I}(t, \mathcal{P}_r) := \int_{\mathcal{P}(t)} \rho \underline{v}\, dx = \int_{\mathcal{P}_r} \rho_r \dot{\underline{x}}$$

   and

2. the *angular momentum* of $\mathcal{P}$ about $\underline{0} \in \mathbb{R}^d$ is

$$\mathbf{a}(t, \mathcal{P}_r) := \int_{\mathcal{P}(t)} (\underline{x} - \underline{0}) \times \rho \underline{v}\, dx$$

Q: What forces act on $\mathcal{P}(t)$? First, there are *external forces* per unit volume, denoted by $\underline{b}(t,x)$. The external force acting on $\mathcal{P}(t)$ is $\int_{\mathcal{P}(t)} \underline{b}(t,x)\, dx$. What about the force that $\mathcal{B}(t) \setminus \mathcal{P}(t)$ exerts upon $\mathcal{P}(t)$? To answer this, we follow the ideas of Cauchy.

Cauchy's Hypotheses

1. The force exerted on a part $\mathcal{P}(t)$ of a body $\mathcal{B}(t)$ by the complement $\mathcal{B}(t) \setminus \mathcal{P}(t)$ can be represented as a surface attraction (force per unit area) acting on $\partial \mathcal{P}(t)$, so that the force is $\int_{\partial \mathcal{P}(t)} \underline{s}$

2. The surface traction at $x \in \partial \mathcal{P}(t)$ can be expressed as a function of the form $\underline{s}(t, \underline{x}, \underline{n})$, so that $\underline{s}$ only depends on $\partial \mathcal{P}(t)$ via the normal vector $\underline{n}$.

Note: If $\mathcal{P}_1(t), \mathcal{P}_2(t)$ are two parts of the body with a point $x \in \partial \mathcal{P}_1 \cap \partial \mathcal{P}_2$ with common normal vector, then the traction that $\mathcal{B} \setminus \mathcal{P}_1$ exerts on $\mathcal{P}_1$ at x is *equal* to the traction that $\mathcal{B} \setminus \mathcal{P}_2$ exerts on $\mathcal{P}_2$ at x. Notation from Gurtin: A force "system" for $\mathcal{B}_r$ is a pair $(\underline{b}(t, \underline{x}), \underline{s}(t, \underline{x}, \underline{n}))$ of body forces and tractions.

1.2.1 Classical statements of balance of momentum

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\underline{v}\,dx = \int_{\mathcal{P}(t)} \underline{b}\,dx + \int_{\partial\mathcal{P}(t)} s\,du$$

$$\Rightarrow \quad \frac{d}{dt}\int_{\mathcal{P}(t)} (x-\underline{0})\times(\rho\underline{v})\,dx = \int_{\mathcal{P}(t)} (x-\underline{0})\times\underline{b}\,dx$$

$$= \int_{\partial\mathcal{P}(t)} (x-\underline{0})\times s\,da$$

Remark 1.6. If the balance of linear momentum holds and the balance of angular momentum about $\underline{0}$ holds, then the balance of angular momentum holds about any $\underline{0}' \in \mathbb{R}^d$

Proof. Observe that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{P}(t)} (x-\underline{0}')\times\rho\underline{v} - \int_{\mathcal{P}(t)} (x-\underline{0}')\times b - \int_{\partial\mathcal{P}(t)} (x-\underline{0}')\times s \\
&= \frac{d}{dt}\int_{\mathcal{P}(t)} (x-\underline{0})\times\rho\underline{v} - \int_{\mathcal{P}(t)} (x-\underline{0})\times b - \int_{\partial\mathcal{P}(t)} (x-\underline{0})\times s \\
&\quad - (0-\underline{0}')\times\left(\frac{d}{dt}\int_{\mathcal{P}(t)} \rho v - \int_{\mathcal{P}(t)} b - \int_{\partial\mathcal{P}(t)} s\right) \\
&= \underline{0}+\underline{0}=\underline{0}
\end{aligned}$$

□

We summarize here the classical statements for Linear Momentum

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\underline{v} = \int_{\mathcal{P}(t)} \underline{b} + \int_{\partial\mathcal{P}(t)} \underline{s} \quad \forall\mathcal{P}(t)=\mathcal{X}(t,\mathcal{P}_r), \mathcal{P}_r\subseteq\mathcal{B}_r \tag{2}$$

and Reynolds' form thereof,

$$\int_{\partial\mathcal{P}(t)} \rho\dot{\underline{v}} = \int_{\mathcal{P}(t)} \underline{b} + \int_{\partial\mathcal{P}(t)} \underline{s} \tag{3}$$

as well as Angular Momentum

$$\frac{d}{dt}\int_{\mathcal{P}(t)} (\underline{x}\times\underline{v})\rho = \int_{\mathcal{P}(t)} \underline{x}\times\underline{b} + \int_{\partial\mathcal{P}(t)} \underline{x}\times\underline{s} \tag{4}$$

and Reynolds' form thereof

$$\int_{\mathcal{P}(t)} (\underline{x}\times\dot{\underline{v}})\rho = \int_{\mathcal{P}(t)} \underline{x}\times\underline{b} + \int_{\partial\mathcal{P}(t)} \underline{x}\times\underline{s} \tag{5}$$

since $(x\times v)^{\cdot} = \dot{x}\times v + x\times\dot{v} = 0 + x\times\dot{v}$.

Theorem 1.7 (Cauchy)**.** *Suppose $\mathcal{B}_r$ undergoes a classical motion (i.e. smooth diffeomorphisms) subjected to a (smooth) force system (b, s) and assume the postulates of Cauchy hold (i.e. $s = s(t, x, n)$). Then a necessary and sufficient condition for the Balance of Linear Momentum to hold is the existence of a stress tensor $T = T(t, x)$ for which*

1. *$s(t, x, n) = T(t, x)n$, and*
2. *$\rho\dot{v} = \operatorname{div}(T) = b$ where*

$$\operatorname{div}(T)_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij} = \sum_j T_{ij,j}$$

In this situation, the Balance of Angular Momentum holds $\iff T = T^T$.

Proof. ($\Rightarrow$) Fix $t > 0$ and $x \in \mathcal{B}(t)$ and write $s(n) = s(t, x, n)$.

Step 1: Let $\{e_i\}_{i=1}^3$ be a basis for $\mathbb{R}^3$ and let $k \in S^2$ be a unit vector such that $k \cdot e_i > 0$. Let K_ε be the right tetrahedron with center x and force normal K with volume ε^3. Select $\mathcal{P}(t) = K_\varepsilon$. (Note that the normal to the xz, xy, yz coordinate planes of the tetrahedron are $-e_2, -e_3, -e_1$ respectively, and the normal to the skew plane is k.) Now, the Balance of Linear Momentum states

$$\frac{1}{\varepsilon^2} \int_{\partial K_\varepsilon} s(n) = \frac{1}{\varepsilon^2} \int_{K_\varepsilon} (\rho\dot{v} - b) = O\left(\frac{|K_\varepsilon|}{\varepsilon^2} \|\rho\dot{v} - b\|_{L^\infty}\right) = O(\varepsilon)$$

Note the following fact:

$$\frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y)\, dy \xrightarrow[\varepsilon \to 0]{} f(x)$$

So then,

$$\begin{aligned}
\frac{1}{\varepsilon^2} \int_{\partial K_\varepsilon} s(n) &= \frac{1}{\varepsilon^2} \left(\sum_{i=1}^{3} \int_{A_i} s(-e_i) + \int_{A_0} s(k) \right) \\
&= \frac{1}{\varepsilon^2} \left(\sum_{i=1}^{3} |A_i| s(-e_i) + |A_0| s(k) + |\partial K| o(1) \right) \\
&= \frac{|A_0|}{\varepsilon^2} \left(\sum_{i=1}^{3} \frac{|A_i|}{|A_0|} s(-e_i) + s(k) + o(1) \right)
\end{aligned}$$

since the normals are constant on the faces, $s(t, \cdot, n)$ is continuous, and both $|A_0|, |\partial K_\varepsilon| = O(\varepsilon^2)$. Also, note that $\frac{|A_i|}{|A_0|} = k \cdot e_i$. This tells us

$$O(\varepsilon) = \frac{1}{\varepsilon^2} \int_{\partial K_\varepsilon} s(n) = C \cdot \left(\sum_{i=1}^{3} (k \cdot e_i) s(-e_i) + s(k) + o(1) \right)$$

Letting $\varepsilon \to 0$, we find

$$s(k) = \sum_{i=1}^{3} -(k \cdot e_i)s(-e_i) \quad \text{ for } k \cdot e_i > 0 \quad , i = 1, 2, 3$$

Step 2: Note that $s(e_i) = -s(-e_i)$. This follows because $s(t, x, n)$ is continuous in $k \in S^2$, so we may let $k \to e_i$ and so

$$s(e_i) = \lim_{k \to e_i} s(k) = \lim_{k \to e_i} \sum_{j=1}^{3} -(k \cdot e_j)s(-e_j) = -s(-e_i)$$

Thus,

$$s(k) = \sum_{i=1}^{3} (k \cdot e_i)s(e_i) \quad \forall k \in S^2 \text{ with } k \cdot e_i \geq 0$$

where we have applied continuity to relax the condition to ≥ 0.

Step 3: Let $k \in S^2$ be arbitrary and define $\bar{e}_i = \text{sgn}(k \cdot e_i)e_i = \pm e_i$. Then $\{\bar{e}_i\}_{i=1}^3$ is an orthonormal basis for $\mathbb{R}^3$ and $k \cdot \bar{e}_i \geq 0$ for $i = 1, 2, 3$. Thus,

$$s(k) = \sum_{i=1}^{3} (k \cdot \bar{e}_i)s(\bar{e}_i) = \sum_{i=1}^{3} \text{sgn}(k \cdot e_i)^2 (k \cdot e_i)s(e_i) = \sum_{i=1}^{3} (k \cdot e_i)s(e_i)$$

for *every* $k \in S^2$. So we have shown that $s(k)$ is linear in k, and thus it must be a matrix. Define

$$T := \sum_{i=1}^{3} s(e_i) \otimes e_i$$

Then

$$Tk = \sum_{i=1}^{3} (k \cdot e_i)s(e_i) = s(k)$$

since $(a \otimes b)c = (b \cdot c)a$. $\square$

Recall that the Balance of Momentum holds $\iff s(n) = Tn$, in which case

$$s(n) = \int_{\partial \mathcal{P}(t)} Tn = \int_{\mathcal{P}(t)} d\omega(t)$$

and all integrands are continuous. Also, note that

$$\int_{\mathcal{P}} \text{div}(T)_i = \int_{\mathcal{P}} T_{ij,j} = \int_{\partial \mathcal{P}} T_{ij} \cdot n_j = \int_{\partial \mathcal{P}} (Tn)_i$$

i.e. $\int_{\mathcal{P}} \text{div}(T) = \int_{\partial \mathcal{P}} Tn$. Then the Balance of Momentum implies $\int_{\mathcal{P}(t)} \rho \dot{v} - \text{div}(T) = \int_{\mathcal{P}(t)} b$, and so

$$\rho \dot{v} - \text{div}(T) = b$$

Furthermore, this implies

$$\int_{\mathcal{P}(t)} \rho\dot{v} = \int_{\mathcal{P}(t)} b + \int_{\partial\mathcal{P}(t)} Tn$$

so if $s(n) = Tn$ then the Balance of Momentum (M) holds. Thus, we have the equivalency

$$(M) \iff s(n) = TN, \rho\dot{v} - \operatorname{div}(T) = b$$

Given $s(n) = Tn$, then the Balance of *Angular* Momentum holds $\iff T = T^T$. Recall the Balance of Angular Momentum

$$\int_{\mathcal{P}(t)} \rho(x \times \dot{v}) = \int_{\mathcal{P}(t)} x \times b + \int_{\partial\mathcal{P}(t)} x \times s(n)$$

We compute (using the Levi-Civita symbol ε_{ijk})

$$(x \times Tn)_i = \varepsilon_{ijk} x_j (Tn)_k = \varepsilon_{ijk} x_j T_{k\ell} n_\ell$$

and so

$$\begin{aligned}
\int_{\partial\mathcal{P}(t)} (x \times Tn)_i &= \int_{\mathcal{P}} \left(\varepsilon_{ijk} x_j T_{k\ell}\right)_{,\ell} \\
&= \int_{\mathcal{P}} \varepsilon_{ijk} \left(\delta_{j\ell} T_{k\ell} + x_j T_{k\ell,\ell}\right) \\
&= \int_{\mathcal{P}} \varepsilon_{ijk} T_{kj} + x_j \operatorname{div}(T)_k
\end{aligned}$$

Define $\operatorname{Rot}(T)_i = \varepsilon_{ijk} T_{jk}$. Then

$$\int_{\partial\mathcal{P}} x \times (Tn) = \int_{\mathcal{P}} -\operatorname{Rot}(T) + x \times \operatorname{div}(T)$$

and then

$$\int_{\mathcal{P}} x \times (\rho\dot{v}) = \int_{\mathcal{P}} x \times b + \int_{\partial\mathcal{P}} x \times (Tn)$$

which holds $\iff$

$$\int_{\mathcal{P}} x \times (\rho\dot{v}) = \int_{\mathcal{P}} x \times b + \int_{\mathcal{P}} -\operatorname{Rot}(T) + x \times \operatorname{div}(T)$$

and so

$$\int_{\mathcal{P}} x \times \underbrace{(\rho\dot{v} - \operatorname{div}(T) - b)}_{\text{linear momentum}} + \operatorname{Rot}(T) = 0$$

which finally implies

$$\int_{\mathcal{P}} \operatorname{Rot}(T) = 0 \; \forall \mathcal{P} \subseteq \mathcal{B}(t) \;\Rightarrow\; \operatorname{Rot}(T) = 0$$

Observe that

$$\operatorname{Rot}(T) = \begin{bmatrix} T_{23} - T_{32} \\ T_{31} - T_{13} \\ T_{12} - T_{21} \end{bmatrix} = 0 \iff T^T = T$$

1.2.2 Classical Configurations

1. Given a surface S with normal n,
   (a) the normal traction is $Tn \cdot n = n^T Tn$ or $(n^T Tn)n = (n \otimes n)Tn$, and
   (b) the shearing traction is $(I - n \otimes n)Tn$
2. A "hydrostatic" stress tensor is one of the form $T = -pI$, so then $Tn = -pn$ for all normals n. Note that $T' = T - \frac{1}{d}\text{tr}(T)I$ which is *trace-free.*

1.2.3 Alternative Forms of the Momentum Equation

The standard statement + Leibniz's Rule gives us

$$\int_{\mathcal{P}(t)} (\rho v)_t + \text{div}(\rho v \otimes v) - \text{div}(T) = \int_{\mathcal{P}(t)} b$$

and localizing yields

$$(\rho v)_t + \text{div}(\rho v \otimes v) - \text{div}(T) = b$$

This is the *conservation form* of the equation. We also have the *skew symmetrized* form, which is used in numerical codes:

$$\frac{1}{2}\left(\rho \dot{v} + (\rho v)_t + \text{div}(\rho v \otimes v)\right) - \text{div}(T) = b$$

Lemma 1.8. *Suppose the Balance of Mass, Linear Momentum and Angular Momentum hold. Then,*

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\frac{|v|^2}{2} + \int_{\mathcal{P}(t)} T : D(v) = \int_{\mathcal{P}(t)} b \cdot v + \int_{\partial\mathcal{P}(t)} Tn \cdot v$$

This is called the "principal of virtual work". The first term represents kinetic energy, the middle term is some kind of dissipation, and the right hand side represents power.

Proof. Apply Reynolds' formula to write

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\frac{|v|^2}{2} &= \int_{\mathcal{P}(t)} \rho\left(\frac{|v|^2}{2}\right)^{\cdot} = \int_{\mathcal{P}(t)} \rho v \cdot \dot{v} = \int_{\mathcal{P}(t)} (\text{div}(T) + b) \cdot v \\ &= \int_{\mathcal{P}(t)} b \cdot v - T : \nabla v + \int_{\partial\mathcal{P}(t)} Tn \cdot v\end{aligned}$$

Thus,

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\frac{|v|^2}{2} + \int_{\mathcal{P}(t)} T : \nabla v = \int_{\mathcal{P}(t)} b \cdot v + \int_{\partial\mathcal{P}(t)} Tn \cdot v$$

If $T = T^T$, then $T : \nabla v$ reduces to the desired form, according to the identities $A : B = \frac{1}{2}A : B + \frac{1}{2}A^T : B^T = A : \frac{1}{2}(B + B^T)$ for any *symmetric* matrix A and arbitrary B, and recalling that $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$. □

Before moving on to study fluids, we note the following properties of incompressible materials:

$$\det F - 1 \iff \operatorname{div}(v) = 0 \iff \rho = \text{const.}$$

Proof.

$$\dot{\rho} + \rho \operatorname{div}(v) = 0 \Rightarrow (\dot{\rho} = 0 \iff \operatorname{div}(v) = 0 \iff \rho(t, x(t,p)) = \rho_r(p))$$

and

$$\rho(t, x(t,p)) = \frac{\rho_r(p)}{\det(F(t,p))} \Rightarrow (\rho = \text{const.} \iff \det F = \text{const.}$$
$$\iff \det F = \det F(0) = \det I = 1)$$

□

2 Classical Fluids

Inviscid fluids have a stress tensor given by $T(t,x) = -p(t,x)I$. Then,

$$\operatorname{div}(T)_i = T_{ij,j} = (-p\delta_{ij})_{,j} = -p_{,j}\delta_{ij} - p\delta_{j,j} = -p_{,i} = -(\nabla p)_i$$

Take $\rho\dot{v} - \nabla p = b$. Then

$$\dot{v} = v_t + (v.\nabla)v = v_t + \nabla\left(\frac{|v|^2}{2}\right) - v \times \operatorname{curl}(v)$$

which implies

$$v_t + \nabla\left(\frac{|v|^2}{2}\right) - v \times \operatorname{curl}(v) = \frac{1}{\rho}\nabla p = \frac{1}{\rho}b$$

1. If the fluid is incompressible then ρ =const.
2. If $p = p(\rho)$, then

$$\frac{1}{\rho}\nabla p = \nabla \int^{\rho} \frac{p'(\xi)}{\xi}\, d\xi = \frac{p'(\rho)}{\rho}\nabla p = \frac{1}{\rho}\nabla p = \nabla\left(\frac{1}{\rho}p\right) =: \nabla P$$

3. If the force per unit mass $f = \frac{1}{\rho}b$ is the gradient of a potential, i.e. $f = \nabla F$, then we obtain

$$v_t + \nabla\left(\frac{|v|^2}{2} + P(\rho) - F\right) - v \times \operatorname{curl}(v) = 0$$

If $v = \nabla\varphi$ for some scalar φ, then $\operatorname{curl}(v) = 0$ and $v_t = \nabla\varphi_t$, so

$$\nabla\left(\varphi_t + \frac{|v|^2}{2} + P(\rho) - F\right) = 0$$

(from classical fluid study), so then

$$\varphi_t + \frac{|v|^2}{2} + P(\rho) - F = \text{const.}$$

This is Bernoulli's Equation!

