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Dynamical Scaling in Smoluchowski's Coagulation Equations: Uniform Convergence*

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Abstract. Smoluchowski's coagulation equation is a fundamental mean-field model of clustering dynamics. We consider the approach to self-similarity (or dynamical scaling) of the cluster size distribution for the "solvable" rate kernels $K(x,y) = 2, x+y$, and xy. In the case of continuous cluster size distributions, we prove uniform convergence of densities to a self-similar solution with exponential tail, under the regularity hypothesis that a suitable moment have an integrable Fourier transform. For discrete size distributions, we prove uniform convergence under optimal moment hypotheses. Our results are completely analogous to classical local convergence theorems for the normal law in probability theory. The proofs rely on the Fourier inversion formula and the solution for the Laplace transform by the method of characteristics in the complex plane.

Key words. coarsening, dynamic scaling, self-similar solutions, complex characteristics

AMS subject classifications. 70F99, 82C28, 45M10, 35L65, 35Q99

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1. Introduction.

Smoluchowski's coagulation equation

$$\partial_t n(t,x) = \frac{1}{2}\int_0^x K(x-y,y)n(t,x-y)n(t,y)dy - \int_0^\infty K(x,y)n(t,x)n(t,y)dy \tag{1.1}$$

is a widely studied model for cluster growth [4, 11, 25]. We study the evolution of $n(t,x)$, the number of clusters of size or mass x per unit volume at time t. Clusters of mass x and y coalesce by binary collisions with a rate proportional to $K(x,y)n(t,x)n(t,y)$, where $K(x,y)$ is a symmetric rate kernel. Integrating over y yields the loss term in (1.1), and coalescence of clusters of mass $x-y$ and y produces the gain term. All details of the mechanism of coalescence are subsumed into the form of $K(x,y)$, and we make the mean-field assumption that the sizes of coalescing pairs are independent and occur with frequency proportional to the overall population of clusters of the same size. As time proceeds, one expects the total number of clusters to decrease and the typical cluster size to grow.

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Equation (1.1) has been used as a model of cluster growth in a surprisingly diverse range of fields such as physical chemistry, astrophysics, and population dynamics. For example, it has been used to model the coagulation of colloids [25], the formation of clouds and smog [11], the clustering of planets, stars, and galaxies [24], the kinetics of polymerization [27], and even the schooling of fishes [22] and the formation of "marine snow" [14]. In addition, over the past few years a rich mathematical theory has been developed for these equations. Aldous [1] provides an excellent introduction.

Many kernels in applications are homogeneous; that is, $K(\alpha x, \alpha y) = \alpha^\gamma K(x,y)$, $x, y, \alpha > 0$, for some exponent γ [4]. A mathematical problem of scientific interest is to study self-similar or dynamical scaling behavior for homogeneous kernels. There is an extensive scientific literature on the subject, especially formal asymptotics and numerics [17, 18, 26]. It is known that the degree of homogeneity γ plays a crucial role. On physical grounds, we expect solutions to (1.1) to conserve the total mass $\int_0^\infty xn(t,x)dx$. When $K(x,y) \leq 1 + x + y$ (corresponding to $0 \leq \gamma \leq 1$), mass-conserving solutions exist globally in time under suitable moment hypotheses on initial data [5]. It is then typical in applications to assert that the solutions approach "scaling form" [18, 26]. There is little mathematical justification for this in general. The existence of self-similar profiles and their asymptotics for general kernels was established only recently by Fournier and Laurençot [9] and Escobedo, Mischler, and Rodriguez Ricard [7]. It is still not known if "typical" solutions approach these profiles as $t \to \infty$.

For a large class of kernels satisfying $(xy)^{\gamma/2} \leq K(x,y)$ with $1 < \gamma \leq 2$, it is known that there is no solution that preserves mass for all time. This breakdown phenomenon is known as *gelation*. It was first demonstrated by McLeod [19] with an explicit solution for the kernel $K = xy$. The first result for a general class of kernels was proved probabilistically by Jeon [12]. Simple analytical proofs have since been found; see in particular [6]. It is natural to ask whether the blow-up is self-similar, but there are no general results on this problem yet.

