Carnegie Mellon University
Department of Mathematical Sciences
21-260 Differential Equations
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Solutions of First Order ODEs – Summary

1. **Simplest Equations:** $y' = f(x)$.

$$y(x) = \int f(x)\,dx + c.$$

2. **Linear Equations:** $y' + p(x)y = g(x)$

$$y(x) = \frac{1}{\mu(x)}\left[\int g(x)\mu(x)\,dx + c\right],$$

where $\mu(x) = \exp \int p(x)\,dx$.

3. **Separable Equations:** $y' = g(x)h(y)$

   (a) If $h(y^*) = 0$ for some constant y^*, then $y = y^*$.

   (b) If $h(y) \neq 0$, then y is given by

$$\int \frac{dy}{h(y)} = \int g(x)\,dx + c.$$

4. **Exact Equations:** $M(x,y) + N(x,y)y' = 0$, where $M_y(x,y) = N_x(x,y)$.
The solution is given in an implicit form by $\psi(x,y) = c$, where

$$\psi(x,y) = \int M(x,y)\,dx + h(y). \tag{1}$$

(Here, $h(y)$ is determined by differentiating (1) with respect to y and using the relation $\psi_y = N$.)

5. **Substitutions:** Here are some types of equations that can be solved by making a suitable substitution.

   (a) **Bernoulli Equations:** $\frac{dy}{dx} + p(x)y = q(x)y^n$, $n \neq 0, 1$.
   Set $v = y^{1-n}$. The equation becomes

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x) \text{ (linear)}.$$

   (b) **Homogeneous Equations:** $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$
   Set $v = \frac{y}{x}$. The equation becomes

$$x\frac{dv}{dx} + v = F(v) \text{ (separable)}.$$

(c) **Miscellaneous Equations:**

i. $\frac{dy}{dx} = F(x \pm y)$
Set $v = x \pm y$. The equation becomes

$$\pm\left(\frac{dv}{dx} - 1\right) = F(v) \text{ (separable).}$$

ii. $\frac{dy}{dx} = f(x, y)$
It might be easier to solve

$$\frac{dx}{dy} = \frac{1}{f(x, y)}.$$

(I.e. find $x(y)$ instead of $y(x)$.)