Example 2.1. Consider a steady flow on an incompressible fluid. Then

$$\frac{|v|^2}{2} + \frac{p}{\rho} - g = C$$

for some constant C, where $f = gz$ and $g \approx 9.81$ (gravity). Consider the setup of a Pitot tube, with pressure p_0 at $v = 0$. Then

$$\frac{|v|^2}{2} + \frac{P}{\rho} - gz = 0 + \frac{p_0}{\rho} - gz \Rightarrow \frac{|v|^2}{2} = \frac{p - p_0}{\rho}$$

2.1 Inviscid Fluids

We assume $T = -pI$. For a *barotropic* fluid, $p = p(\rho)$. Bernoulli's equation states that if $v = \nabla\varphi$ and $f = \frac{b}{\rho} = \nabla F$, then

$$\varphi_t + \frac{|v|^2}{2} + P(p) = F$$

We have two natural questions: Why should $v = \nabla\varphi$? And why should $p = p(\rho)$ and not $p = p(\rho, \theta)$, where θ is the temperature in the gas law $\frac{p}{\rho} = R\theta$.

Theorem 2.2 (Velocity transport theorem)**.** *Assume $T = -pI$, where $p = p(\rho)$, and $f = \frac{b}{\rho} = \nabla F$. Then $\left(F^{-1}\left(\frac{\omega}{\rho}\right)\right)^{\cdot} = \underline{0}$.*

Proof. Recall

$$(v.\nabla)v = \nabla\left(\frac{|v|^2}{2}\right) - v \times \omega \quad , \text{ where } \omega = \text{curl}(v)$$

Then the momentum equation becomes

$$v_t + \nabla\left(\frac{|v|^2}{2} + P(\rho) - F\right) - v \times \omega = 0$$

where

$$P(\rho) = \int^{\rho} \frac{p'(r)}{r}\, dr$$

Take the curl of both sides to get

$$\omega_t - \text{curl}(v \times \omega) = 0$$

since $\operatorname{curl}(\nabla H) = 0$ for any smooth H. Now, we use the identity

$$\operatorname{curl}(v \times \omega) = (\omega.\nabla)v - (v.\nabla)\omega - \operatorname{div}(v)\omega$$

to write

$$\omega_t + (v.\nabla)\omega + \operatorname{div}(v)\omega = (\omega.\nabla)v$$

which simplifies to

$$\dot{\omega} + \operatorname{div}(v)\omega = (\nabla v)\omega \tag{6}$$

since $(\omega.\nabla)v_i = \omega_j v_{i,j} = v_{i,j}\omega_j = [(\nabla v)\omega]_i$. Observe that

$$\left(\frac{1}{\rho}\right)^{\cdot} = -\frac{1}{\rho^2}\dot{\rho} = \frac{1}{\rho}\operatorname{div}(v)$$

which we will write as

$$\left(\frac{1}{\rho}\right)^{\cdot} - \frac{1}{\rho}\operatorname{div}(v) = 0 \tag{7}$$

Now, we take the sum of $\frac{1}{\rho}$ times (6) and ω times (7) and apply the product rule to write

$$\left(\frac{\omega}{\rho}\right)^{\cdot} = (\nabla v)\frac{\omega}{\rho}$$

Now, recall that $F = \left[\frac{\partial x_i}{\partial p_\alpha}\right]$ and $\dot{F} = (\nabla v)F$, and notice that

$$0 = \dot{I} = (FF^{-1}) \ \Rightarrow \ \left(F^{-1}\right)^{\cdot} = -F^{-1}\dot{F}F^{-1} = -F^{-1}\nabla v F F^{-1} = -F^{-1}\nabla v$$

Then,

$$\begin{aligned}\left(F^{-1}\left(\frac{\omega}{\rho}\right)\right)^{\cdot} &= \left(F^{-1}\right)^{\cdot}\frac{\omega}{\rho} + F^{-1}\left(\frac{\omega}{\rho}\right)^{\cdot} \\ &= -F^{-1}\nabla v\frac{\omega}{\rho} + F^{-1}\nabla v\frac{\omega}{\rho} = \underline{0}\end{aligned}$$

□

Corollary 2.3. *If every particle in the flow originates from a region with zero vorticity (and the flow is smooth), then $\omega = \operatorname{curl}(v) = \underline{0}$.*

Example 2.4. Designing an aerofoil.

2.2 Balance of Energy

Assumptions:

1. The energy per unit mass is $e + \frac{|v|^2}{2}$, where e is the "internal energy" (i.e. inherent to the material)
2. $\exists(r, \underline{q})$ where r is the "energy source" and $\underline{q}$ is the "energy flux". Specifically, $r : \mathcal{B}(t) \to \mathbb{R}$ and $\underline{q} : \partial\mathcal{P}(t) \to \mathbb{R}^d$ for all parts $\mathcal{P}(t) \subseteq \mathcal{B}(t)$.

3. The Balance of Energy equation holds:

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\left(e + \frac{|v|^2}{2}\right) = \int_{\mathcal{P}(t)} r + \underline{b}\cdot\underline{v} + \int_{\partial\mathcal{P}(t)} -\underline{q}\cdot\underline{n} + \underline{s}\cdot\underline{v}$$

Using Reynolds' Formula and Gauss' Divergence Theorem (plus Balance of Mass) and the fact that $S = Tn \Rightarrow s \cdot v = (T^T v)\cdot n$, we can prove that the Balanceof Energy equation above implies

$$\int_{\mathcal{P}(t)} \rho(\dot{e} + \dot{v}\cdot v) = \int_{\mathcal{P}(t)} r + b\cdot v - \operatorname{div}(q) + \operatorname{div}(T^T v)$$

Note that

$$\begin{aligned}\operatorname{div}(T^T v) &= \sum_i (T^T v)_{i,i} = \sum_j\sum_i (T_{ji}v_j)_{,i} = \sum_j\sum_i T_{ji,i}v_j + T_{ji}v_{j,i} \\ &= \sum_j \operatorname{div}(T)_j v_j + T_{ji}(\nabla v)_{ji} = \operatorname{div}(T)\cdot v + T : \nabla v\end{aligned}$$

This allows us to write

$$\int_{\mathcal{P}(t)} \rho\dot{e} + \underbrace{(\rho\dot{v} - \operatorname{div}(T) - b)}_{=0 \text{ by Momentum Eqn}} \cdot v + \operatorname{div}(q) = \int_{\mathcal{P}(t)} r + T : \nabla v$$

and localizing shows that

$$\rho\dot{e} + \operatorname{div}(q) = r + T : \nabla v$$

This is an example of the phenomenon of the "decoupling" of kinetic and thermal/internal energy. The equation in the linea above has something to do with "mechanized heating".

Also, when $T = T^T$, then $T : \nabla v = T : D(v)$ where $D(v) = (\nabla v)_{\text{sym}}$.

Example 2.5. Let θ be temperature, and $e = c\theta$ for some $c \in \mathbb{R}^+$, a specific heat, and $q = -k\nabla\theta$ for some $k \in \mathbb{R}^+$, a conductivity. This is where heat flows down a temperature gradient. Suppose $\underline{v} = \underline{0}$ (a rigid solid). Then $\nabla v = 0$ and $\dot{\theta} = \theta_t + v\nabla\theta = \theta_t$. Thus,

$$c\theta_t - k\Delta\theta = r$$

which is the classical heat equation!

Let's return to the case of an inviscid fluid and try to convince ourselves why $p = p(\rho)$ and *not* $p = p(\rho, \theta)$. We assume $T = -pI$ and $p = p(e, \rho)$. Suppose, for example, we have an ideal gas, so $e = c\theta$ and $p(e,\rho) = \rho\theta = \frac{R}{c}\rho e$. Suppose further that the fluid is non-heat conducting, so $\underline{q} = 0$. Finally, suppose $r = 0$. Then the energy equation becomes

$$\rho\dot{e} = T : \nabla v = -p\operatorname{div}(v) \;\Rightarrow\; \rho\dot{e} + p\operatorname{div}(v) = 0 \tag{8}$$

Also, we have

$$\dot{\rho} + \rho \operatorname{div}(v) = 0 \quad (\iff \quad \rho_t + \operatorname{div}(\rho v) = 0) \tag{9}$$

We take ρ times Equation (8) and subtract p times Equation (9) to get

$$\rho^2 \dot{e} - p\dot{p} = 0$$

which we write as

$$\dot{e} - \frac{p}{\rho^2}\dot{p} = 0$$

Suppose, now, that $p = p(e, \rho)$ and we can construct $\eta(e, \rho)$ (which is like *entropy*) such that

$$\frac{d\eta}{d\rho} = -\frac{p(e,\rho)}{\rho^2} \cdot \frac{d\eta}{de}$$

Then η is constant along curves of the form $\cdot e(\rho) = -\frac{p(e,\rho)}{\rho^2}$ (see method of characteristics). This implies

$$\frac{d\eta}{de}\dot{e} + \frac{d\eta}{d\rho}\dot{\rho} = 0 \qquad \text{i.e. } \dot{\eta} = 0$$

Thus, if the flow originates from a state where η_∞ =const., then

$$\eta\left(e(t,x), \rho(t,x)\right) = \eta_\infty \ \forall (t,x)$$

That is, $\eta(e,\rho) = \eta_\infty$ implies that we could write $e = e(\rho)$, which in turn implies that $p(e(\rho), p) = \tilde{p}(\rho)$, by the Implicit Function Theorem (since $e(\rho)$ is increasing). This shows that, indeed, pressure is a function of density only, and not temperature. Note that for an ideal gas, we typically have $\eta = \ln\left(\frac{p}{\rho^\gamma}\right)$ for some constant γ. Let's look at this example more specifically:

Example 2.6 (Ideal Gas). Set

$$\eta(p,e) = \eta_\infty + C\ln\left(\frac{p}{\rho^\gamma}\right) = \eta_\infty + C\ln\left(e\rho^{1-\gamma}\right)$$

where the second equality follows from the ideal gas law $\frac{pc}{Re} = \rho$. Also. we are still assuming $e = c\theta$. Then,

$$\frac{d\eta}{de} = \frac{c}{e} = \frac{1}{\theta}$$

and

$$\begin{aligned}\frac{d\eta}{d\rho} &= \frac{C}{e\rho^{1-\gamma}}(1-\gamma)\rho^{-\gamma} \\ &= \frac{C(\gamma-1)}{\rho} \cdot \left(\frac{-p}{R\rho\theta}\right) = \left(\frac{d\eta}{de}\right) \cdot \left(-\frac{p}{\rho^2}\right)\end{aligned}$$

provided $\gamma = 1 + \frac{R}{c}$. Note: $\eta(p,e)$ represents the (specific) entropy of the ideal gas (where "specific" indicates "per unit mass").

Bernoulli's Formula

$$\varphi_t + \frac{|v|^2}{2} + P(\rho) = F \quad , \quad v = \nabla\varphi$$

follows as a corollary to this example.

Theorem 2.7. *Let* $\underline{\omega} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *vector field, and suppose* $\underline{\omega}$ *does not vanish in a neighborhood* $\mathbb{R}^2 \supseteq U \ni x_0$ *with* U *open. Then* $\exists \eta : V \subseteq U \to \mathbb{R}$ *differentiable that is constant on* trajectories *of* $\underline{\omega}$ *and* $\nabla\eta \neq \underline{0}$ *on* V*.*

Example 2.8. As in the previous example (ideal gas), for entropy we define $\underline{\omega} = [-p, \rho^2]^T$. Then having η constant on trajectories of $\underline{\omega}$ means

$$0 = \nabla\eta \cdot \underline{\omega} = \frac{d\eta}{d\rho}(-p) + \frac{d\eta}{de}(\rho^2)$$

which is true, as we have seen.

Proof. We sketch the proof of the theorem above. Select coordinates so that $\underline{x}_0$ lies at the origin and $\underline{\omega}(\underline{x}_0)$ lies along the x-axis. Consider the system of ODEs

$$\begin{aligned} \dot{\underline{x}}(t;\eta) &= \underline{\omega}(\underline{x}(t;\eta)) \\ \underline{x}(0;\eta) &= \begin{bmatrix} 0 \\ \eta \end{bmatrix} \end{aligned}$$

Show that the mapping $(t, \eta) \mapsto (x(t;\eta), y(t;\eta))$ is a bijection (by Implicit Function Theorem). Thus, $\eta = \eta(x, y)$ is the required function. □

2.3 Frame-Indifference

To distinguish materials, we have the quantities $b, s = Tn, r, q$. We still wonder about the properties of T, and *frame-indifference* (a.k.a. "change of observer") will dictate certain properties of T.

Definition 2.9. *Given a reference body* $\mathcal{B}_r \subseteq \mathbb{R}^d$ *and two motions* $x = \mathcal{X}(t, p)$ *and* $x^\star = \mathcal{X}^\star(t, p)$*, we say* x *and* $x^\star$ *are related by a change of observer provided*

$$x^\star(t, p) = \underline{y}(t) + Q(t)x(t, p)$$

for some $y : (0, T) \to \mathbb{R}^d$ *and* $Q : (0, T) \to Orth^+$ *(i.e.* $Q(t)$ *is orthogonal and* $\det(Q(t)) = +1\ \forall t$*).*

If $f(t, x) := y(t) + Q(t)x$, this just says $x^\star(t, p) = f \circ x(t, p)$.

Remark 2.10. A cynical aside: Gurtin's book uses the term $Q(t)(\underline{x} - \underline{0})$ to "vectorize" the point x. But really, these quantities are interchangeable because there is a canonical isomorphism and a *linear* map between tangent spaces to manifolds.

Consider the quantities

$$F = \left[\frac{\partial x_i}{\partial p_\alpha}\right] \quad \text{and} \quad F^\star = \left[\frac{\partial x_i^\star}{\partial p_\alpha}\right]$$

Then $x = y + Qx = y_i + \sum_j Q_{ij}x_j$, and so

$$\frac{\partial x_i^\star}{\partial p_\alpha} = Q_{ij}\frac{\partial x_j}{\partial p_\alpha}$$

which implies $F^\star = QF$.

Theorem 2.11 (Polar Decomposition)**.** *Given $F \in \mathbb{R}^{d\times d}$, $\exists R, U, V \in \mathbb{R}^{d\times d}$, with R orthogonal and U, V symmetric and positive semi-definite, such that $F = RU = VR$. (This decomposition is unique when F is nonsingular.)*

Proof. We leave the full proof as an exercise and sketch the idea here. We would guess that we need to satisfy

$$F^T = U^T R^T = UR^T \;\Rightarrow\; F^T F = UR^T RU = U^2$$

so it would make sense to set $U = \sqrt{F^T F}$. This is okay since $F^T F$ is symmetric and positive semi-definite. Defininf V is similar. □

We use this theorem to write

$$F^\star = R^\star U^\star = V^\star R^\star \;,\; F = RU = VR \;\Rightarrow\; F^\star = QF$$

and furthermore

$$(U^\star)^2 = (F^\star)^T F^\star = F^T Q^T Q F = F^T F = U^2$$

so $U^\star = U$ is invariant under a change of observer! Also,

$$R^\star U^\star = F^\star = QF = QRU \;\Rightarrow\; R^\star = QR$$

since $\det F \neq 0$. Finally, we also observe

$$V^\star R^\star = F^\star = QF = QVR \;\Rightarrow\; V^\star QR = QVR \;\Rightarrow\; V^\star = QVQ^T$$

so V is *not* invariant.

Notation: The matrix $C = F^T F = U^2$ is sometimes called the *right Cauchy-Green tensor* and $B = FF^T = V^2$ is sometimes called the *left Cauchy-Green tensor* or finger tensor. These are defined to be such that

$$C^\star = (F^\star)^T F^\star = F^T F = C$$

is invariant, but

$$B^\star = F^\star (F^\star)^T = FF^T = QBQ^T$$

is not. Also, notice that $C_{\alpha\beta} = F_{i\alpha} F_{j\beta}$ so C uses only Greek indices and is thus invariant, whereas $B_{ij} = F_{i\alpha} F_{j\alpha}$ uses Latin indices and so it depends on the frame, i.e. the "spectacles" with which we view our experiment.

Start from

$$x^\star(t,p) = y(t) + Q(t)x(t,p)$$

and take a t derivative to get

$$\underbrace{\dot{x}^\star(t,p)}_{v^\star(t,x^\star)} = \dot{y}(t) + \dot{Q}(t)x(t,p) + Q\,\underbrace{\dot{x}(t,p)}_{v(t,x)}$$

and then take $\frac{\partial}{\partial x_j}$ of both sides using the chain rule to get

$$\frac{\partial v_i^\star}{\partial x_k^\star} \cdot \frac{\partial x_k^\star}{\partial x_j} = 0 + \dot{Q}\delta_{ij} + Q_{ik}\frac{\partial v_k}{\partial x_j}$$

where we have used the fact that $\dfrac{\partial x_k^\star}{\partial x_i} = Q_{ki}$, which follows from the equation for $x^\star$ a few lines above. We now write this derivative equation as

$$\nabla^\star v^\star Q = \dot{Q} + Q\nabla v \;\Rightarrow\; \nabla^\star v^\star = \dot{Q}Q^T + Q\nabla v Q^T$$

Note that $Q^T Q = I \Rightarrow \dot{Q}Q$ is skew. Thus,

$$\left(\nabla^\star v^\star\right)_{\text{sym}} = Q\left(\nabla v\right)_{\text{sym}} Q^T$$

which we write as

$$D^\star(v^\star) = QD(v)Q^T$$

where $D(v) = \frac{1}{2}\left(\nabla v + \nabla v^T\right)$.

2.3.1 Normals to Surfaces

Consider a surface $\mathcal{S}_r$ of $\mathcal{B}_r$ with normal $n_r(p)$, and the corresponding surface $\mathcal{S}$ of $\mathcal{B}$ with normal $n(x)$. Given $p \in \mathcal{S}_r$, we construct (locally) the function $\varphi : \mathcal{B}_r \to \mathbb{R}$ for which

$$\mathcal{S}_r = \{p : \varphi_r(p) = 0\} \equiv \text{ the zero level set of } \varphi_r$$

Then

$$n_r = \frac{\nabla_p \varphi_r}{|\nabla_p \varphi_r|}$$

since $\nabla\varphi_r$ is $\perp$ to level sets. Taking $\mathcal{S} = \mathcal{X}(\mathcal{S}_r)$ and $\varphi(x) = \varphi_r(p)$, then (locally)

$$\mathcal{S} = \{x : \varphi(x) = 0\} \quad \text{and} \quad \frac{\partial \varphi_r}{\partial p_\alpha} = \frac{\partial \varphi}{\partial x_i} \cdot \underbrace{\frac{\partial x_i}{\partial p_\alpha}}_{=F_{i\alpha}}$$

Thus, $\nabla_p \varphi_r = F^T \nabla_x \varphi$ and $\nabla_p \varphi_r \parallel n_r$ and $\nabla_x \varphi \parallel n$. Accordingly, $n_r = cF^T n$ or $n = cF^{-T} n_r$ for some constant. It follows that $c(F^\star)^T n^\star = n_r = kF^T n$, and since we can write $F^\star = QF$, we can reduce this to $cn^\star = kQn$. But, $|n^\star| = |n| = |Qn| = 1$ so$k = c = 1$. Thus, $n^\star = Qn$!