There are a number of results, however, for the "solvable" kernels $K = 2$, $x + y$, and xy, for which $\gamma = 0$, 1, and 2, respectively. A remarkable feature of these kernels is that the problem of dynamical scaling can be understood quite deeply by analogy with classical limit theorems in probability theory. Some examples are as follows:

(a) Smoluchowski's equation defines a continuous dynamical system on the space of probability measures on $(0, \infty)$ with the weak-$*$ topology. For any initial number measure ν_0 with finite γth moment, there is a unique solution in a suitable weak sense, $t \mapsto \nu_t$ (corresponding to $n(x,t)\,dx$ if ν_t has a density), to which is associated a natural probability distribution function

$$F_t(x) = \int_{(0,x]} y^\gamma \nu_t(dy) \Big/ \int_0^\infty y^\gamma \nu_t(dy),$$

and $t \mapsto F_t(x)$ is continuous [20]. This optimal well-posedness theorem also holds for a wide class of kernels [10]. It is akin to the simple and basic fact that addition of independent, identically distributed random variables generates discrete dynamics on the space of probability measures.

(b) There is a one-parameter family of self-similar solutions to (1.1). Of these, only one has a finite $(\gamma+1)$st moment, and its profile decays exponentially as $x \to \infty$. All others have algebraic decay. This is analogous to the classification of the *Lévy stable laws* in probability theory—the normal distribution is the only stable law with finite variance; all the others have algebraic tails.

(c) The domains of attraction of self-similar solutions are classified completely in terms of the tails of the initial data: Each $\alpha \in (1,2]$ corresponds to a domain of attraction (or universality class) that consists of all ν_0 with the property that $\int_0^x y^{\gamma+1}\nu_0(dy) \sim x^{2-\alpha}L(x)$ as $x \to \infty$, where L is a slowly varying function in the sense of Karamata. This is completely analogous to the central limit theorem and the characterization of the domains of attraction of the stable laws in probability theory. See [20] for a more precise and general assertion.

(d) Bertoin showed that all eternal solutions for the kernel $K = x + y$ can be classified by a Lévy–Khintchine formula [2], just like the infinitely divisible distributions in probability theory. For this kernel, solutions are eternal if they are defined for all $t \in (-\infty, \infty)$, meaning that they model coagulation processes "infinitely divisible" under Smoluchowski dynamics. Such a Lévy–Khintchine formula also holds for the other solvable kernels, and may be used to characterize the attractor of this dynamical system modulo scaling [21].

The probabilistic analogy may also be used as a basis for refined convergence theorems. A general theme in probabilistic limit theorems is the interplay between moment and regularity hypotheses and the topology of convergence. In this article, we develop one aspect of this idea. Under stronger regularity hypotheses, the weak convergence results of [20] will be strengthened to obtain uniform convergence of densities using the Fourier transform. This method is classical in probability theory and is used to prove uniform convergence of densities in the central limit theorem [8, Theorem XV.5.2]. Feller's argument in [8] is simple and robust, and our main contribution is to show that it extends naturally to Smoluchowski's equation. The key new idea is to use the method of characteristics in the right half of the complex plane to obtain strong decay estimates on the Laplace transform. A broader contribution of this work and [20] is to show that the analytical methods used to prove classical limit theorems in probability apply to a wider range of problems involving scaling phenomena for integral equations of convolution type.

Let us briefly connect our results to earlier and later work. The only uniform convergence theorems in the literature are those of Kreer and Penrose for the kernel $K = 2$ [15] and closely related work of da Costa [3]. In this article, for $K = 2$ and $x + y$ we present theorems on uniform convergence to the self-similar solutions with exponential tails for the continuous and discrete Smoluchowski equations. For $K = xy$, we prove uniform convergence of densities to self-similar form as t approaches the gelation time $T_{\rm gel}$. For $K = 2$, we strengthen the result of Kreer and Penrose and simplify the proof. Their decay hypothesis on the initial data ($n_0(x) \leq Ce^{-ax}$) is weakened to an (almost) optimal moment hypothesis, and their regularity hypothesis ($n_0 \in C^2$) is weakened to a little bit more than continuity. For $K = x + y$ the convergence theorem is new. Study of the kernel $K = xy$ is reduced to $K = x + y$ by a well-known change of variables [4]. Uniform convergence to the self-similar solutions with "fat" or "heavy" tails is a more delicate issue, which will not be considered. All these results (including ours) rely on the solution via the Laplace transform. In a different approach, Laurençot and Mischler proved weak convergence to the self-similar solution with exponential tails for $K = 2$ by constructing Lyapunov functions [16].