This shows that a vector-valued quantity in the reference configuration is "transported" by the maps $x = \mathcal{X}(p)$ and $x^\star = \mathcal{X}^\star(p)$ corresponding to a change of observer related by $n^\star = Qn$.

Thus, if $s^\star = T^\star n^\star$ and $s = Tn$ and the surface forces correspond to the same "experiment" then $s^\star = Qs$. Thus,

$$T^\star n^\star = Q(Tn) \;\Rightarrow\; T^\star Q n = QTn$$

This should hold for *all* n, so $T^\star Q = QT$ and therefore $T^\star = QTQ^T$. This now rules out many possibilities for what T can be.

Traditionally, a formula for T is *frame-indifferent* if under a change of observer given by $x^\star = y + Qx$ we have $T^\star = QTQ^T$.

Example 2.12.

1. If $T = -pI$ then $T^\star = -pI = -pQQ^T = Q(-pI)Q^T = QTQ^T$ which works.
2. If $T = \mu D(v)$ then we have shown $D^\star(v^\star) = QD(v)Q^T$ so $T^\star = QTQ^T$ which works.
3. If $T = \mu B = \mu FF^T$ then $F^\star = QF$ implies $T^\star = F^\star(F^\star)^T = QFF^TQ^T = QTQ^T$ which works.
4. However, if $T = \mu C = \mu F^TF$ then $(F^\star)^TF^\star = (QF)^T(QF) = F^TF \neq Q(F^TF)Q^T$ so C is $T = \mu C$ is not frame-indifferent.

2.4 Newtonian Fluids

How do we distinguish a solid from a fluid? Thinking about forces, we can say that solids resist (static) shear whereas a fluid will deform and "relax" to a state of zero stress.

For instance, consider

$$x = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} p \equiv \mathcal{X}(p) \quad \text{and} \quad F = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$$

For a fluid, T doesn't depend upon F. A fluid will resist a state of shear, e.g. a nontrivial velocity gradient. A solid resists a deformation gradient.

We consider constitutive relations of the form

$$T = -\pi I + \mathcal{C}(L)$$

where π denotes the *pressure* (so as not to coincide with the coordinate p), $L \equiv \nabla v$, and $\mathcal{C} : \mathrm{Lin} \to \mathrm{Lin}$ (i.e. $\mathcal{C} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$) is linear. Think of $\mathcal{C}$ as the 1st term in some Taylor expansion:

$$T = T(L) = -\pi I + \mathcal{C}(L) + o(|L|^2)$$

1. If $L = \nabla v = 0$ then $T = -\pi I$ is a hydrostatic stress.

2. There is a degeneracy

$$T = -\pi I + \mathcal{C}(L) = -\tilde{\pi} I + \tilde{\mathcal{C}}(L)$$

with

$$\tilde{\pi} = \pi + \beta(L) \quad \text{and} \quad \tilde{\mathcal{C}}(L) = \mathcal{C}(L) + \beta(L) I$$

where $\beta : \mathrm{Lin} \to \mathbb{R}$ is linear, i.e. $\beta \in \mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R})$. By convention, we select the representation for which $\mathrm{tr}(\mathcal{C}(L)) = 0$, so then

$$\mathcal{C}(L) = \tilde{\mathcal{C}}(L) - \frac{1}{d}\mathrm{tr}(\tilde{\mathcal{C}}(L)) I$$

where d is the dimension.

3. With this convention, $\mathrm{tr}(T) = -\pi \mathrm{tr}(I) = -d\pi$.

4. If the fluid is incompressible, $0 = \mathrm{div}(v) = \mathrm{tr}(L)$. In general,

$$\mathcal{C} : \mathrm{Lin} \to \mathrm{Lin}_0 := \left\{ A \in \mathbb{R}^{d\times d} : \mathrm{tr}(A) = 0 \right\}$$

but when the fluid is incompressible, we have $\mathcal{C} : \mathrm{Lin}_0 \to \mathrm{Lin}_0$ since $L \in \mathrm{Lin}_0$.

5. If the Balance of Angular Momentum is to hold, we should have $T = T^T$; i.e. we need

$$\mathcal{C} : \mathrm{Lin} \to \mathrm{Sym}_0 := \{A \in \mathrm{Lin} : A = A^T\} \cap \mathrm{Lin}_0$$

Note $\dim(\mathrm{Sym}_0) = \frac{d(d+1)}{2} - 1$ which is 5 in 3D. For incompressible fluids, $\dim(\mathrm{Lin}_0) = d^2 - 1$ which is 8 in 3D.

6. The quantity $T_0 = T - \frac{1}{d}\mathrm{tr}(T) I$ is called the *deviatoric stress*; it is, in some sense, the "amount" that we are "away" from being incompressible. We write $T = -\pi I + T_0$. (Note: T_0 is sometimes denoted by T'.) This is closely related to the shear.

Definition 2.13. *A* Newtonian fluid *is one for which $T_0 = \mathcal{C}(L)$ (where $L = \nabla v$) and $\mathcal{C} : Lin \to Sym_0$.*

Theorem 2.14. *A necessary and sufficient condition for a Newtonian fluid to be frame-indifferent is*

$$T_0 = 2\mu D(v) = \mu(L + L^T)$$

where $\mu = \mu(p)$ is a scalar (that is frame-indifferent, i.e. $\mu^\star = \mu$).

Proof. ($\Leftarrow$) Suppose $T_0 = 2\mu D(v)$ and $x(t,p)$ and $x^\star(t,p)$ are related by a change in observer (so that $x^\star = y + Qx$). We have shown that $D^\star = QDQ^T$. Thus,

$$T_0^\star = 2\mu D^\star = 2\mu Q D Q^T = Q T_0 Q^T$$

Then

$$T^\star = -\pi^\star I + T_0^\star = -\pi^\star QQ^T + QT_0Q^T = Q\left(-\pi^\star I + T_0\right)Q^T = QTQ^T$$

using the fact that $\pi^\star = \pi$ since π is a scalar, so $\pi^\star(x^\star) \equiv \pi(x)$. $\square$

Note: If a response function $T = \hat{T}(\cdots)$ is independent of observer, then

$$\mathrm{tr}(T^\star) = \mathrm{tr}(QTQ^T) = \mathrm{tr}(T)$$

For Newtonion fluids,

$$T_0 \equiv T - \frac{1}{3}\mathrm{tr}(T)I = \mathcal{C}(L_0)$$

where $\mathcal{C} : \mathrm{Lin}_0 \to \mathrm{Sym}_0$ is *linear* (and $L = \nabla v$, $L_0 =$ trace-free part of L). This should tell us how to make the "correct" statement in the theorem below.

Theorem 2.15. *The response function of a Newtonian fluid is independent of observer* $\iff$ $\mathcal{C}(L_0) = 2\mu D_0$ *and the trace of the stress tensor is a "scalar" (i.e. independent of observer).*

Proof. ($\Leftarrow$) Let $x, x^\star : (0,T) \times \mathcal{B}_r \to \mathbb{R}6d$ be related by a change of observer. Then

$$L^\star = QLQ^T + \underbrace{\dot{Q}Q^T}_{\text{skew}} \quad \text{and} \quad D^\star = QDQ^T$$

Now,

$$\begin{aligned} T^\star &= \frac{1}{3}\mathrm{tr}(T^\star)I + 2\mu^\star D_0^\star \\ &= \frac{1}{3}\mathrm{tr}(T)I + 2\mu D_0^\star \quad \text{(by hypothesis)} \\ &= Q\left(\frac{1}{3}\mathrm{tr}(T)I + 2\mu D_0\right)Q^T = QTQ^T \end{aligned}$$

i.e. frame-indifference.

($\Rightarrow$) Suppose the response function $T = -\pi I + \mathcal{C}(L)$ is independent of observer. If $x, x^\star$ are related by a change of observer and $T^\star = -\pi^\star I + \mathcal{C}(L^\star)$, then $\mathrm{tr}(T) = \mathrm{tr}(T^\star) \Rightarrow \pi = \pi^\star$ since $\mathcal{C} : \mathrm{Lin} \to \mathrm{Sym}_0$. Thus,

$$T^\star = -\pi I + \mathcal{C}(L^\star) = -\pi I + \mathcal{C}\left(QLQ^T + \dot{Q}Q^T\right)$$

and frame-indifference requires

$$Q\mathcal{C}(L)Q^T = \mathcal{C}\left(QLQ^T + \dot{Q}Q^T\right) \tag{10}$$

since $\mathcal{C}$ is linear. We complete the proof in the following steps.

1. If $L \in \mathrm{Lin}$ is fixed and $F(t) = \exp(Lt)$, then set $x(t,p) = F(t)p$, so that

$$\dot{x} = \dot{F}p = L\exp(Lt)p = LFp = Lx \;\Rightarrow\; \nabla v = L$$

Thus, (10) holds for all (fixed) L and arbitrary Q, since we can define $x^\star(t,p) = Q(t)x(t,p) = QFp$.

2. Also, for L fixed, let $Q(t) = \exp(-Wt)$, where $W = \frac{1}{2}(L - L^T)$. Then $\dot{Q} = -WQ$ so $\dot{Q}Q^T = -W$ and $Q(0) = I$. Evaluating (10) with this choice of L and $Q(0)$ gives

$$\mathcal{C}(L) = \mathcal{C}(L - W) = \mathcal{C}(D) \quad \text{where } D = \frac{1}{2}(L + L^T)$$

i.e. $\mathcal{C}$ depends *only* upon the symmetric part of L.

3. Since $\mathrm{tr}(T) = \mathrm{tr}(T^\star)$, it follows that $T_0^\star = QT_0Q^T$, i.e.

$$\mathcal{C}(QDQ^T) = \mathcal{C}(D^\star) = \mathcal{C}(D)$$

We now make the **claim**:

$$\left.\begin{array}{r} \mathcal{C} : \mathrm{Sym} \to \mathrm{Sym} \\ \mathcal{C}(QDQ^T) = \mathcal{C}(D) \\ Q \in \mathrm{Orth} \\ \mathcal{C} \text{ linear} \end{array}\right\} \Rightarrow \mathcal{C}(D) = \lambda \mathrm{tr}(D) I + 2\mu D$$

for some scalars λ, μ. This claim then tells us $\mathrm{tr}(\mathcal{C}(D)) = 0 \Rightarrow \mathcal{C}(D) = 2\mu D_0$.

□

2.5 Isotropic Functions

Definition 2.16. *A function $\varphi : Lin \to \mathbb{R}$ is* isotropic *provided $\varphi(A) = \varphi(QAQ^T)$ for all $Q \in Orth$.*

A function $G : Lin \to Lin$ is isotropic provided $QG(A)Q^T = G(QAQ^T)$ for all $Q \in Orth$.

Theorem 2.17 (Representation of scalar-valued isotropic functions)**.** *A function $\varphi : \mathcal{A} \subseteq Sym \to \mathbb{R}$ is isotropic $\iff \exists \psi : \mathcal{I}_\mathcal{A} \to \mathbb{R}$ such that $\varphi(A) = \psi(I_A)$ where $I_A \in \mathbb{R}^d$ are the invariants of A and $\mathcal{I}_\mathcal{A} = \{I_A : A \in \mathcal{A}\}$.*

Proof. ($\Rightarrow$) It suffices to show that $\mathcal{I}_A = \mathcal{I}_B \Rightarrow \varphi(A) = \varphi(B)$. If $\mathcal{I}_A = \mathcal{I}_B$ then A and B have the same set of eigenvalues (since they have the same characteristic polynomial), call them $\{\omega_i\}$. Then write

$$A = \sum_i \omega_i e_i \otimes e_i \quad \text{and} \quad B = \sum_i \omega_i f_i \otimes f_i$$

where $\{e_i\}, \{f_i\}$ are orthonormal bases of eigenvectors for A and B, respectively. (Remember A, B are symmetric.) Let $Qe_i = f_i$ with $QQ^T = I$. Specifically, we can define $Q = \sum_i e_i \otimes f_i$. Then,

$$\begin{aligned} \varphi(A) = \varphi(QAQ^T) &= \varphi\left(\sum_i \omega_i Q(e_i \otimes e_i) Q^T\right) \\ &= \varphi\Big(\sum_i \omega_i \underbrace{Qe_i \otimes Qe_i}_{=f_i \otimes f_i}\Big) = \varphi(B) \end{aligned}$$

($\Leftarrow$) Note $I_A = I_{QAQ^T}$. □

2.5.1 Tensor-Valued Functions

Lemma 2.18. *Let $G : \mathcal{A} \subseteq Sym \to Lin$ be isotropic. Then each eigenvector of $A \in \mathcal{A}$ is an eigenvector of $G(A)$.*

Proof. Let $e = e_1$ be an eigenvector of $A \in \mathcal{A}$. By the Spectral Theorem, $\exists \{e_i\}_{i=1}^3$ an orthonormal basis of eigenvectors of A. Let

$$Q = -e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

so that $QQ^T = I$. **Claim**: $QAQ^T = A$. To see why, notice that

$$(QAQ^T)e_1 = QA(-e_1) = Q(-\lambda_1 e_1) = \lambda_1 e_1 = Ae_1$$
$$(QAQ^T)e_i = QA(e_i) = Q(\lambda_i e_i) = \lambda_i e_i = Ae_i \text{ , } i = 2, 3$$

i.e. $QAQ^T x = Ax$ for a basis, and hence for all x, which proves the claim. Next,

$$QG(A)Q^T = G(QAQ^T) = G(A)$$

and applying all of these tensors e_1, we have $QG(A)e_1 = -G(A)e_1$, so $Q(x) = -x$, i.e. Qx is parallel to (with opposite sign of) x, where $x = G(A)e_1$. It follows that $G(A)e_1$ is parallel to e_1, i.e. $G(A)e_1 = \omega e_1$. □

Lemma 2.19. *Let $A \in Sym$ and set $A = \sum_i \omega_i e_i \otimes e_i$ to be the spectrail decomposition.*

1. *If A has 3 dsitrinct eigenvalues, then $\{I, A, A^2\}$ are linearly independent and span$\{I, A, A^2\}$ = span$\{e_i \otimes e_i\}_{i=1}^3$.*
2. *If A has 2 distinct eigenvalues, then write $A = \omega_1 e \otimes e + \omega_2(I - e \otimes e)$. Then $\{I, A\}$ are linearly independent and span$\{I, A\}$ = span$\{e \otimes e\}$ = span$\{e \otimes e, I - e \otimes e\}$.*
3. *If A has 1 distinct eigenvalue, then $A = \lambda I$ for $\lambda \in \mathbb{R}$.*

Proof. 1. Suppose $\alpha A^2 + \beta A + \gamma I = [0]$. Multiply by e_i to get

$$(\alpha\omega_i^2 + \beta\omega_i + \gamma)e = 0 \Rightarrow p(\omega_i) = \alpha\omega_i^2 + \beta\omega_i + \gamma = 0 \text{ for } i = 1, 2, 3$$

i.e. $p(\omega)$ is a quadratic with 3 roots, which means $p(\omega) \equiv= 0$ and hence $\alpha = \beta = \gamma = 0$. Thus, $\{I, A, A^2\}$ are, indeed, linearly independent. Next,

$$A^\alpha = \sum_i \omega_i^\alpha (e_i \otimes e_i) \text{ for } \alpha = 0, 1, 2 \Rightarrow \{I, A, A^2\} = \text{span}\{e_i \otimes e_i\}_{i=1}^3$$

We know $\dim(e_i \otimes e_i) = 3$ and $\dim\{I, A, A^2\} = 3$ (since they're linearly independent), so the spaces must agree.

The other two statements are similar.

□

Theorem 2.20 (Representation of isotropic tensor-valued functions)**.** *The function $G : \mathcal{A} \subseteq Sym \to Sym$ is isotropic $\iff$ $\exists \varphi_0, \varphi_1, \varphi_2 : \mathcal{I}_\mathcal{A} \to \mathbb{R}$ isotropic such that*

$$G(A) = \varphi_0(I_A)I + \varphi_1(I_A)A + \varphi_2(I_A)A^2$$

Proof. ($\Leftarrow$) Suppose $G(A)$ takes the form shown. Then

$$\begin{aligned} G(QAQ^T) &= \varphi_0(I_{QAQ^T})I + \varphi_1(I_{QAQ^T})QAQ^T + \varphi_2(I_{QAQ^T})Q\overbrace{AQ^TQA}^{=A^2}Q^T \\ &= Q\left(\varphi_0(I_A)I + \varphi_1(I_A)A + \varphi_2(I_A)A^2\right)Q^T = QG(A)Q^T \end{aligned}$$

($\Rightarrow$) Suppose A has 3 distinct eigenvalues and write $A = \sum_i \omega_i e_i \otimes e_i$. We showed that $G(A)e_i = \beta_i(A)e_i$ for some $\beta_i(A)$ and since $G(A) \in \text{Sym}$, it follows that

$$G(A) = \sum_i \beta_i(A)e_i \otimes e_i = \alpha_0(A)I + \alpha_1(A)A + \alpha_2(A)A^2$$

Claim: $\alpha_i : \text{Sym} \to \mathbb{R}$ are isotropic. To see why, notice that

$$\begin{aligned} 0 = QG(A)Q^T - G(QAQ^T) &= \left(\alpha_0(A) - \alpha_0(QAQ^T)\right)I \\ &+ \left(\alpha_1(A) - \alpha_1(QAQ^T)\right)\underbrace{QAQ^T}_{=A} + \left(\alpha_2(A) - \alpha_2(QAQ^T)\right)\underbrace{QA^2Q^T}_{=A^2} \end{aligned}$$

Since A has three *distinct* eigenvalues, $\{I, A, A^2\}$ are linearly independent, so

$$\underbrace{\alpha_0(A) = \alpha_0(QAQ^T)}_{\text{isotropic scalar}} = \varphi_0(I_A)$$

The other two cases are similar. □

Remark 2.21. If A is invertible, then

$$A^3 + i_1A^2 + i_2A + i_1I = 0 \;\Rightarrow\; A^2 = -i_1 - i_2I - i_3A^{-1}$$

Thus, on invertible matrices,

$$G(A) = \psi_0(I_A)I + \psi_1(I_A)A + \psi_{-1}(I_A)A^{-1}$$

where $\psi_0 = \varphi_0 - i_2\varphi_2$, etc.

Corollary 2.22. *A* linear *function $G : Sym \to Sym$ is isotropic $\iff$ $G(A) = \lambda tr(A)I + 2\mu A$ for constants $\lambda, \mu \in \mathbb{R}$.*

Proof. ($\Leftarrow$) Trivial.
($\Rightarrow$) Let e be a unit vector and set $A = e \otimes e$. Then $\sigma(A) = \{0, 0, 1\}$ so $I_A = (1, 0, 0) = (i_1, i_2, i_3)$. Also, $A^2 = (e \otimes e)^2 = e \otimes e$. Thus,

$$\begin{aligned} G(e \otimes e) &= \varphi_0(1,0,0)I + \varphi_1(1,0,0)(e \otimes e) + \varphi_2(1,0,0)(e \otimes e) \\ &= \underbrace{\varphi_0(1,0,0)}_{:=\lambda}I + \underbrace{(\varphi_1(1,0,0) + \varphi_2(1,0,0))}_{:=2\mu}(e \otimes e) \end{aligned}$$

Given $A \in \text{Sym}$, write $A = \sum_i \omega_i e_i \otimes e_i$ and use the linearity of G to write

$$\begin{aligned} G(A) = \sum_i \omega_i G(e_i \otimes e_i) &= \lambda(\sum \omega_i) I + 2\mu \sum_i \omega_i e_i \otimes e_i \\ &= \lambda \text{tr}(A) I + 2\mu A \end{aligned}$$

□

Corollary 2.23. *1. A linear function $G : Sym_0 \to Sym$ is isotropic $\iff$ $G(A) = 2\mu A$ for some $\mu \in \mathbb{R}$.*

2. A linear function $G : Sym \to Sym_0$ is isotropic $\iff$ $G(A) = 2\mu(A - \frac{1}{3} tr(A) I)$.

Proof. 1. Given $G : \text{Sym}_0 \to \text{Sym}$ isotropic, define

$$\hat{G} : \text{Sym} \to \text{Sym} \quad \text{by} \quad \hat{G}(A) = G\left(A - \frac{1}{3}\text{tr}(A) I\right)$$

Then $\hat{G}(A) = G(A)$ for $A \in \text{Sym}_0$ and $\hat{G}(A) = \lambda \text{tr}(A) + 2\mu A$ for $\lambda, \mu \in \mathbb{R}$.

2. Exercise.

□

Theorem 2.24. *Let $\mathcal{U} \subseteq Lin$ be a linear subspace and let $\mathcal{A} \subseteq \mathcal{U}$ be open. Let $\mathcal{G} \subseteq Orth$ be any subset. Suppose $G : \mathcal{A} \to Lin$ is invariant under $\mathcal{G}$, i.e.*

$$G(QAQ^T) = QG(A)Q^T \quad \forall Q \in \mathcal{G}$$

Then

$$QDG(A)(U)Q^T = DG(QAQ^T)(QUQ^T) \quad \forall A \in \mathcal{A}, \forall U \in \mathcal{U}, \forall Q \in \mathcal{G}$$

Proof. Note: the definition of invariance requires $Q\mathcal{A}Q^T = \mathcal{A}$ for any $Q \in \mathcal{G}$. **Claim**: $Q\mathcal{U}Q^T = \mathcal{U}$. To see why, fix $U \in \mathcal{U}$; since $\mathcal{A} \subseteq \mathcal{U}$ is open then for any $A \in \mathcal{A}$, $\exists \varepsilon > 0$ such that $A + \varepsilon\mathcal{U} \subseteq \mathcal{A}$. Since $Q\mathcal{A}Q^T = \mathcal{A}$, then

$$\underbrace{QAQ^T}_{\in \mathcal{A} \subseteq \mathcal{U}} + \varepsilon QUQ^T \in \mathcal{A} \subseteq \mathcal{U} \;\Rightarrow\; QUQ^T \in \mathcal{U}$$

Next,

$$\begin{aligned} G\left(Q(A+U)Q^T\right) &= G\left(QAQ^T + QUQ^T\right) \\ &= G(QAQ^T) + DG(QAQ^T)(QUQ^T) + o(U) \\ &= QG(A)Q^T + DG(QAQ^T)(QUQ^T) + o(U) \end{aligned}$$

and

$$\begin{aligned} G\left(Q(A+U)Q^T\right) = QG(A+U)Q^T &= Q\left(G(A) + DG(A)(U) + o(U)\right)Q^T \\ &= QG(A)Q^T + QDG(A)(U)Q^T + o(U) \end{aligned}$$

so the last lines of these are equal. □

2.6 Navier-Stokes Equations

Classical Incompressible Navier-Stokes Fluid:

$$T = -\pi I + 2\mu D(v) \quad , \text{ where } D(v) = \frac{1}{2}(\nabla v + \nabla v^T), \text{ and } \operatorname{div}(v) = 0$$

The "homogeneous" case is where $\rho = \rho_0 =$const. and $\mu = \mu_0 =$const. Then

$$\rho_0 \dot{v} - \operatorname{div}(-\pi I + 2\mu D(v)) = \rho_0 f \;\Rightarrow\; \rho_0 \dot{v} + \nabla \pi - \mu \Delta v = \rho_0 f$$

Lemma 2.25. *Let v satisfy the Navier-Stokes Equations with conservative forces. Then*

1. $\dot{W} + D(v)W + WD(v) = \Delta W$ *with* $W = \frac{1}{2}(\nabla v - \nabla v^T)$.
2. *For any closed material curve,*

$$\frac{d}{dt}\oint_{c(t)} v \cdot dx = \nu \oint_{c(t)} \Delta v \cdot dx$$

   with $\nu = \frac{\mu}{\rho}$.

3. *In two dimensions,* $\dot{W} = \nu \Delta W$.

Proof. To prove (1), we write the N-S equations as

$$\dot{v} = \nu \nabla v + \nabla (F - \pi)$$

where $f = \nabla F$. Then

$$\nabla \dot{v} = \nu \Delta (\nabla v) + D^2(F - \pi)$$

where $D^2(\cdot)$ is the Hessian, so then

$$(\nabla \dot{v})_{\text{skew}} = \nu \Delta W + 0$$

Now,

$$(\nabla \dot{v}) = (\nabla v)^{\cdot} + \nabla v \nabla v$$

and so

$$(\nabla \dot{v})_{\text{skew}} = (\nabla v)^{\cdot}_{\text{skew}} + \frac{1}{2}(\nabla v \nabla v - \nabla v^T \nabla v^T) = \nu \Delta W$$

Note that

$$\begin{aligned} DW + WD &= \frac{1}{4}(\nabla v + \nabla v^T)(\nabla v - \nabla v^T) + \frac{1}{4}(\nabla v - \nabla v^T)(\nabla v + \nabla v^T) \\ &= \frac{1}{2}(\nabla v \nabla v - \nabla v^T \nabla v^T) \end{aligned}$$

The proof of (2) is left as an exercise; the trick is to show

$$\frac{d}{dt}\oint_{c(t)} v \cdot dx = \oint_{c(t)} \dot{v} \cdot dx$$

so then
$$\frac{d}{dt}\oint_{c(t)} v\cdot dx = \oint_{c(t)} (\nu\Delta v + \nabla(F-\pi))\cdot dx = \oint_{c(t)} \nu\Delta v$$
To prove (3), note that in 2D
$$W = \pm w \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
so
$$WD + DW = \pm w \operatorname{div}(v) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0$$
□

Claim: $\omega_i = \frac{1}{2}\varepsilon_{ijk}W_{kj}$. **Proof**:
$$\begin{aligned}(\omega\times a)_i &= \frac{1}{2}\varepsilon_{ijk}\omega_j a_k = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{jmn}W_{nm}a_k \\ &= \frac{1}{2}(W_{ik}a_k - W_{ki}a_k) = (Wa)_i\end{aligned}$$

2.6.1 Stability/Comparison of Solutions

Suppose
$$\begin{aligned}\dot v_1 - \operatorname{div}(-p_1 I + 2\nu D(v_1)) = f_1 \\ \dot v_2 - \operatorname{div}(-p_2 I + 2\nu D(v_2)) = f_2\end{aligned}$$
with $\operatorname{div}(v_1) = \operatorname{div}(v_2) = 0$ and $v_1 \restriction_{\partial\Omega} = v_2 \restriction_{\partial\Omega}$, and assume $\Omega \subset \mathbb{R}^d$ is bounded. Write $v = v_2 - v_1$, so that $v\restriction_{\partial\Omega} = 0$, and $p = p_2 - p_1$ and $f = f_2 - f_1$. Subtracting the two equations yields
$$v_t + (v_2\cdot\nabla)v_2 - (v_1\cdot\nabla)v_1 - \operatorname{div}(-pI + 2\nu D(v)) = f$$
Take the dot product with w which vanishes on $\partial\Omega$, yielding
$$\int v_t\cdot w - p\operatorname{div}(w) + 2\nu D(v) : D(w) = \int_\Omega f\cdot w - [(v_2\cdot\nabla)v_2 - (v_1\cdot\nabla)v_1]\cdot w$$
Put $w = v$, and recall $\operatorname{div} v = 0$. Then
$$\frac{d}{dt}\int_\Omega \frac{|v|^2}{2} + \int_\Omega 2\nu|D(v)|^2 = \int_\Omega f\cdot v - [(v_2\cdot\nabla)v_2 - (v_1\cdot\nabla)v_1]\cdot v$$
We write
$$\begin{aligned}(v_2\cdot\nabla)v_2 - (v_1\cdot\nabla)v_1 &= ((v_2-v_1)\cdot\nabla)\,v_2 + (v_1\cdot\nabla)(v_2-v_1) \\ &= (v\cdot\nabla)v_2 + (v_1\cdot\nabla)v \\ &= (v\cdot\nabla)v_2 + (v_2\cdot\nabla)v - (v\cdot\nabla)v\end{aligned}$$

so now we have

$$
\begin{aligned}
\int_\Omega ((v\cdot\nabla)v_2 - (v_1\cdot\nabla)v_1)\,v &= \int_\Omega ((v\cdot\nabla)v_2 + (v_2\cdot\nabla)v)\cdot v - v\cdot\nabla\left(\frac{|v|^2}{2}\right)\\
&= \int_\Omega ((v\cdot\nabla)v_2 + (v_2\cdot\nabla)v)\cdot v + \left(\frac{|v|^2}{2}\right)\operatorname{div} v\\
&\le \|\nabla v_2\|_\infty\|v\|^2 + \|v_2\|_\infty\|\nabla v\|\|v\|
\end{aligned}
$$

where $\|\cdot\| = \|\cdot\|_{L^2}$. Returning to the equation above, we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + 2\nu\|D(v)\|^2 \le \|f\|\|v\| + \|\nabla v_2\|_\infty\|v\|^2 + \|v_2\|_\infty\|\nabla v\|\|v\|$$

We now apply the inequality

$$\|f\|\|v\| \le \frac{1}{2}\|f\|^2 + \frac{1}{2}\|v\|^2$$

and Korn's Inequality and Young's ε-inequality with $\varepsilon = 2\nu$

$$\|v_2\|_\infty\|\nabla v\|\|v\| \le C\|v_2\|_\infty\|v\|\|D(v)\| \le \frac{1}{4\nu}\left(C_K\|v_2\|_\infty\|v\|\right)^2 + \nu\|D(v)\|^2$$

Using these in the line above and absorbing terms, we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu\|D(v)\|^2 \le \frac{1}{2}\|f\|^2 + \frac{1}{2}C\|v\|^2$$

where, for completeness, we note

$$C = 1 + \frac{C_k^2}{2\nu}\|v_2\|_\infty^2 + 2\|\nabla v_2\|_\infty$$

Multiply through by e^{-ct} to write

$$\frac{d}{dt}\left(\mathrm{e}^{-ct}\|v\|^2\right) + \mathrm{e}^{-ct}\nu\|D(v)\|^2 = \mathrm{e}^{-ct}\|f\|^2$$

and then

$$\mathrm{e}^{-ct}\|v(t)\|^2 + \nu\int_0^t \mathrm{e}^{-cs}\|D(v(s))\|^2\,ds \le \|v(0)\|^2 + \int_0^t \mathrm{e}^{-cs}\|f(s)\|^2\,ds$$

Thus, if

- $\|\nabla v_2\|_\infty, \|v_2\|_\infty < +\infty$ (assumed) and
- $\|v_1(t)\| \le C$ and $\int_0^T \|D(v)\|^2 < \infty$ (from the PDE)

then for $f_2 = f_1$ and $v_1(0) = v_2(0)$, it follows that

$$\mathrm{e}^{-ct}\|(v_2 - v_1)(t)\|^2 + \int_0^t \nu\mathrm{e}^{-cs}\|D(v_2 - v_1)(s)\|^2\,ds \le 0$$

i.e. $v_2(t) = v_1(t)$. Notice how we have *assumed* $\|v_2\|, \|\nabla v_2\| < \infty$; being able to prove this would answer a million dollar question!

3 Elastic Materials

Definition 3.1. *An elastic body is one for which the stress at each point $p \in \mathcal{B}_r$ takes the form $T(t,x) = \hat{T}(F(t,p),p)$ where $x = \mathcal{X}(t,p)$.*

Proposition 3.2. *An elastic response function $\hat{T} : Lin^+ \to Sym$ is independent of observer $\iff Q\hat{T}(F)Q^T = \hat{T}(QF)$ for all $F \in Lin^+$ and $Q \in Orth$.*

Proof. Recall that if

$$x^\star = y(t) + Q(t)(x - 0) \qquad \text{for } Q(t) \in \text{Orth}$$

then T is independent of observer $\iff T^\star = QTQ^T$ and $F^\star = QF$. Then we know

$$T^\star = QTQ^T \iff \hat{T}(F^\star) = Q\hat{T}(F)Q^T \iff \hat{T}(QF) = Q\hat{T}(F)Q^T$$

□

Recall: We write the polar decomposition of F as $F = RU$ where $R \in \text{Orth}^+$ and $U \in \text{Sym}^+$. Also, $U = C^{1/2} = (F^TF)^{1/2}$.

Corollary 3.3. *The response function of an elastic material is determined by restriction to Sym^+. Specifically, if $F = RU$ then*

$$\hat{T}(F) = \hat{T}(RU) = R\hat{T}(U)R^T$$

Moreover, there are functions $T_1, T_2, T_3 : Sym^+ \to Sym$ such that

$$\begin{aligned} \hat{T}(F) &= FT_1(U)F^T \\ \hat{T}(F) &= RT_2(C)R^T \\ \hat{T}(F) &= FT_3(C)F^T \end{aligned}$$

Proof. Write $F = RU$, so

$$\hat{T}(F) = \hat{T}(RU) = R\hat{T}(U)R^T = F\overbrace{U^{-1}\hat{T}(U)U^{-1}}^{:=T_1(U)}F^T \equiv FT_1(U)F^T$$

and

$$\hat{T}(F) = R\hat{T}(U)R^T = R\overbrace{\hat{T}(C^{1/2})}^{:=T_2(C)}R^T \equiv RT_2(C)R^T$$

and

$$\hat{T}(F) = RT_2(C)R^T = F\overbrace{U^{-1}T_2(C)U^{-1}}^{:=T_3(C)}F^T \equiv FT_3(C)F^T$$

knowing that $U^{-1} = C^{-1/2}$. □

3.1 Material Symmetry

Consider conducting Experiment 1

$$x_1 = p_0 + F(p - p_0) \quad , \quad \nabla x_1 = F$$

and then somebody rotates your symmetric material by Q and you conduct Experiment 2

$$x_2 = p_0 + FQ(p - p_0) \quad \text{(total deformation)}$$

If for *some* $Q \in \text{Orth}^+$ we have that $\hat{T}(F) = \hat{T}(FQ)$, then we call Q a *symmetry transformation* (at $p_0 \in \mathcal{B}_r$).

Lemma 3.4. *Let $\hat{T}$ be a (frame-indifferent) elastic response function. Then*

$$G_p = \left\{ Q \in Orth^+ : \hat{T}(F) = \hat{T}(FQ)\ \forall F \in Lin^+ \right\}$$

is a subgroup *of $Orth^+$.*

Proof. Clearly, $I \in G_p$. Next, if $Q \in G_p$ then selecting $F \sim FQ^{-1}$ shows that $\hat{T}(FQ^{-1}) = \hat{T}(FQ^{-1}Q) = \hat{T}(F)$ for all F, so $Q^{-1} \in G_p$. Finally, if $Q, R \in G_p$ then $\hat{T}(F) = \hat{T}(FQ) = \hat{T}((FQ)R) = \hat{T}(F(QR))$ so $QR \in G_p$. □

Recall: We say $\hat{T} : \text{Lin} \to \text{Lin}$ is *invariant* under $Q \in \text{Orth}$ if $\hat{T}(QFQ^T) = Q\hat{T}(F)Q^T$.

Lemma 3.5. *Let $\hat{T}$ be an elastic response function. Then $\hat{T}$ is invariant under G_p, as are T_1, T_2, T_3 (as defined above).*

Proof. Let $Q \in G_p$. Then

$$\hat{T}(QFQ^T) = \hat{T}(QF) = Q\hat{T}(F)Q^T$$

by the facts that $Q^T \in G_p$ and $\hat{T}$ is independent of observer, respectively. Next,

$$\begin{aligned} T_1(QUQ^T) &= (QUQ^T)^{-1}\hat{T}(QUQ^T)(QUQ^T)^{-1} \\ &= (QU^{-1}Q^T)Q\hat{T}(U)Q^T(QU^{-1}Q^T) = QU^{-1}\hat{T}(U)U^{-1}Q^T = QT_1(U)Q^T \end{aligned}$$

The other two are similar. □

Definition 3.6. *An elastic material/reponse function is* isotropic *if $G_p = Orth^+$.*

Recall the notation $F = RU = VR$ and $C = F^TF = U^2$ and $B = FF^T = V^2$. Then

$$\hat{T}(F) = RT_2(F^TF)R^T = T_2(RF^TFR^T) = T_2(VV) = T_2(B) = T_2(FF^T)$$

if $\hat{T}$ is isotropic. Remember $T_2 : \text{Sym}^+ \to \text{Sym}$ and it is isotropic if $G_p = \text{Orth}^+$.

Corollary 3.7. *The reponse of an elastic isotropic material (at $p \in \mathcal{B}_r$) takes the form*

$$T = \beta_0(\mathcal{I}_B)I + \beta_1(\mathcal{I}_B)B + \beta_{-1}(\mathcal{I}_B)B^{-1}$$

where $B = FF^T$ and $\mathcal{I}_B = \{\iota_0(B), \iota_1(B), \iota_2(B)\}$ are the invariants and $\beta_i : \mathbb{R}^3 \to \mathbb{R}$.

Moving on,

$$\int_{\partial \mathcal{P}} v \cdot n \, da(x) = \int_{\partial \mathcal{P}_r} v_r \cdot \operatorname{cof}(F) n_r \, da(p)$$

Note $n = C \operatorname{cof}(F) n_r \approx C F^{-T} n_r$, so $n = \frac{F^{-T} n_r}{|F^{-T} n_r|}$.

Piola Stress: Also known as Piola-Kirchoff or 1st Piola Stress.

$$\int_{\partial \mathcal{P}} Tn \, da(x) = \int_{\partial \mathcal{P}_r} T \operatorname{cof}(F) n_r \, da(p)$$

Definition 3.8. *The Piola stress is $s = T \operatorname{cof}(F) = \det(F) T F^{-T}$. So $T = \frac{1}{\det(F)} S F^T$.*

Make the change of variables $x = \mathcal{X}(t.p)$, $dx = \det(F) \, dp$ in the balance of linear momentum

$$\frac{d}{dt} \int_{\mathcal{P}(t)} \rho v = \int_{\mathcal{P}(t)} \rho f + \int_{\partial \mathcal{P}(t)} Tn \, da$$

and recall that $\rho(t,x) \det(F(t,p)) = \rho_r(p)$. We obtain

$$\frac{d}{dt} \int_{\mathcal{P}_r} \rho_r \dot{x} = \int_{\mathcal{P}_r} f(t, x(t,p)) + \int_{\partial \mathcal{P}(r)} S n_r$$

and this is rewritten as

$$\int_{\mathcal{P}_r} \rho_r \ddot{x} - \operatorname*{div}_p(S) = \int_{\mathcal{P}_r} f$$

Localizing tells us

$$\rho_r \ddot{x} - \operatorname*{div}_p(S) = f$$

in $(0,T) \times \mathcal{B}_r$ where $f = f(t, x(t,p))$.