Our uniform convergence theorems may be stated in a unified manner as follows for the continuous Smoluchowski equations with kernels $K(x, y) = 2$, $x + y$, and xy, corresponding to $\gamma = 0, 1, 2$, respectively. Presuming that the γth and $(\gamma + 1)$st moments are finite, we may scale x and n so both moments are initially 1. For the multiplicative kernel this ensures that the gelation time $T_{\rm gel} = 1$. Let $T_\gamma = \infty$ for

$\gamma = 0, 1$ and $T_\gamma = T_{\text{gel}} = 1$ for $\gamma = 2$. The self-similar solutions with exponential tails have the explicit form [1, 20]

$$n(t,x) = \frac{m_\gamma(t)}{\lambda_\gamma(t)^{\gamma+1}}\,\hat{n}_{*,\gamma}\left(\frac{x}{\lambda_\gamma(t)}\right), \tag{1.2}$$

where the self-similar profiles $\hat{n}_{*,\gamma}$ are given by

$$\hat{n}_{*,0}(\hat{x}) = e^{-\hat{x}}, \qquad \hat{x}\hat{n}_{*,1}(\hat{x}) = \hat{x}^2\hat{n}_{*,2}(\hat{x}) = \frac{1}{\sqrt{2\pi}}\hat{x}^{-1/2}e^{-\hat{x}/2} \tag{1.3}$$

for $\hat{x} \geq 0$, and the moments m_γ and scaling factors λ_γ are given by

$$m_0(t) = t^{-1}, \quad m_1(t) = 1, \quad m_2(t) = (1-t)^{-1}, \tag{1.4}$$

$$\lambda_0(t) = t, \quad \lambda_1(t) = e^{2t}, \quad \lambda_2(t) = (1-t)^{-2}. \tag{1.5}$$

Our sufficient conditions for uniform convergence to these self-similar solutions for the continuous Smoluchowski equations are summarized by the following result.

THEOREM 1.1. *Let* $n_0 \geq 0$, $\int_0^\infty x^\gamma n_0(x)dx = \int_0^\infty x^{1+\gamma} n_0(x)dx = 1$. *Assume that the Fourier transform of* $x^{1+\gamma}n_0$ *is integrable, and let* $n(t,x)$ *be the solution to Smoluchowski's equation with initial data* $n_0(x)$ *and* $K = 2$, $x+y$, *or* xy *for* $\gamma = 0, 1$, *or* 2. *Then the rescaled solution*

$$\hat{n}(t,\hat{x}) = \frac{\lambda_\gamma(t)^{1+\gamma}}{m_\gamma(t)} n(t, \hat{x}\lambda_\gamma(t))$$

satisfies

$$\lim_{t\to T_\gamma} \sup_{\hat{x}>0} \hat{x}^{1+\gamma}\left|\hat{n}(t,\hat{x}) - \hat{n}_{*,\gamma}(\hat{x})\right| = 0.$$

It has been traditional to treat the discrete Smoluchowski equations separately from the continuous equations. Yet, within the framework of measure valued solutions [20, 23], the discrete Smoluchowski equations simply correspond to the special case of a lattice distribution, a measure-valued solution supported on the lattice $h\mathbb{N}$ and taking the form

$$\nu_t = \sum_{l=1}^\infty n_l(t)\delta_{hl}(x),$$

where $\delta_{hl}(x)$ is a Dirac delta at hl. If h is maximal we call ν_t a lattice measure with *span* h. The coefficients n_l satisfy the discrete Smoluchowski equations

$$\partial_t n_l(t) = \frac{1}{2}\sum_{j=1}^{l-1}\kappa_{l-j,j}n_{l-j}(t)n_j(t) - \sum_{j=1}^\infty \kappa_{l,j}n_l(t)n_j(t), \tag{1.6}$$

where $\kappa_{l,j} = K(lh, jh)$. Physically, this case is of importance since some mass aggregation processes (e.g., polymerization) have a fundamental unit of mass (e.g., a monomer). The uniform convergence theorems for the continuous Smoluchowski equations have a natural extension to this case.