The balance of angular momentum: $T = T^T$, $\frac{1}{J} S F^T = \frac{1}{J} (S F^T)^T$ where $J = \det(F)$, then $SF^T = FS^T$; i.e. S is *not* symmetric.

Energy Estimate: Take the dot product of the linear monetum equation with $\dot{x}$ and integrate by parts:

$$\frac{d}{dt} \int_{\mathcal{P}_r} \rho_r \frac{|\dot{x}|^2}{2} + \int_{\mathcal{P}_r} S : \nabla_p \dot{x} = \int_{\mathcal{P}_r} f \cdot \dot{x} + \int_{\partial \mathcal{P}_r} S n_r \cdot \dot{x}$$

$$\frac{d}{dt} \int_{\mathcal{P}_r} \rho_r \frac{|\dot{x}|^2}{2} + \int_{\mathcal{P}_r} S : \dot{F} = \int_{\mathcal{P}_r} f \cdot \dot{x} + \int_{\partial \mathcal{P}_r} S n_r \cdot \dot{x}$$

Recall: $\dot{F} = \nabla v F$ and $S = \det(F) T F^{-T}$, so $S : \dot{F} = \det(F) T F^{-T} : \nabla v F$. Thus,

$$\int_{\mathcal{P}_r} S : \dot{F} = \int_{\mathcal{P}} T : \nabla v$$

Independence of Observer: Recall, if $T = \hat{T}(F)$, then $Q\hat{T}(F)Q^T = \hat{T}(QF)$ for $Q \in \text{Orth}^+$. Define

$$\hat{S}(F) = \det(F)\hat{T}(F)F^{-T}$$

Then

$$\begin{aligned}
\hat{S}(QF) &= \det(QF)\hat{T}(QF)(QF)^{-T} \\
&= \det(F)Q\hat{T}(F)Q^T Q F^{-T} \\
&= \det(F)Q\hat{T}(F)F^{-T} = Q\hat{S}(F)
\end{aligned}$$

Thus, $S = \hat{S}(F)$ is frame-indifferent $\iff$ $\hat{S}(QF) = Q\hat{S}(F)$.

Recall: If $C = F^T F$, then $\hat{T}(F) = F T_3(C) F^T$. Then

$$\begin{aligned}
\hat{S}(F) &= \det(F)\hat{T}(F)F^{-T} = \det(F) F T_3(C) \\
&= F\left(\sqrt{\det(C)}T_3(C)\right) = F S_3(C)
\end{aligned}$$

Then $SF^T = FS^T$ implies

$$F S_3(C) F^T = F S_3(C)^T F^T$$

i.e. $S_3(C) = S_3(C)^T$. Thus, $S_3 : \text{Sym}^+ \to \text{Sym}$.

3.2 Hyperelastic Bodies

Motivation: Suppose $f = 0$ and $Sn_r = 0$. Then

$$\int_{\mathcal{P}_r} \rho_r \frac{|\dot{x}(t)|^2}{2} + \underbrace{\int_0^t \int_{\mathcal{P}_r} S : \dot{F}}_{\geq 0} = \int_{\mathcal{P}_r} \rho_r \frac{|\dot{x}(0)|^2}{2}$$

i.e. we expect things to *slow down.*

Definition 3.9. *A (mechanical) process* (x, T, f) *(or* (x, S, f)*) is* closed *on* $[t_0, t_1]$ *if* $x(t_0) = x(t_1)$ *and* $\dot{x}(t_0) = \dot{x}(t_1)$.

For a closed process on $[t_0, t_1]$,

$$\int_{t_0}^{t_1} \int_{\mathcal{P}_r} S : \dot{F} = \int_{t_0}^{t_1} \int_{\mathcal{P}_r} f\dot{x} + \int_{t_0}^{t_1} \int_{\partial\mathcal{P}_r} Sn_r \cdot \dot{x}$$

or

$$\int_{t_0}^{t_1} \int_{\mathcal{P}(t)} T : \nabla v = \int_{t_0}^{t_1} \int_{\mathcal{P}(t)} \rho f \dot{x} + \int_{t_0}^{t_1} \int_{\partial\mathcal{P}(t)} Tn \cdot v$$

Definition 3.10. *The work is nonnegative in a closed process if for every part $\mathcal{P}_r \subseteq \mathcal{B}_r$,*

$$\int_{t_0}^{t_1} \int_{\mathcal{P}_r} S : \dot{F} \geq 0$$

for every closed process.

Note: localize, then this is equivalent to

$$\int_{t_0}^{t_1} S(t,p) : \dot{F}(t,p) \geq 0 \quad \forall p \in \mathcal{B}_r$$

Definition 3.11. *An elastic body is* hyperelastic *if there exists a (strain energy) function $\hat{\sigma} : Lin^+ \times \mathcal{B}_r \to \mathbb{R}$ such that*

$$\hat{S}(F,p) = D\hat{\sigma}(F,p)$$

i.e. $S_{i\alpha} = \frac{\partial \hat{\sigma}}{\partial F_{i\alpha}}$.

Theorem 3.12. *An elastic body is hyperelastic $\iff$ the work is nonnegative for every closed process.*

Proof. ($\Rightarrow$) Notice that

$$\frac{d}{dt}\hat{\sigma}(F) = \frac{\partial \hat{\sigma}}{\partial F_{i\alpha}} \frac{\partial F_{i\alpha}}{\partial t} = D\hat{\sigma} : \dot{F} = \hat{S}(F) : \dot{F}$$

Then,

$$[\hat{\sigma}(F)]_{t_0}^{t_1} = \int_{t_0}^{t_1} \hat{S}(F) : \dot{F}$$

and the LHS is zero for a closed process. Since $x(t_0,p) = x(t_1,p)$ implies $F(t_0) = \nabla_p x(t_0) = \nabla_p x(t_1) = F(t_1)$, then we're done.

($\Leftarrow$) Assume nonnegativity of work during closed processes.

Step 1: Let $F : [t_0,t_1] \to \mathrm{Lin}^+$ be smooth and satisfy $F(t_0) = F(t_1)$ and $\dot{F}(t_0) = \dot{F}(t_1)$. Then

$$\int_{t_0}^{t_1} \hat{S}(F(t)) : \dot{F}(t)\, dt = 0$$

Proof: Define $x(t,p) = p_0 + F(t)(p - p_0)$, so $\nabla x(t) = F(t)$ and $x(t_0) = x(t_1)$ and $\dot{x}(t_0,p) = \dot{F}(t_0)p = \dot{F}(t_1)p = \dot{x}(t_1,p)$. Thus, x is closed on $[t_0,t_1]$, which implies

$$\int_{t_0}^{t_1} \hat{S}(F) : \dot{F}\, dt \geq 0$$

Next, define the "reversal", $x^\star(t,p) = p_0 + F(t_0+t_1-t)(p-p_0)$. Then $\nabla x^\star(t,p) = F(t_0+t_1-t)$ and

$$x^\star(t_0,p) = p_0 + F(t_1)(p-p_0) = p_0 + F(t_0)(p-p_0) = x^\star(t_1,p)$$

Similarly, $\dot{x}^\star(t_0) = \dot{x}^\star(t_1)$. Then,

$$(\nabla x^\star)^{\cdot}(t) = -\dot{F}(t_0 + t_1 - t)$$

and so

$$-\int_{t_0}^{t_1} \hat{S}(F) : \dot{F}\, dt = \int_{t_0}^{t_1} \hat{S}(\nabla x^\star) : (\nabla x^\star)^{\cdot}(t_0 + t_1 - t)\, dt$$

Change variables by letting $t^\star = t_0 + t_1 - t$, so $dt^\star = -dt$. Then the RHS above is

$$\int_{t^\star = t_1}^{t^\star = t_0} \hat{S}(\nabla x^\star) : (\nabla x^\star)^{\cdot}(t^\star)(-dt^\star) = \int_{t_0}^{t_1} \cdots\, dt^\star \geq 0$$

This proves the claim from Step 1.
Step 2: Let $F : [t_0, t_1] \to \mathrm{Lin}^+$ be continuous and piecewise smooth and satisfy $F(t_0) = F(t_1) \equiv A$. Then,

$$\int_{t_0}^{t_1} \hat{S}(F) : \dot{F} = 0$$

Proof sketch: Extend the domain of F to $\mathbb{R}$ by $F(t) = A$ for $t \notin [t_0, t_1]$, so $F : \mathbb{R} \to \mathrm{Lin}^+$ is continuous. Mollify F to obtain a smooth function $F_\varepsilon = F \star \varphi_\varepsilon$. Note $F_\varepsilon(t) = A$ on $\mathbb{R} \setminus [t_0 - \varepsilon, t_1 + \varepsilon]$, so $\dot{F}_\varepsilon = 0$ off $[t_0 - \varepsilon, t_1 + \varepsilon]$. We know $F_\varepsilon \to F$ uniformly on $\mathbb{R}$ and $\dot{F}_\varepsilon \to \dot{F}$ in $L^1(\mathbb{R})$. Since we assume $\hat{S}$ is continuous, then

$$\int_{t_0}^{t_1} \hat{S}(F) : \dot{F} = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \hat{S}(F_\varepsilon) : \dot{F}_\varepsilon = \lim_{\varepsilon \to 0} \int_{t_0 - \varepsilon}^{t_1 + \varepsilon} \hat{S}(F_\varepsilon) : \dot{F}_\varepsilon$$

but $F_\varepsilon(t_0 - \varepsilon) = A = F_\varepsilon(t_1 + \varepsilon)$ and $\dot{F}_\varepsilon(t_0 - \varepsilon) = 0 = \dot{F}_\varepsilon(t_1 + \varepsilon)$ and F_ε is smooth, so Step 1 is applicable, and taking a limit in ε tells us what we want.
Step 3: Construct $\sigma : \mathrm{Lin}^+ \to \mathbb{R}$. Given $F \in \mathrm{Lin}^+$, let $\tilde{F} : [0,1] \to \mathrm{Lin}^+$ be a smooth curve satisfying $\tilde{F}(0) = I$ and $\tilde{F}(1) = F$, and define

$$\sigma(F) := \int_0^1 \hat{S}(\tilde{F}) : (\tilde{F})^{\cdot}\, dt$$

First, $\sigma(F)$ is "well-defined" since if $\tilde{\tilde{F}}$ is another path for which $\tilde{\tilde{F}}(1) = F$ and $\tilde{\tilde{F}}(0) = I$ then

$$\int_0^1 \hat{S}(\tilde{F}) : (\tilde{F})^{\cdot}\, dt - \int_0^1 \hat{S}(\tilde{\tilde{F}}) : (\tilde{\tilde{F}})^{\cdot}\, dt = \int_0^2 \hat{S}(P) : \dot{P}\, dt$$

where $P : [0,2] \to \mathrm{Lin}^+$ is the map $\tilde{F}$ followed by $\tilde{\tilde{F}}$ reversed. Then $P(0) = P(2) = I$ so $\int \hat{S}(P) : \dot{P} = 0$ and thus

$$\int_0^1 \hat{S}(\tilde{F}) : (\tilde{F})^{\cdot}\, dt = \int_0^1 \hat{S}(\tilde{\tilde{F}}) : (\tilde{\tilde{F}})^{\cdot}\, dt$$

To compute $\frac{\partial \hat{\sigma}}{\partial F_{i\alpha}}$, let

$$J_{j\beta} = \begin{cases} 0 & \text{if } (j,\beta) \neq (i,\alpha) \\ 1 & \text{if } (j,\beta) = (i,\alpha) \end{cases}$$

Then
$$\frac{\partial\hat{\sigma}}{\partial F_{i\alpha}} = \lim_{\varepsilon\to 0}\frac{\sigma(F+\varepsilon J)-\sigma(F)}{\varepsilon}$$
Let P be any smooth path from I to F in Lin^+. Then,
$$\sigma(F+\varepsilon J) = \int_0^1 \hat{S}(P) : \dot{P} + \int_0^\varepsilon \hat{S}(F+tJ) : J\,dt = \sigma(F) + \int_0^\varepsilon \hat{S}_{i\alpha}(F+tJ)\,dt$$
Then
$$\frac{\partial\hat{\sigma}}{\partial F_{i\alpha}} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon S_{i\alpha}(F+tJ)\,dt = \hat{S}_{i\alpha}(F)$$
since $\hat{S}(\cdot)$ is continuous and the quantity inside the limit is just the average value of $\hat{S}_{i\alpha}$ on $[F, F+\varepsilon J]$. $\square$

Recall
$$\frac{d}{dt}\int_{\mathcal{P}_r}\rho_r\frac{|\dot{x}|^2}{2} + \int_{\mathcal{P}_r}\hat{S}(F):\dot{F} = \int_{\mathcal{P}_r} f\cdot\dot{x} + \int_{\partial\mathcal{P}_r} Sn_r\cdot\dot{x}$$
which we write as
$$\frac{d}{dt}\int_{\mathcal{P}_r}\Big(\underbrace{\rho_r\frac{|\dot{x}|^2}{2}}_{\text{kinetic energy}} + \underbrace{\sigma(F)}_{\text{stored elastic energy}}\Big) = \int_{\mathcal{P}_r} f\cdot\dot{x} + \int_{\partial\mathcal{P}_r} Sn_r\cdot\dot{x}$$
In particular, if $f=0$ and either $\dot{x}=0$ on $\partial\mathcal{P}_r$ or $Sn_r=0$ on $\partial\mathcal{P}_r$, then
$$\int_{\mathcal{P}_r}\rho_r\frac{|\dot{x}|^2}{2} + \sigma(\nabla_p x) = \text{ const.}$$

3.3 Independence of Observer

Recall $\hat{S}(QF) = Q\hat{S}(F)$ for $Q\in\mathrm{Orth}^+$. Then
$$\frac{\partial}{\partial F_{i\alpha}}\sigma(QF) = \frac{\partial\sigma}{\partial F_{j\beta}}\frac{\partial(QF)_{j\beta}}{\partial F_{i\alpha}} = D\sigma(QF)_{j\beta}Q_{ji}\delta_{\alpha\beta} = Q^T D\sigma(QF)$$
so $D_F\sigma(QF) = Q^T(D\sigma)(QF) \iff QD_F\sigma(QF) = (D\sigma)(QF)$. Given $Q=\mathrm{e}^W$, let $Q(T)=\mathrm{e}^{tW}$, so $\dot{Q}=WQ$. Then
$$\begin{aligned}\sigma(QF)-\sigma(I) &= \int_0^1\frac{d}{dt}\sigma(Q(t)F) = \int_0^1 (D\sigma)(Q(t)F) : (Q(t)F)^{\cdot} \\ &= \int_0^1 (D\sigma)(QF) : WQF = \int_0^1 (D\sigma)(QF)(QF)^T : W \\ &= \int_0^1 \underbrace{\hat{S}(Q(t)F)(Q(t)F)^T}_{\text{if BAM holds}} : W\end{aligned}$$
Now, $\hat{S}F^T = (\hat{S}F^T)^T$ and $W=-W^T$ is skew, so $\hat{S}(QF)(QF)^T : W = \mathrm{sym} : \mathrm{skew} = 0$, and thus $\sigma(QF)=\sigma(F)$.

Lemma 3.13. *The response function of a hyperelastic material is independent of observer* $\iff$ $\sigma(QF) = \sigma(F)$ *for every* $Q \in$ *Orth*$^+$*.*

Writing $C = F^T F$, then

$$\tilde{\sigma}(Q^T C Q) = \tilde{\sigma}((FQ^T)(FQ)) = \hat{\sigma}(FQ)$$

and material symmetry under Q (i.e. Q is in the symmetric group of the material) implies that

$$\tilde{\sigma}(Q^T C Q) = \hat{\sigma}(FQ) = \hat{\sigma}(F) = \tilde{\sigma}(C)$$

Thus, if the material is isotropic, i.e. $\hat{\sigma}(FQ) = \hat{\sigma}(F)$ for all $Q \in \text{Orth}^+$ or $\tilde{\sigma}(Q^T C Q) = \tilde{\sigma}(C)$ for all such Q, then $\tilde{\sigma}(C) = \tilde{\tilde{\sigma}}(\mathcal{J}_C)$.

Write

$$S = 2F\tilde{\tilde{\sigma}}(F^T F) = 2F(\sigma_1 D_{\iota_1}(C) + \sigma_2 D_{\iota_2}(C) + \sigma_3 D_{\iota_3}(C))$$

where

$$\sigma_p(\iota_1, \iota_2, \iota_3) = \frac{\partial}{\partial \iota_p} \tilde{\tilde{\sigma}}(\iota_1, \iota_2, \iota_3) \quad \text{for } p = 1, 2, 3$$

Now, we calculate these derivatives:

1. $\iota_1(C) = \text{tr}(C)$, so $D_{\iota_1}(C) = I$.
2. $\iota_2(C) = \frac{1}{2}(\text{tr}(C)^2 - \text{tr}(C^2)) = \text{tr}(C)^2 - |C|^2$ since $C = C^T$, so $D_{\iota_2}(C) = \text{tr}(C)I - C$.
3. $\iota_3(C) = \det(C)$, so $D_{\iota_3}(C) = \text{cof}(C) = \det(C)C^{-1}$.

Thus,

$$S = 2F\left(\sigma_1 I + \sigma_2(\text{tr}(C)I - C) + \sigma_3 \det(C)C^{-1}\right)$$

and

$$\hat{S}(F) = 2\left((\sigma_1 + \sigma_2|F|^2)F - \sigma_2 F F^T F + \sigma_3 \det(F)^2 F^{-T}\right)$$

Let $T = \frac{1}{\det(F)} S F^T$. Then, using $FF^T = B$, we have

$$T = \frac{1}{\sqrt{\det(B)}}\left((\sigma_1 + \sigma_2 \text{tr}(B))B - \sigma_2 B^2 + \sigma_3 \det(B) I\right)$$

Note: $\mathcal{J}_C \equiv \mathcal{J}_B$. Thus, we think of $\sigma_p = \sigma_p(\mathcal{J}_B)$ when computing T and $\sigma_p = \sigma_p(\mathcal{J}_C)$ when computing S. Or, notice

$$\begin{aligned}
\iota_1(C) &= \text{tr}(C) = |F|^2 \\
\iota_2(C) &= \frac{1}{2}\left(\text{tr}(C)^2 - \text{tr}(C^2)\right) = \frac{1}{2}\left(|F|^4 - |F^T F|^2\right) \\
\iota_3(C) &= \det(F)^2
\end{aligned}$$

Example 3.14. Let

$$\hat{\sigma}(F) = \frac{1}{2}\left(\alpha(|F|^2 - |I|^2) - \beta \ln[\det(F)^2]\right)$$

Then $\hat{S}(F) = D\hat{\sigma}(F) = \alpha F - \beta F^{-T}$. So then

$$\begin{aligned} T &= \frac{1}{\det(F)} S F^T = \frac{1}{\det(F)}(\alpha F F^T - \beta I) \\ &= \frac{1}{\sqrt{\det(B)}}(\alpha B - \beta I) \end{aligned}$$

Frequently in applications, $\alpha = \beta$, so that T vanishes when $B = I$, otherwise there is a "residual" stress. Also,

$$\begin{aligned} \tilde{\sigma}(C) &= \frac{1}{2}\left(\alpha(\mathrm{tr}(C) - \mathrm{tr}(I)) - \beta \ln(\det(C))\right) \\ &= \frac{1}{2}\left(\alpha(\pi_1 - 1) + \alpha(\pi_2 - 1) + \alpha(\pi_3 - 1) - \beta \ln(\pi_1) - \beta \ln(\pi_2) - \beta \ln(\pi_3)\right) \\ &= \sum_{\lambda} \alpha(\pi - 1) - \beta \ln(\lambda) \end{aligned}$$

which is a convex function of the λs, and where $\lambda_i = \lambda_i(C)$.

3.4 Linear Elasticity

If $\hat{S}(F) = \hat{S}(p, F)$, then

$$\hat{S}(F) = \hat{S}(I) + D\hat{S}(I)(F - I) + O(|F - I|^2)$$

where

$$D\hat{S}(I)(F - I)|_{i\alpha} = \sum_{j\beta} \frac{\partial \hat{S}_{i\alpha}(I)}{\partial F_{j\beta}}(F - I)_{j\beta} = \sum_{j\beta} C_{i\alpha j\beta}(F - I)_{j\beta}$$

Define $C : \mathrm{Lin} \to \mathrm{Lin}$ by $C(H) = D\hat{S}(I)(H)$. Note $C(H)_{i\alpha} = C_{i\alpha j\beta} H_{j\beta}$ (sum on $j\beta$, using Einstein notation).

Remark 3.15.

- C is called the elasticity tensor (at p).
- $\hat{S}(I)$ is called the "residual" stress.

Lemma 3.16. *For an elastic material, $T = \hat{T}(F)$. Then*

$$D\hat{T}(I)(H) = \hat{S}(I)\left(-tr(H)I + H^T\right) + C(H)$$

Proof. Recall $\hat{T}(F) = \frac{1}{\det(F)}\hat{S}(F)$, and $D\det(F)(H) = \det(F)(F^{-T} : H)$, so then

$$\begin{aligned} D\hat{T}(F)(H) &= -\frac{1}{\det(F)^2} D(\det(F))(H)\hat{S}(F)F^T \\ &\quad + \frac{1}{\det(F)} D\hat{S}(F)(H)F^T + \frac{1}{\det(F)}\hat{S}(F) \\ &= -\frac{1}{\det(F)}(F^{-T} : H)\hat{S}(F)F^T + \frac{1}{\det(F)} D\hat{S}(F)(H)F^T \\ &\quad + \frac{1}{\det(F)}\hat{S}(F)H^T \\ &= \hat{S}(I)\left(-\mathrm{tr}(H)I + H^T\right) + C(H) \end{aligned}$$

□

Remark 3.17.

1. $\hat{S}(QF) = Q\hat{S}(F)$, frame indifference, $C_{i\alpha j\beta} = C_{i\alpha\beta j}$.
2. $\hat{S}(F) = D\hat{\sigma}(F)$, 2nd law for hyperelastic materials, $C_{i\alpha j\beta} = C_{j\beta i\alpha}$.
3. (1) and (2) $\Rightarrow C_{i\alpha j\beta} = C_{\alpha i j\beta}$.

To prove (2), notice that

$$\hat{S}(F)_{i\alpha} = \frac{\partial\hat{\sigma}(F)}{\partial F_{i\alpha}} \Rightarrow C_{i\alpha j\beta} = \frac{\partial^2\hat{\sigma}}{\partial F_{i\alpha}\partial F_{j\beta}}(I) = C_{j\beta i\alpha}$$

To prove (1), note that $\hat{S}(QF) = \hat{S}(F) \Rightarrow \hat{S}(Q) = \hat{S}(I)$ for any $Q \in \mathrm{Orth}^+$. Set $Q(t) = \exp(tW)$. Then

$$\begin{aligned} \hat{S}(I) = \hat{S}(Q(t)) \Rightarrow \mathbf{0} &= \frac{d}{dt}\hat{S}(Q(t)) = D\hat{S}(Q(t))(\dot{Q}(t)) \\ &= D\hat{S}(Q(t))(WQ(t)) \end{aligned}$$

and evaluating at $t = 0$ tells us $\mathbf{0} = D\hat{S}(I)(W)$, so $C(W) = 0$ for all $W \in \mathrm{Skw}$. Thus, $0 = C_{i\alpha j\beta}W_{j\beta}$ for all $W \in \mathrm{Skw}$. If $C_{i\alpha j\beta} \neq C_{i\alpha j\beta}$ for some $j\beta$, then select W such that $W_{j\beta} = -1 = -W_{\beta j}$ to get $0 = \mathbf{0}_{i\alpha} = C_{i\alpha j\beta} - C_{i\alpha\beta j}$. Thus,

$$C(H) = C(H_{\mathrm{sym}} + H_{\mathrm{skw}}) = C(H_{\mathrm{sym}}) + \underbrace{C(H_{\mathrm{skw}})}_{=0} \Rightarrow C(H) = C(H_{\mathrm{sym}})$$

Recall: The function $C : \mathrm{Lin} \to \mathrm{Lin}$ is invariant under $Q \in \mathrm{Orth}$ if $C(Q^T HQ) = Q^T C(H)Q$.

Lemma 3.18. *If a material is (hyper)elastic at $p \in \mathcal{B}_r$, then the elasticity tensor is invariant under the symmetry group at p, i.e.*

$$C(Q^T HQ) = Q^T C(H)Q \quad \text{for } Q \in \mathcal{G}_p < \mathit{Orth}^+$$

where $\mathcal{G}_p$ denotes the symmetry group of the material, which is a subgroup ($<$) of Orth$^+$.

Proof. Note $\hat{S}(FQ) = \hat{S}(F)Q$ for $Q \in \mathcal{G}_p$. Set $F \mapsto Q^T F$ to get

$$\hat{S}(Q^T FQ) = \hat{S}(Q^T F)Q = Q^T \hat{S}(F)Q$$

by frame indifference. Then

$$D\hat{S}(Q^T FQ)(Q^T HQ) = Q^T D\hat{S}(F)(H)Q$$

Evaluating at $F = I$ tells us $C(Q^T HQ) = Q^T C(H)Q$. $\square$

Corollary 3.19. *If an elastic material is isotropic at p, then* $C(E) = \lambda tr(E)I + 2\mu E$ *for all* $E \in Sym$*, for some scalars* $\lambda = \lambda(p)$ *and* $\mu = \mu(p)$*.*

Define $(\cdot,\cdot)_C : \text{Lin} \times \text{Lin} \to \mathbb{R}$ by

$$(G, H)_C = C(G) : H \equiv G : C(H)$$

where the equivalence follows because $C(G) : H = C_{i\alpha j\beta} G_{i\alpha} H_{j\beta}$, and we know we can swap the indices. Thus, $(G, H)_C = (H, G)_C$ and

$$(\alpha G_1 + \beta G_2, H) = \alpha(G_1, H) + \beta(G_2, H)$$

and $(G, W)_C = 0$ for all $W \in \text{Skw}$.

Recall: $T = \frac{1}{\det(F)} \hat{S}(F)F^T$ and $T = T^T$, so $\hat{S}(I) = \hat{T}(I)$, so $\hat{S}(I)$ is symmetric. Then

$$\hat{T}(I + H) = \hat{S}(I)(H^T - \text{tr}(H)I) + C(H) + o(H)$$

Symmetries: $S_{i\alpha} = \dfrac{\partial \sigma}{\partial F_{i\alpha}}$ (always), and then

$$C_{i\alpha j\beta} = \frac{\partial S_{i\alpha}}{\partial F_{j\beta}} \restriction_{F=I} = \frac{\partial^2 \sigma}{\partial F_{i\alpha} \partial F_{j\beta}}(I)$$

so $C_{i\alpha j\beta} = C_{j\beta i\alpha}$. If $\hat{S}(I) = 0$, then $D\hat{T}(I) = C(H)$ and $T \in \text{Sym}$ gives $C_{i\alpha j\beta} + C_{\alpha i j\beta}$, and hence

$$\begin{array}{ccc} C_{i\alpha j\beta} & = & C_{i\alpha\beta j} \\ \downarrow & & \uparrow \\ C_{j\beta i\alpha} & = & C_{\beta j i\alpha} \end{array}$$

For isotropic materials with $\hat{S}(I) = 0$, then

$$C(E) = \lambda \text{tr}(E)I + 2\mu E$$

where $H = F - I$ and $E = \frac{1}{2}(H + H^T)$. Assume $\hat{S}(I) = 0$. Define

$$(\cdot,\cdot)_C : \text{Lin} \times \text{Lin} \to \mathbb{R}$$

by $(G, H)_C = C(G) : H = G : C(H) = C(H) : G = (H, G)_C$. Also, notice that

$$(\alpha G_1 + \beta G_2, H)_C = \alpha(G_1, H)_C + \beta(G_2, H)_C$$

Thus, if $C(H) : H \geq 0$ for all H, then $(\cdot,\cdot)_C$ is a *semi-inner product.*

Lemma 3.20. *Let $C(E) = \lambda tr(E) + 2\mu E$. Then $C(E) : E > 0$ for all non-zero symmetric matrices $\iff$ $\mu > 0$ and $2\mu + 3\lambda > 0$.*

Proof. Consider the more general case on $\mathbb{R}^{d\times d}$. Given $E \in \text{Lin}$, write $E = \frac{1}{d}\text{tr}(E)I + E_0$, so that $\text{tr}(E_0) = 0$. Then,

$$
\begin{aligned}
|E|^2 = E : E &= \left|\frac{1}{d}\text{tr}(E)I\right|^2 + |E_0|^2 \\
&= \frac{1}{d^2}\text{tr}(E)^2|I|^2 + |E_0|^2 \quad (\text{since } I : E_0 = 0) \\
&= \frac{1}{d^2}\text{tr}(E)^2 + |E_0|^2
\end{aligned}
$$

Then,

$$
\begin{aligned}
C(E) : E = (\lambda\text{tr}(E)I + E) : E &= \lambda\text{tr}(E)^2 + 2\mu|E|^2 \\
&= \left(\lambda + \frac{2\mu}{d}\right)\text{tr}(E)^2 + 2\mu|E_0|^2 > 0
\end{aligned}
$$

for all $E \neq 0 \iff \mu > 0$ and $\lambda + \frac{2}{d}\mu > 0$. ☐

Example 3.21. Let $\hat{\sigma}(F) = \frac{\alpha}{2}(|F|^2 - |I|^2) - \frac{\beta}{2}\ln(\det(F)^2)$. Then

$$\hat{S}_{i\alpha} = \frac{\partial\sigma}{\partial F_{i\alpha}} = \alpha F_{i\alpha} - \beta\frac{1}{\det(F)}\det(F)(F^{-T})_{i\alpha}$$

and so

$$\hat{S}(F) = \alpha F - \beta(F^{-T})$$

and then

$$\hat{T}(F) = \frac{1}{\det(F)}\hat{S}(F)F^T = \frac{1}{\det(F)}(\alpha FF^T - \beta I)$$

Note $\hat{S}(I) = 0 \iff \alpha = \beta$. We need to compute

$$C_{i\alpha j\beta} = \frac{\partial S_{i\alpha}(I)}{\partial F_{j\beta}} = \frac{\partial^2\sigma(I)}{\partial F_{i\alpha}\partial F_{j\beta}}$$

First,

$$\frac{\partial}{\partial F_{j\beta}}(F_{i\alpha}) = \delta_{ij}\delta_{\alpha\beta}$$

and second,

$$F^{-T}F^T = I \Rightarrow (\delta F^{-T}F^T + F^{-T}\delta F^T = 0 \Rightarrow \delta F^{-T} = -F^{-T}\delta F^T F^{-T}$$

so

$$(\delta F^{-T})_{i\alpha} = -(F^{-T})_{i\beta}\delta F_{j\beta}(F^{-T})_{j\alpha} = -(F^{-T})_{i\beta}(F^{-T})_{j\alpha}\delta F_{j\beta}$$

i.e.

$$\frac{\partial(F^{-T})_{i\alpha}}{\partial F_{j\beta}}$$

and

$$\frac{\partial (F^{-T})_{i\alpha}}{\partial F_{j\beta}} \restriction_{F=I} = -\delta_{i\beta}\delta_{j\alpha}$$

Then,

$$C_{i\alpha j\beta} = \alpha\delta_{ij}\delta_{\alpha\beta} + \beta\delta_{i\beta}\delta_{j\alpha}$$

or

$$C(H)_{i\alpha} = C_{i\alpha j\beta}H_{j\beta} = \alpha H_{i\alpha} + \beta H_{\alpha i}$$

so $C(H) = \alpha H + \beta H^T$. Thus, if $\alpha = \beta$ then $C(H) = 2\alpha H_{\text{sym}}$.

Cauchy Stress:

$$\begin{aligned} T(F) &= \frac{1}{\det(F)}\hat{S}(F)F^T \\ &= \frac{1}{\det(I+H)}\hat{S}(I+H)(I+H^T) \\ &= \frac{1}{\det(I+H)}(\hat{S}(I) + C(H) + o(|H|))(I+H)^T \end{aligned}$$

Now, $\det(I+H) = 1 + \text{tr}(H) + o(|H|)$ so $\frac{1}{\det(I+H)} = 1 - \text{tr}(H) + o(|H|)$. Then,

$$\begin{aligned} T &\approx (1-\text{tr}(H))\left((\alpha-\beta)I + \alpha H + \beta H^T\right)(I+H^T) \\ &\approx (1-\text{tr}(H))\left[(\alpha-\beta)I + (\alpha-\beta)H^T + \alpha H + \beta H^T\right] \\ &\approx (1-\text{tr}(H))\left[(\alpha-\beta)I + \alpha(H+H^T)\right] \\ &\approx (1-\text{tr}(H))(\alpha-\beta)I + \alpha(H+H^T) + o(|H|) \end{aligned}$$

Notation: Given a motion, write $x = \mathcal{X}(t,p)$ and

- the displacement is $u(t,p) = x(t,p) - p$
- $\nabla u = \nabla x - I = F - I = H$
- the "infinitesimal" strain is $E = \frac{1}{2}(H + H^T)$.

Note:

- $F^TF = (I+H)^T(I+H) = I + (H+H^T) + O(H^2)$ and so $F^TF - I = 2E + O(\nabla u^2)$
- If $\hat{S}(I) = 0$ then

  $$\hat{S}(F) = C(F-I) + o(|F-I|^2) = C(E) + o(|\nabla u|^2)$$

  where $E = (\nabla u)_{\text{sym}}$ and $C(E) = C(\nabla u)$.

Recall: for an elastic material

$$
\begin{aligned}
\rho_r \ddot{x} - \operatorname{div}(\hat{S}(F)) &= b \\
u = \mathcal{X}(t,p) - p \quad , \quad \ddot{u} &= \ddot{x} \\
F = \nabla x &= \nabla u - I \\
\rho_r \ddot{u} - \operatorname{div}(C(\nabla u)) &= b + O(|Du|^2)
\end{aligned}
$$

The last line above is the *equation of linear elasticity*. Note: if

$$C(\nabla u) = \lambda \operatorname{tr}(\nabla u) + \mu(\nabla u + \nabla u^T)$$

then

$$
\begin{aligned}
\operatorname{div}(C(\nabla u))_i &= C(\nabla u)_{ij,j} \\
&= (\lambda u_{k,k}\delta_{ij} + \mu u_{i,j} + \mu u_{j,i})_{ij} \\
&= \lambda u_{k,kj}\delta_{ij} + \mu u_{i,jj} + \mu u_{j,ji} \\
&= ((\lambda+\mu)\nabla \operatorname{div}(u) + \mu \Delta u)_i
\end{aligned}
$$

One often sees the equations of isotropic, linear elasticity (with zero residual stress) written as

$$\ddot{u} - (\lambda+\mu)\nabla \operatorname{div}(u) - \mu\Delta u = b$$

3.4.1 Stability

$$\rho_r \ddot{u} - \operatorname{div}(C(\nabla u)) = b$$

Take the dot product with v and integrate by parts to get

$$\int_{\mathcal{B}_r} \rho_r \ddot{u} \cdot v + C(\nabla u) : \nabla v = \int_{\mathcal{B}_r} b \cdot v + \int_{\partial \mathcal{B}_r} C(\nabla u) n \cdot v$$

Typically, we have $\partial \mathcal{B}_r = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ with $u \restriction_{\Gamma_0} = u_0$ is specified (like a displacement) and $C(\nabla u) n \restriction_{\Gamma_1} = \hat{s}$ (the "traction" boundary condition).

Example 3.22. Consider the unit square in $\mathbb{R}^2$. Let Γ_0 be the x-axis and Γ_1 be the remaining 3 sides. Specify $u \restriction_{\Gamma_0} = 0$ and $\hat{s} = T\vec{e}_1$ on the top side and $\hat{s} = 0$ on the other two sides. Then

$$\int_{\mathcal{B}_r} \rho_r \ddot{u} + C(\nabla u) : \nabla v = \int_{\mathcal{B}_r} b \cdot v + \int_{\Gamma_1} \hat{s} \cdot v$$

for all v with $v \restriction_{\Gamma_0} = 0$.