THEOREM 1.2. *Let* $\nu_0 \geq 0$ *be a lattice measure with span* h *such that* $\int_0^\infty x^\gamma \nu_0(dx) = \int_0^\infty x^{1+\gamma}\nu_0(dx) = 1$. *Then with*

$$\hat{l} = \frac{lh}{\lambda_\gamma(t)}, \quad \hat{n}_l(t) = \frac{1}{h}\frac{\lambda_\gamma(t)^{1+\gamma}}{m_\gamma(t)}n_l(t),$$

we have

$$\lim_{t\to T_\gamma} \sup_{l\in\mathbb{N}} \hat{l}^{1+\gamma} \left| \hat{n}_l(t) - \hat{n}_{*,\gamma}(\hat{l}) \right| = 0.$$

Let us comment on the hypotheses and rescaling in Theorems 1.1 and 1.2. The moment hypotheses in both theorems are essentially the same. $\int_0^\infty x^\gamma \nu_0(dx) = 1$ is the natural hypothesis for existence and uniqueness of solutions [20]. The other moment condition $\int_0^\infty x^{1+\gamma}\nu_0(dx) = 1$ is of a different character. It implies that n_0 or ν_0 is in the weak domain of attraction of the self-similar solution with exponential tail under a rescaling $n(t,x) \longrightarrow \hat{n}(\hat{t},\hat{x})$ that fixes both moments:

$$\int_0^\infty \hat{x}^\gamma \hat{n}(\hat{t},\hat{x}) d\hat{x} = \int_0^\infty \hat{x}^{\gamma+1}\hat{n}(\hat{t},\hat{x})d\hat{x} = 1 \quad \text{for all } 0 \le \hat{t} < T_\gamma. \tag{1.7}$$

The hypothesis that the $(\gamma+1)$st moment is finite is almost optimal. The weak domain of attraction under a broader class of rescalings is a bit bigger, as it allows for a weak divergence $\int_0^y x^{1+\gamma}\nu_0(dx) \sim L(y)$ as $y \to \infty$ for a slowly varying function $L(y)$ [20]. Thus, Theorem 1.2 shows that within the class of lattice measures, the weak convergence of measures almost implies uniform convergence of the coefficients.

The rescaling (1.7) demands more explanation in the case of the kernel $K = xy$, for which our results establish a self-similar approach to gelation. In this case, the normalization $\int_0^\infty x^2 n_0(dx) = 1$ ensures the time of gelation $T_{\text{gel}} = 1$. The rescaling (1.7) corresponds to the similarity variables

$$\hat{x} = (1-t)^2 x, \quad \hat{n}(t,\hat{x}) = \frac{n(t,\hat{x}(1-t)^{-2})}{(1-t)^5} = \frac{n(t,x)}{(1-t)^5}, \tag{1.8}$$

and the self-similar profile is

$$\hat{n}_{*,2}(\hat{x}) = \frac{1}{\sqrt{2\pi\hat{x}^5}} e^{-\hat{x}/2}. \tag{1.9}$$

This rescaling *does not* preserve mass. Even if $\int_0^\infty x n_0(dx) < \infty$ we have

$$\int_0^\infty \hat{x}\hat{n}(t,\hat{x})d\hat{x} = \frac{1}{1-t}\int_0^\infty x n(t,x)\,dx = \frac{1}{1-t}\int_0^\infty x n_0(dx) \to \infty.$$

Instead, the rescaling preserves the second moment:

$$\int_0^\infty \hat{x}^2 \hat{n}(t,\hat{x})d\hat{x} = (1-t)\int_0^\infty x^2 n(t,x)\,dx = 1, \quad t \in [0,1).$$

The explanation is that the scaling in (1.8) is designed to capture the behavior of the distribution of large clusters as t approaches T_{gel}—the average cluster size is $(1-t)^{-1}$. Correspondingly, the mass of the self-similar solution itself is infinite.

Theorem 1.1 requires an additional hypothesis on the integrability of a suitable Fourier transform. This is a regularity hypothesis that is the analogue of the hypothesis for uniform convergence to the normal law used by Feller [8]. One may heuristically understand the role of regularity as follows. Equation (1.1) is hyperbolic and discontinuities in the initial data persist for all finite times. On the other hand, the self-similar solutions in (1.3) are analytic. Thus, one expects that some regularity of the initial data is necessary to obtain uniform convergence to a self-similar solution. Loosely speaking, regularity of the initial data $n_0(x)$ translates into a decay hypothesis on its Fourier transform. We need only the weak decay implied by integrability.