3.4.2 Uniqueness

Theorem 3.23. *Suppose u_1 and u_2 satisfy the same elasticity equation with the same boundary conditions and the initial conditions $u_1(0,p) = u_2(0,p)$ and $\dot{u}_1(0,p) = \dot{u}_2(0,p)$. Then $u_1(t,x) = u_2(t,x)$ for all $(t,x) \in (0,T) \times \mathcal{B}_r$.*

Proof. Note $u = u_2 - u_1$ satisfies $\rho_r \ddot{u} - \operatorname{div}(C(\nabla u)) = 0$ with $u \restriction_{\Gamma_0} = 0$ and $C(\nabla u) \restriction_{\Gamma_1} = 0$. Thus,

$$\int_{\mathcal{B}_r} \rho_r \ddot{u} \cdot v + C(\nabla u) : \nabla v = 0 \qquad \forall v \text{ s.t. } v \restriction_{\Gamma_0} = 0$$

Set $v = \dot{u}$. Then

$$\ddot{u} \cdot \dot{u} = \left(\frac{1}{2} \dot{u} \cdot \dot{u}\right)^{\cdot} = \left(\frac{1}{2} |\dot{u}|^2\right)^{\cdot}$$

and

$$C(\nabla u) : \nabla \dot{u} = \left(\frac{1}{2} |\nabla u|_C^2\right)^{\cdot}$$

since

$$\frac{d}{dt} \frac{1}{2} (\nabla u, \nabla u)_C = \frac{1}{2} (\nabla \dot{u}, \nabla u)_C + \frac{1}{2} (\nabla u, \nabla \dot{u})_C = (\nabla u, \nabla \dot{u})_C = C(\nabla u) : \nabla \dot{u}$$

Also, $\rho_r = \rho_r(p)$ is independent of t, as is $\mathcal{B}_r$. Then

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{B}_r} \underbrace{\rho_r |\dot{u}|^2}_{\text{kinetic energy}} + \underbrace{|\nabla u|_C^2}_{\text{elastic energy}} = 0$$

and so

$$\left(\int_{\mathcal{B}_r} \rho_r |\dot{u}|^2 + |\nabla u|_C^2\right) \restriction_{t=0} = \int_{\mathcal{B}_r} \rho_r |\dot{u}|^2 + |\nabla u|_C^2 \restriction_{t=0} = 0$$

since $u_1 = u - 2$ and $\dot{u}_1 = \dot{u}_2$ when $t = 0$. Since $\rho_r > 0$ on $\mathcal{B}_r$ then $\dot{u} = 0$ so $u(t, p) = u(0, p) = 0$. □

3.5 Elastostatics

Consider $-\operatorname{div}(C(\nabla u)) = b$ with $u \restriction_{\Gamma_0} = u_0$ and $C(\nabla u) n \restriction_{\Gamma_1} = \hat{s}$. Then

$$\int_{\mathcal{B}_r} C(\nabla u) : \nabla v = \int_{\mathcal{B}_r} b \cdot v + \int_{\Gamma_1} \hat{s} \cdot v \qquad \forall v \text{ s.t. } v \restriction_{\Gamma_0} = 0$$

Suppose u_1, u_2 are solutions with the same b and boundary conditions. Then $u_2 - u_1$ satisfies $-\operatorname{div}(C(\nabla u)) = 0$ and $u \restriction_{\Gamma_0} = 0$. Thus,

$$\int_{\mathcal{B}_r} C(\nabla u) : \nabla v = 0 \qquad \forall v \text{ s.t. } v \restriction_{\Gamma_0} = 0$$

so select $v = u$ to get

$$\int_{\mathcal{B}_r} |\nabla u|_C^2 = 0 \;\Rightarrow\; |\nabla u|_C^2 = 0 \text{ on } \mathcal{B}_r \;\Rightarrow\; (\nabla u)_{\text{sym}} = 0$$

Recall $\nabla u = 0 \iff u(x) = u_0 \in \mathbb{R}^d$ is constant.
Exercise 1: If $\Omega \subseteq \mathbb{R}^d$ is a connected domain and $u : \Omega \to \mathbb{R}^d$ is smooth, then

$$(\nabla u)_{\text{sym}} = 0 \iff u(p) = u_0 + Wp \qquad \text{for some } u_0 \in \mathbb{R}^d, W \in \text{Skw}$$

Exercise 2: If $\partial\Omega$ is Lipschitz and $\Gamma \subseteq \partial\Omega$ has nonzero measure and $u \restriction_\Gamma = 0$, then $u = 0$.

****

$$\int_{\mathcal{B}_r} C(\nabla u) : \nabla v = \int_{\mathcal{B}_r} b \cdot v + \int_{\Gamma_1} \hat{s} \cdot v \quad \forall v \restriction_{\Gamma_0} = 0$$

Given two solutions u_1, u_2 of the same elastostatic problem, the difference $u = u_2 - u_1$ satisfies

$$\int_{\mathcal{B}_r} C(\nabla u) : \nabla v = 0 \quad \forall v \restriction_{\Gamma_0} = 0$$

and setting $v = u$ gives

$$\int_{\mathcal{B}_r} |\nabla u|_C^2 = 0$$

Recall $(G, H)_C = C(G) : H = G : H(C)$. Note that this integral condition does *not* necessarily imply $\nabla u = 0$.

If we assume $(G, H)_C$ is an inner product on Sym (e.g. $\mu > 0$ and $2\lambda + 3\mu > 0$ in isotropic case), then $(\nabla u)_{\text{Sym}} = 0$.

We're thinking of $u = [u, v]^T$, so

$$\nabla u = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \Rightarrow (\nabla u)_{\text{Sym}} = \begin{bmatrix} u_x & \frac{u_y + v_x}{2} \\ 0 & v_y \end{bmatrix}$$

By Exercise 1 above, it follows that $u(t, p) = u_2 - u_1 = u_0 + Wp$. Thus, if $u \restriction_{\Gamma_0} = u_2 - u_1 \restriction_{\Gamma_0} = 0$ is "sufficient" to eliminate rigid body motions, then $u(p) = 0$.

Pure Traction Problem: $(\Gamma_0 = 0)$ We want $C(\nabla u)n = \hat{s}$ on $\mathcal{B}_r$.

$$\int_{\mathcal{B}_r} (\nabla u, \nabla v)_C = \int_{\mathcal{B}_r} b \cdot v = \int_{\partial\mathcal{B}_r} \hat{s} \cdot v \qquad \forall v \text{ smooth}$$

Set $v = v_0 \in \mathbb{R}^d$ constant, so then

$$0 = \left(\int_{\mathcal{B}_r} b + \int_{\partial\mathcal{B}_r} \hat{s} \right) \cdot v_0 \Rightarrow \int_{\mathcal{B}_r} b + \int_{\partial\mathcal{B}_r} \hat{s} = 0$$

Next, set $v = Wp$, so that $(\nabla u, \nabla v)_C = \nabla u : C(W) = 0$, and then

$$0 = \int_{\mathcal{B}_r} b \cdot Wp + \int_{\partial\mathcal{B}_r} \hat{s} \cdot Wp$$

If $W = W(\omega)$ has axial vector $\omega \in \mathbb{R}^d$, then

$$b \cdot W(\omega)p = b \cdot (\omega \times p) = \omega \cdot (p \times b)$$

and so

$$0 = \omega \cdot \left(\int_{\mathcal{B}_r} p \times b + \int_{\partial\mathcal{B}_r} p \times \hat{s} \right)$$

for all ω, and thus

$$\int_{\mathcal{B}_r} p \times b + \int_{\partial\mathcal{B}_r} p \times \hat{s} = 0$$

These are necessary and sufficient conditions for existence of a solution to the pure traction elastostatics problem. Solutions may be found by minimizing

$$I(u) = \int_{\mathcal{B}_r} \frac{1}{2}|\nabla u|_C^2 - b \cdot u - \int_{\partial\mathcal{B}_r} \hat{s} \cdot u$$

over the set

$$u \in \mathcal{U} := \left\{ u \in H^1(\mathcal{B}_r)^d : u \restriction_{\Gamma_0} = u_0 \right\}$$

3.6 Wave Propagation

Consider isotropic elasticity $C(H) = \lambda \mathrm{tr}(H) I + \mu(H + H^T)$,

$$\rho_0 \ddot{u} - (\lambda + \mu)\nabla \operatorname{div} u - \mu \Delta u = 0$$

We seek a solution of the form

$$u(t, x) = \vec{a} \exp(i(\omega t - \vec{k} \cdot x))$$

so we do some calculus! Notice $\ddot{u} = -\omega^2 u$ and

$$\nabla u = -i(a \otimes k) \exp(i(\omega t - k \cdot x)) = -iu \otimes k$$

and

$$\operatorname{div} u = \mathrm{tr}(\nabla u) = -ia \cdot k \exp(i(\omega t - k \cdot x))$$

Then

$$\Delta u = -|k|^2 u = -|k|^2 a \exp(i(\omega t - k \cdot x))$$

and

$$\nabla \operatorname{div} u = -(a \cdot k) k \exp(i(\omega t - k \cdot x))$$

so

$$\left(-\rho_0 \omega^2 I + (\lambda + \mu) k \otimes k + \mu |k|^2 I\right) a \exp(i(\omega t - k \cdot x)) = 0$$

Divide by $|k|^2$ and write $c^2 = \frac{\omega^2}{|k|^2}$ and $\hat{k} = \frac{1}{|k|}$ to get

$$(\lambda + \mu)(\hat{k} \otimes \hat{k}) a = (\rho_0 c^2 - \mu) a$$

Thus, a must be an eigenvector of $(\lambda + \mu)\hat{k} \otimes \hat{k}$ with eigenvalue $\rho c^2 - \mu$. The eigenpairs are

$$a = \hat{k} \text{ with } c^2 = \frac{\lambda + 2\mu}{\rho_0} \quad \text{and} \quad a \in \{\hat{k}^\perp\} \text{ with } c^2 = \frac{\mu}{\rho_0}$$

where $\{\hat{k}^\perp\}$ is a 2-D null space.

The solution

$$u(x,t) = C\exp(ic(t - \hat{k}\cdot x))\hat{k}$$

is a longitudinal wave with speed $\sqrt{\frac{\lambda+2\mu}{\rho_0}}$ and

$$u(t,x) = C\exp(ic(t - \hat{k}\cdot x))\ell \quad \text{for } \ell \in \hat{k}^{\perp}$$

is a transverse wave with speed $\sqrt{\frac{\mu}{\rho_0}}$.

4 Thermomechanics

1. **Statistical mechanics**: for an *ideal gas*,

$$\begin{aligned} \text{density}: \ \rho &= \frac{\#\text{ molecules}}{\text{volume}} \\ \text{temperature}: \ \theta &= \text{ average of } \left(\frac{v^2}{2}\right) \geq 0 \\ \text{pressure}: \ p &= \frac{\text{force}}{\text{unit wall area}} \\ &\approx \text{ change in momentum of molecules bouncing off walls} \end{aligned}$$

   and $\frac{p}{\rho} = R\theta$ for some constant R.

2. **Theory of heat engines**: important concepts are heat in Q and work out W

   Example 4.1. Ideal gas in a cylinder. $W =$ "force $\times$ distance", but really

$$W = \int_{x_1}^{x_2} p \underbrace{A\,dx}_{d\text{Volume}} = \int_{V_1}^{V_2} p\,dV$$

   and $Q \propto T\theta$ ($T =$ temperature).

First Law. For a cyclic process, $Q = 0 \Rightarrow W =$).

Reversibility: using an ideal gas, we can construct a reversible machine, with W in and JQ out.

Lemma 4.2 (Joule's relation)**.** *Let J be the constant for the ideal gas. Then $Q = JW$ for all cyclic processes.*

Proof. Suppose otherwise, so $W = (1+\alpha)JQ$ for some process. Contradicts First Law. ***** insert picture ***** □

Second law. **********

"**Zeroth law**". If two bodies are in contact, heat flows from one to another $\iff$ the one is hotter than the other.

So in our concept of "temperature", we just need an "order" of hotness.

Reversible machine. Using an ideal gas, one can construct a reversible machine for which heat is added at a constant temperature θ_{in} and heat is removed at a constant temperature θ_{out} and $\theta_{\text{in}} > \theta_{\text{out}}$ when $W > 0$.

***** diagram *****

With this cycle ($J = 1$),

$$\underbrace{W = Q_{\text{in}} - Q_{\text{out}}}_{\text{1st Law}} = \underbrace{\left(\frac{\theta_{\text{in}} - \theta_{\text{out}}}{\theta_{\text{in}}}\right)}_{\text{efficiency}} Q_{\text{in}}$$

Lemma 4.3. *No cyclic process operating between temperatures θ_1 and θ_2 can be "more efficient" than the Carnot cycle constructed with an ideal gas.*

Proof. Suppose $W = Q_{\text{in}} - Q_{\text{out}} = \eta Q_{\text{in}}$ for some device with $\eta > \frac{\theta_{\text{in}} - \theta_{\text{out}}}{\theta_{\text{in}}}$.

******** diagram ******

□

If we assume work W and heat Q are "basic" or "fundamental" then the 1st law gives $\oint dW - dQ = 0$. We can then define

$$e(W, Q) = e_0 + \int_{(W_0, Q_0)}^{(W,Q)} dW - dQ$$

and this is well-defined. The 2nd law gives an inequality; if we knew $\oint v(x) \cdot dx \leq 0$ for closed loops, then $\exists$ "lower potential" $\eta(x)$ such that

$$\int_{x_0}^{x} v \cdot dx **** \substack{\leq \\ \geq} ***** \eta(x) - \eta(x_0)$$

Coleman-Noll procedure ($\approx$ 1960s)

- Kinematics: $x = \mathcal{X}(t, p)$, $v = \dot{x}$, $F = \nabla x$ etc.
- Balance of mass: $\rho_t + \operatorname{div}(\rho v) = 0$ or $\rho = \frac{\rho_r}{\det F}$
- Balance of momentum: $\rho \dot{v} - \operatorname{div} T = b$, $T = T^T$ where b is known/specified and T is constitutive
- Balance of energy: $\rho \dot{e} + \operatorname{div} q = r + T : \nabla v$ where r = heat source is known/specified and e, q are constitutive (and θ = temperature, in the background)
- Clausius Duhem inequality:

$$\frac{d}{dt} \int_{\mathcal{P}(t)} \rho \eta \geq \int_{\mathcal{P}(t)} \frac{r}{\theta} + \int_{\partial \mathcal{P}(t)} \frac{q \cdot n}{\theta}$$

for all parts $\mathcal{P}(t) = \mathcal{X}(t, \mathcal{P}_r)$ with $\mathcal{P}_r \subseteq \mathcal{B}_r$, and where θ = temperature is fundamental and η = specific entropy is constitutive.

Using the Reynolds transport theorem 1 and the divergence theorem gives

$$\int_{\mathcal{P}(t)} \rho\dot{\eta} \geq \int_{\mathcal{P}(t)} \frac{r}{\theta} - \operatorname{div}\left(\frac{1}{\theta}q\right)$$

and localizing gives

$$\rho\dot{\eta} + \operatorname{div}\left(\frac{1}{\theta}q\right) \geq \frac{r}{\theta} \iff \rho\dot{\eta}\theta + \operatorname{div} q \geq r + \frac{1}{\theta}q \cdot \nabla\theta$$

Ingredients: $(T, e, q, \eta, \mathcal{X}, \theta, b, r)$
Given (T, e, q, η) functions of $(\mathcal{X}, \theta)$ then given any motion $\mathcal{X}$ and $\theta > 0$ we can construct a process with motion $\mathcal{X}$ and temperature θ by selecting b, r to be the RHS of the momentum and energy equations, respectively.

That is, T, e, q, η are *constitutive* (functions of $\mathcal{X}, \theta$), we assume $\mathcal{X}, \theta$ can be specified arbitrarily, and these specifications give us b, r.

Second Law: A constitutive law $(T, e, q, n) = \mathcal{F}(\mathcal{X}, \theta)$ satisfies the Clausius Duhem inequality for all admissible motions $\mathcal{X}$ and $\theta > 0$.

Helmholz Free Energy: First, recall $\rho\dot{e} = r - \operatorname{div} q + T : \nabla v$ so the Clausius Duhem inequality becomes

$$\rho\dot{\eta} \geq \frac{1}{\theta}\rho\dot{e} - \frac{1}{\theta}T : \nabla v + \frac{1}{\theta^2}q \cdot \nabla\theta$$

and so

$$\rho(\dot{e} - \theta\dot{\eta}) - T : \nabla v + \frac{1}{\theta}q \cdot \nabla\theta \leq 0$$

Note: this eliminates r. Define $\psi = e - \theta\eta$, so $\dot{\psi} = \dot{e} - \dot{\theta}\eta - \theta\dot{\eta}$. Thus,

$$\rho(\dot{\psi} + \eta\dot{\theta}) - T : \nabla v + \frac{1}{\theta}q \cdot \nabla\theta \leq 0$$

Elastic Material with Linear Viscosity

$$T = \hat{T}_e(F, \theta) + \hat{T}_v(F, \theta)(\nabla v)$$

where $\hat{T}_v(F, \theta) : \mathrm{Lin}^+ \to \mathrm{Sym}$ is *linear*, and e = elastic and v = viscous. We assume $e = \hat{e}(F, \theta, \nabla\theta)$ and $q = \hat{q}(F, \theta, \nabla\theta)$ and $\eta = \hat{\eta}(F, \theta, \nabla\theta)$. We often write $g = \nabla\theta$. Then $\hat{\psi} = \hat{e} - \theta\hat{\eta}$ is a derived quantity.

$$\dot{\psi} = D_F\hat{\psi} : \dot{F} + \hat{\psi}_\theta\dot{\theta} + \nabla_g\hat{\psi} \cdot \nabla\dot{\theta} \qquad (\dot{F} = \nabla_v F)$$

so then

$$\rho(\hat{\psi}_\theta + \hat{\eta})\dot{\theta} + \rho(D_F\hat{\psi}F^T - \hat{T}_e) : \nabla v - \hat{T}_v(\nabla v) : \nabla v + \frac{1}{\theta}q \cdot \nabla\theta + \rho\nabla_g\hat{\psi} \cdot \nabla\dot{\theta} \leq 0$$

Entropy Relation. $x = \mathcal{X}(t,p) = F_0(p - \vec{0})$, for $F_0 \in \text{Lin}^+$ and $v = 0$ etc.; $\theta(t,x) = \theta_0 + \alpha t$ for $\theta_0 > 0$ and t small, $\nabla\theta = 0$ etc.

$$\rho(\hat{\psi}_\theta\,(F_0, \theta_0, g_0) + \hat{\eta}(F_0, \theta_0, g_0))\,\alpha \leq 0 \quad \text{evaluate at } t = 0$$

Since α may take arbitrary sign, then $\hat{\eta} = -\frac{\partial\hat{\psi}}{\partial\theta}$. This equation is the *entropy relation*. This removes the first term in the inequality above, so we have

$$(D_F\hat{\psi}F^T - \hat{T}_e) : \nabla v - \hat{T}_v(\nabla v) : \nabla v + \frac{1}{\theta}q \cdot \nabla\theta + \nabla_g\hat{\psi} \cdot \nabla\dot{\theta} \leq 0$$

for this class of materials.

Next, set $x = \mathcal{X}(t,p) = F_0(p - \vec{0})$, so

$$\theta(t,x) = \theta_0 + g_0 \cdot x + tg_1 \cdot x \quad \text{for } |x| + t \text{ small}$$

where $\theta(0,0) = \theta_0$ and $\nabla\theta(0,0) = g_0$ and $\theta\dot{\theta}(0,0) = g_1$. Then

$$\frac{1}{\theta}\hat{q}(F_0, \theta_0, g_0) \cdot g_0 + \nabla_g\hat{\psi}(F_0, \theta_0, g_0) \cdot g_1 \leq 0$$

Selecting g_1 arbitrarily gives

$$\nabla_g\hat{\psi}(F_0, \theta_0, g_0) = 0$$

Thus, $\psi = \hat{\psi}(F, \theta)$. Now, the last term in the inequality above vanishes, as well, and all that remains is

$$(D_F\hat{\psi}F^T - \hat{T}_e) : \nabla v - \hat{T}_v(\nabla v) : \nabla v + \frac{1}{\theta}q \cdot \nabla\theta \leq 0$$

Now, since $\psi = e - \theta\eta$, then $e = \hat{e}(F, \theta)$, as well.

Stress Relation and Dissipation Principle. $x = \mathcal{X}(t,p) = (F_0 + \alpha t L_0)(p - \vec{0})$ for $F_0 \in \text{Lin}^+$ and $L_0 \in \text{Lin}$; $\theta(t,x) = \theta_0 \in \mathbb{R}_+$. Then $F(0,0) = F_0$ and $\dot{F}(0,0) = \alpha L_0$, so

$$\alpha(D_F\hat{\psi}F^T - \hat{T}_e) : L_0 - \alpha^2\hat{T}_v(L_0) : L_0 \leq 0 \quad \forall\alpha$$

Since we can have α small with arbitrary sign, then

$$\hat{T}_e(F, \theta) = \rho\,\underbrace{D_F\hat{\psi}(F, \theta)F^T}_{\text{Piola stress}}$$

is called the *stress relation*, and

$$\hat{T}_v(F, \theta)(\nabla v) : \nabla v \geq 0$$

is called the *dissipation relation*.

Summary of information on elastic materials with linear viscosity: (T, e, q, η) are constitutive, $x = \mathcal{X}(t, p)$ and $\theta(t, x)$ are specified; balance of mass, momentum (linear & angular) can be satisfied by setting

$$\begin{aligned}
\rho &= \frac{\rho_r}{\det F} \\
b &= \rho \dot{v} - \operatorname{div} T \\
T &= T^T \\
r &= \rho \dot{e} + \operatorname{div} q - T : \nabla v \\
T &= \hat{T}_e(F, \theta) + \hat{T}_v(F, \theta)(\nabla v) \\
e, q, \eta &\sim (F, \theta, \nabla\theta)
\end{aligned}$$

It's convenient to introduce $\psi = e - \theta\eta$, so then

$$\rho(\hat{\psi}_\theta + \hat{\eta})\dot{\theta} + \rho \nabla_g \hat{\psi} \cdot \nabla\dot{\theta} + \rho(D_F \hat{\psi} F^T - \hat{T}_e) : \nabla v - \hat{T}_v(\nabla v) : \nabla v \leq 0$$

For the *entropy relation*, write

$$\theta(t, x) = \theta_0 + \alpha t + g_0 \cdot x$$

for $|t| + |x| << 1$, with $\dot{\theta} = \alpha$ and $\nabla\theta = g_0$. Then for any $\alpha \in \mathbb{R}$,

$$\rho(\hat{\psi}_\theta + \hat{\eta}) \restriction_{(F_0, \theta_0, g_0)} \alpha + \frac{1}{\theta_0} \hat{q}(F_0, \theta_0, g_0) \cdot g_0 \leq 0$$

and so $\rho(\hat{\psi}_\theta + \hat{\eta}) = 0$; thus $\hat{\eta} = -\hat{\psi}_\theta$ is the entropy relation.

Also, set $\nabla\theta(0, 0) = g_0$ and $\nabla\dot{\theta} = g_1$, so then

$$\rho \nabla_g \hat{\psi}(F_0, \theta_0, g_0) \cdot g_1 + \frac{1}{\theta_0} \hat{q}(F_0, \theta_0, g_0) \cdot g_0 \leq 0$$

for all g_1, and so by the same argument $\nabla_g \hat{\psi} = 0$. We also get that $\hat{q} \cdot \nabla\theta \leq 0$. Thus,

$$\begin{aligned}
\psi &= \hat{\psi}(F, \theta) \text{ independent of } \nabla\theta \\
\eta &= \hat{\psi}_\theta \text{ independent of } \nabla\theta \\
\psi &= e - \theta\eta \text{ with } e \text{ independent of } \nabla\theta \\
\psi, \eta, e &\sim (F, \theta) \\
q &= \hat{q}(F, \theta, \nabla\theta)
\end{aligned}$$

Evaluating the dissipation inequality at $t = 0$ gives

$$\alpha \left[\rho D_F \hat{\psi} F^T - \hat{T}_e) \restriction_{(F_0, \theta_0)} - \alpha \hat{T}_v(F_0, \theta_0)(L) : L \right] \leq 0 \; \forall \alpha$$

and thus $\hat{T}_e = \rho D_F \hat{\psi}(F, \theta) F^T$ and $\hat{T}_v(F, \theta)(L) : L \geq 0$.

Heat conductivity: since $\frac{1}{\theta}q(F,\theta,\nabla\theta)\cdot\nabla\theta \leq 0$, we define $f(q) = q(F,\theta,g)\cdot g$. Then $f(0) = 0$ is a *max* and $\nabla f(0) = 0$ implies $Dq(F,\theta,0)\cdot 0 + \hat{q}(F,\theta,0) = 0$. Thus, $\hat{q}(F,\theta,0) = 0$; i.e. no temperature gradient $\Rightarrow$ no heat flow.

Fourier heat conductor

$$\hat{q}(F,\theta,\nabla\theta) = \vec{0} - K(F,\theta)\nabla\theta + o(|\nabla\theta|^2)$$

Then

$$\hat{q}\cdot\nabla\theta = -\nabla\theta^T K(F,\theta)\nabla\theta \leq 0$$

so the conductivity matrix is positive-definite (but not necessarily symmetric)!

***** missed class Wed Apr 21

Recall

$$\mathrm{rot}(T)_i = \varepsilon_{ijk}T_{kj}$$

so, e.g. $\mathrm{rot}(T)_1 = T_{32} - T_{23}$.

We write

$$\rho(J\omega)^{\cdot} - \mathrm{div}(C) = m + \mathrm{rot}(T)$$

and

$$\rho\dot{e} + \mathrm{div}(q) = r + T : \nabla v + C : \nabla\omega - \mathrm{rot}(T)\cdot\omega$$

Rod-like molecules (Ericksen). $J = \bar{r}^2(I - d\otimes d)$ with $|d| = 1$ and $d = Qe$ and $\dot{Q} = W(\omega)Q$. Then

$$\dot{d} = \dot{Q}e = W(\omega)Qe = W(\omega)d = \omega\times d$$

and

$$d\times\dot{d} = \omega - (d\cdot\omega)d = (I - d\otimes d)\omega$$

so

$$J\omega = \bar{r}^2(I - d\otimes d)\omega = \bar{r}^2 d\times\dot{d}$$

and

$$(J\omega)^{\cdot} = \bar{r}^2 d\times\ddot{d}$$

To guarantee that $c = Cn \perp d$ we assume $Cn = \hat{C}n$ for any n, so

$$C_{ij}n_j = \varepsilon_{ipq}d_p\hat{C}_{qj}n_j \quad \forall n \iff C_{ij} = \varepsilon_{ipq}d_p\hat{C}_{qj}$$

Now,

$$\mathrm{div}(C)_i = C_{ij,j} = \left(\varepsilon_{ipq}d_p\hat{C}_{qj}\right)_{,j} = \varepsilon_{ipq}\,\underbrace{d_{p,j}\hat{C}_{q,j}}_{(\hat{C}\nabla d^T)_{qp}} + d_p\,\underbrace{\hat{C}_{qj,j}}_{\mathrm{div}(\hat{C})q}$$

so

$$\mathrm{div}\,C = \mathrm{rot}(\hat{C}\nabla d^T) + d\times\mathrm{div}(\hat{C})$$

Then we have

$$d\times\left(\rho\bar{r}^2\ddot{d} - \mathrm{div}(\hat{C})\right) = d\times\hat{m} + \mathrm{rot}\left(T + \hat{C}\nabla d^T\right)$$

This equation can only be satisfied if $\operatorname{rot}(T+\hat{C}\nabla d) \perp d$, i.e. $\operatorname{rot}(T+\hat{C}\nabla d^T) = -d \times g$ for some g. Then we must have

$$\rho \bar{r}^2 \ddot{d} - \operatorname{div}(\hat{C}) + g + \theta d = \hat{m} \quad |d| = 1$$

Observe

$$\begin{aligned} C : \nabla w = C_{ij}\omega_{i,j} &= \varepsilon_{ipq} d_p \hat{C}_{qj} \omega_{i,j} \\ &= \varepsilon_{ipq}[(\underbrace{d_p\omega_i}_{(\omega\times d)_q}),_j - d_{p.j}\omega_i]\hat{C}_{qj} \\ &= (\omega \times d)_{q,j}\hat{C}_{qj} + \varepsilon_{ipq}\omega_i(\hat{C}\nabla d^T)_{qp} \\ &= \nabla(\omega \times d) : \hat{C} + \omega \cdot \operatorname{rot}(\hat{C}\nabla d^T) \\ &= \nabla(\dot{d}) : \hat{C} + \omega \cdot \operatorname{rot}(\hat{C}\nabla d^T) \end{aligned}$$

Thus,

$$\rho\dot{e} + \operatorname{div}(q) = r + T : \nabla v + \hat{C} : \nabla\dot{d} + \omega \cdot \operatorname{rot}(\hat{C}\nabla d^T + T)$$

Recall $d \times g = -\operatorname{rot}(T + \hat{C}\nabla d^T)$, so

$$-\omega \cdot \operatorname{rot}(T + \hat{C}\nabla d^T) = \omega \cdot (d \times g) = (\omega \times d) \cdot g = \dot{d} \cdot g$$

and then

$$\rho\dot{e} + \operatorname{div}(q) = r + T : \nabla v + \hat{C} : \nabla\dot{d} - g \cdot \dot{d}$$

4.1 Invariance Principles

Definition 4.4. *Given a reference body $\mathcal{B}_r$, a* thermodynamic process *is a tuple*

$$\pi(t,p) = \left\{\rho_r, \mathcal{X}, \theta_r, \vec{s}_r, e_r, \hat{q}_r, \eta_r, \vec{b}_r, r_r\right\}(t,p)$$

where

$$\begin{aligned} \rho_r, \theta_r, e_r, \eta_r, r_r &: (0,T) \times \mathcal{B}_r \to \mathbb{R} \\ \mathcal{X}, \vec{b} &: (0,T) \times \mathcal{B}_r \to \mathbb{R}^d \\ \vec{s}_r &: (0,T) \times \mathcal{B}_r \times \mathcal{S}^2 \to \mathbb{R}^d \\ \hat{q}_r &: (0,T) \times \mathcal{B}_r \times \mathcal{S}^2 \to \mathbb{R} \ \ \textit{heat flux} \end{aligned}$$

for which

1. *Balance of Energy*

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\left(e + \frac{1}{2}|v|^2\right) = \int_{\mathcal{P}(t)} (r + b \cdot v) + \int_{\partial\mathcal{P}(t)} \hat{q}(n) + \vec{s}(n) \cdot v$$

2. *and Clausius-Duhem inequality*

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\eta \geq \int_{\mathcal{P}(t)} \frac{r}{\theta} + \int_{\partial\mathcal{P}(t)} \frac{\hat{q}(n)}{\theta}$$

hold for all parts $\mathcal{P}(t) = \mathcal{X}(t, \mathcal{P}_r)$, $\mathcal{P}_r \subseteq \mathcal{B}_r$.

Note: in this case, $e(t, x(t,p)) = e_r(t,p)$ etc. and $\rho(t, x(t,p)) = \frac{\rho_r(p)}{\det(F(t,p))}$ where $F(t,p) = \left[\frac{\partial \mathcal{X}}{\partial p}\right]$ is the Jacobian. For example,

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho\eta = \frac{d}{dt}\int_{\mathcal{P}_r} \rho_r(p)\eta_r(t,p)\,dp$$

etc.

Given a thermodynamics process $\pi(t,p)$ consider

$$\pi^\lambda(t,p) = \left\{\rho_r, \mathcal{X}^\lambda, \theta, s, e, \eta, b^\lambda, r^\lambda\right\}_r (t,p)$$

where

$$\mathcal{X}^\lambda(t,p) = \mathcal{X}(t,p) + t\cdot\lambda \quad \text{for } \lambda \in \mathbb{R}^3$$

Write $x^\lambda = x + \lambda t$, $v^\lambda = v + \lambda$, $p^\lambda(t) = \mathcal{X}^\lambda(t, p_r) = p(t) + \lambda t$ etc. Then we compute, using $\det(F^\lambda) = \det F$,

$$\begin{aligned}
\int_{\mathcal{P}^\lambda(t)} p^\lambda \left(e^\lambda + \frac{1}{2}|v^\lambda|^2\right) &= \int_{\mathcal{P}_r} \rho_r \left(e_r + \frac{1}{2}|\dot{x} + \lambda|^2 \frac{1}{\det(F^\lambda)}\right) dp \\
&= \int_{\mathcal{P}(t)} \rho \left(e + \frac{1}{2}|v + \lambda|^2\right) dx \\
&= \int_{\mathcal{P}(t)} \rho \left(e + \frac{1}{2}|v|^2 + \lambda\cdot v + |\lambda|^2\right)
\end{aligned}$$

Also,

$$\int_{\mathcal{P}^\lambda(t)} r^\lambda + b^\lambda \cdot v^\lambda = \int_{\mathcal{P}(t)} r^\lambda + b^\lambda \cdot (v + \lambda)$$

and

$$\int_{\partial\mathcal{P}^\lambda(t)} \hat{q}^\lambda(n^\lambda) + s^\lambda(n^\lambda)\cdot v^\lambda = \int_{\partial\mathcal{P}(t)} \hat{q}(n) + s(n)\cdot(v+\lambda)$$

Then,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{P}^\lambda(t)} p^\lambda \left(e^\lambda + \frac{1}{2}|v^\lambda|^2\right) - \int_{\mathcal{P}^\lambda(t)} r^\lambda + b^\lambda\cdot v^\lambda - \int_{\partial\mathcal{P}(t)} \hat{q}^\lambda(n^\lambda) + s^\lambda(n^\lambda)\cdot v^\lambda \\
&= \frac{d}{dt}\int_{\mathcal{P}(t)} \rho\left(e + \frac{1}{2}|v|^2 + v\cdot\lambda + |\lambda|^2\right) - \int_{\mathcal{P}(t)} (r + b\cdot v + \lambda\cdot b) \\
&\quad - \int_{\partial\mathcal{P}(t)} \hat{q}(n) + s(n)\cdot v + \lambda\cdot s(n) + \int_{\mathcal{P}(t)} (r - r^\lambda) + (b - b^\lambda)\cdot v^\lambda
\end{aligned}$$

Lemma 4.5. *If $\pi(t,p)$ is a thermodynamic process and $\pi^\lambda(t,p)$ is also a thermodynamic process for all $\lambda \in \mathbb{R}^d$ when $b^\lambda = b$ and $r^\lambda = r$, then*

$$\frac{d}{dt}\int_{\mathcal{P}(t)} \rho v = \int_{\mathcal{P}(t)} b + \int_{\partial\mathcal{P}(t)} s(n)$$

for all parts $\mathcal{P}(t) = \mathcal{X}(t, p_r)$ with $\mathcal{P}_r \subseteq \mathcal{B}_r$.

Proof. Since we have a thermodynamic process, the balance of energy is satisfied

$$\frac{d}{dt}\int_{\mathcal{P}^\lambda(t)} p^\lambda\left(e^\lambda + \frac{1}{2}|v^\lambda|^2\right) - \int_{\mathcal{P}^\lambda(t)} r^\lambda + b^\lambda \cdot v^\lambda - \int_{\partial\mathcal{P}(t)} \hat{q}^\lambda(n^\lambda) + s^\lambda(n^\lambda)\cdot v^\lambda$$

$$= \frac{d}{dt}\int_{\mathcal{P}(t)} \rho(e + \frac{1}{2}|v|^2 + v\cdot\lambda + |\lambda|^2) - \int_{\mathcal{P}(t)} (r + b\cdot v + \lambda\cdot b)$$

$$- \int_{\partial\mathcal{P}(t)} \hat{q}(n) + s(n)\cdot v + \lambda\cdot s(n) + \int_{\mathcal{P}(t)} (r - r^\lambda) + (b - b^\lambda)\cdot v^\lambda$$

************* which implies

$$\left(\frac{d}{dt}\int_{\mathcal{P}(t)} \rho v - \int_{\mathcal{P}(t)} b - \int_{\partial\mathcal{P}(t)} s\right)\cdot\lambda + |\lambda|^2\frac{d}{dt}\int_{\mathcal{P}(t)}\rho = 0 \quad \forall\lambda, \mathcal{P}(t)$$

Note:

$$\frac{d}{dt}\int_{\mathcal{P}(t)}\rho = \frac{d}{dt}\int_{\mathcal{P}_r}\rho_r(p)\,dp = 0$$

implies

$$\frac{d}{dt}\int_{\mathcal{P}(t)}\rho v = \int_{\mathcal{P}(t)} b + \int_{\partial\mathcal{P}(t)} s$$

***************** □

Consider a process $\pi^\star(t,p)$ where

$$\mathcal{X}^\star(t,p) = \mathcal{X}(t, p + Q(t)(x - \vec{0})$$
$$b^\star(t,p) = Qb(t,p) + 2\dot{Q}v + \ddot{Q}(x - \vec{0})$$

and $\theta^\star = \theta$, all others the same. We know the balance of linear momentum would hold.

Also, (i)

$$\frac{d}{dt}\int_{P^\star(t)}\rho^\star\left(e^\star + \frac{1}{2}|v^\star|^2\right)$$
$$= \frac{d}{dt}\int_{\mathcal{P}(t)}\rho\left(e + \frac{1}{2}|\dot{Q}(x-\vec{0}) + Qv|^2\right)$$
$$= \frac{d}{dt}\int_{\mathcal{P}(t)}\rho\left(e + \frac{1}{2}|v|^2 + 2Qv\cdot\dot{Q}(x-\vec{0}) + |\dot{Q}(x-\vec{0})|^2\right)$$
$$= \frac{d}{dt}\int_{\mathcal{P}(t)}\rho\left(e + \frac{1}{2}|v|^2 + 2v\cdot(\omega\times(x-\vec{0})) + |\omega\times(x-\vec{0})|^2\right)$$

where we've used $Q^T\dot{Q} = W(\omega)$ and $v^\star = \dot{Q}(x-\vec{0}) + Q\dot{v}$.

Also, (ii)

$$\int_{\mathcal{P}^\star(t)} \rho^\star(r^\star + b^\star \cdot v^\star)$$

$$= \int_{\mathcal{P}(t)} \rho \left(r + Q^T(Qb + 2\dot{Q}v + \ddot{Q}(x - \vec{0}))\right)$$

$$= \int_{\mathcal{P}(t)} \rho \left[r + \left(b + 2Q^T\dot{Q}v + Q^T\ddot{Q}(x - \vec{0})\right) \cdot \left(v + Q^T\dot{Q}(x - \vec{0})\right)\right]$$

$$= \int_{\mathcal{P}(t)} \rho \Big[r + b \cdot v + b \cdot (\omega \times (x - \vec{0})) + Q^t\ddot{Q}(x - \vec{0}) \cdot v$$

$$+2(\omega \times v) \cdot (\omega \times (x - \vec{0})) + \ddot{Q}(x - \vec{0}) \cdot \dot{Q}(x - \vec{0})\Big]$$

Also, (iii)

$$\int_{\partial\mathcal{P}^\star(t)} \hat{q}^\star(n^\star) + s^\star(n^\star) \cdot v^\star$$

$$= \int_{\partial\mathcal{P}(t)} \hat{q}(n) + Qs(n) \cdot (\dot{Q}(x - \vec{0}) + Qv)$$

$$= \int_{\partial\mathcal{P}(t)} \hat{q}(n) + s(n) \cdot v + s(n) \cdot (\omega \times (x - \vec{0}))$$

Combining these 3 equations, we have

$$(i) - (ii) - (iii) = (\text{Energy Equation})^\star = (\text{Energy Equation}) +$$

$$+ \frac{d}{dt}\int_{\mathcal{P}(t)} \rho\omega \cdot (x - \vec{0}) \times v - \int_{\mathcal{P}(t)} \omega \cdot (x - \vec{0}) \times \rho b - \int_{\partial\mathcal{P}(t)} \omega \cdot (x - \vec{0}) \times s(n)$$