We do not know if this assumption is optimal, or if it may be weakened further. We briefly comment on this issue here; it will not be considered in the rest of the paper. The space of functions with integrable Fourier transforms is of great interest in harmonic analysis. Precisely, for $f \in L^1(\mathbb{R})$, let F be its Fourier transform. Then the space

$$A(\mathbb{R}) = \{f \in L^1(\mathbb{R}) | F \in L^1(\mathbb{R})\}$$

is a closed subalgebra of $L^1(\mathbb{R})$ known as the Wiener algebra [13]. Integrability of F implies that f is continuous. But it also implies more. It is known that functions in $A(\mathbb{R})$ possess some delicate regularity properties. For example, a function in $A(\mathbb{R})$ has a logarithmic modulus of continuity in a neighborhood where it is monotonic. It is definitely not obvious whether this regularity is truly necessary to obtain uniform convergence. If $v_0(ik) = \int_0^\infty e^{-ikx} x^{1+\gamma} n_0(x) dx$ is integrable, it also follows that $v_0 \in H^1(\mathbb{R}) \cap A(\mathbb{R})$, since v_0 is the boundary limit of an analytic function (the Laplace transform of $x^{1+\gamma} n_0$). Here H^1 denotes the classical Hardy space. This in turn means that v_0 has some hidden regularity and integrability properties. It is worth remarking that the precise characterization of $A(\mathbb{R})$ remains an outstanding open problem in harmonic analysis (though several sufficient conditions are known; see [13]).

2. Uniform Convergence of Densities for the Constant Kernel *K* = 2.

2.1. Evolution of the Laplace Transform. Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ and $\bar{\mathbb{C}}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$. We let

$$N(t,z) = \int_0^\infty e^{-zx} n(t,x)\, dx, \quad z \in \bar{\mathbb{C}}_+,$$

denote the Laplace transform of the number density n. We take the Laplace transform of (1.1) with $K = 2$, and take its limit as $z \to 0$, to see that $N(t,z)$ solves

$$\partial_t N = N^2 - 2N(t,0)N, \quad \partial_t N(t,0) = -N(t,0)^2. \tag{2.1}$$

Without loss of generality, we may suppose that the initial time $t = 1$. We will always assume that the initial data is normalized such that

$$\int_0^\infty n(1,x)\, dx = \int_0^\infty x n(1,x)\, dx = 1. \tag{2.2}$$

If the initial number of clusters, $\int_0^\infty n(1,x) dx$, and the mass, $\int_0^\infty x n(1,x) dx$, are finite, we may always assume that (2.2) holds after rescaling x and n. We solve the second equation in (2.1) to see that the total number of clusters decreases according to

$$\int_0^\infty n(t,x)\, dx = N(t,0) = t^{-1}, \quad t \geq 1. \tag{2.3}$$

We hold z fixed and integrate (2.1) in t to obtain the solution

$$N(t,z) = \frac{1}{t} \frac{N(1,z)}{t(1 - N(1,z)) + N(1,z)}. \tag{2.4}$$

The evolution preserves mass. Indeed, if we differentiate (2.4) with respect to z, we find

$$\int_0^\infty x n(t,x)\, dx = -\partial_z N(t,0) = -\partial_z N(1,0) = \int_0^\infty x n(1,x)\, dx = 1. \tag{2.5}$$

2.2. Approach to Self-Similarity. A special case of the weak convergence result of [20], also given by Leyvraz [18], is obtained as follows: Observe that for each fixed $s \in \bar{\mathbb{C}}_+$, equations (2.3), (2.4), and (2.5) imply

$$tN(t, st^{-1}) = \frac{N(1, st^{-1})}{t(1 - N(1, st^{-1})) + N(1, st^{-1})} \underset{t\to\infty}{\longrightarrow} \frac{1}{1+s}. \tag{2.6}$$

It is classical that the pointwise convergence of Laplace transforms is equivalent to weak convergence of measures [8, Theorem XIII.1.2a]. Thus, (2.6) implies that rescaled solutions to Smoluchowski's equations converge weakly. Let us be more precise about the rescaling. We define the similarity variables

$$\tau = \log t, \quad \hat{x} = \frac{x}{t} = e^{-\tau} x, \quad s = tz = e^{\tau} z \tag{2.7}$$

and the rescaled number distribution

$$\hat{n}(\tau, \hat{x}) = e^{2\tau} n(e^{\tau}, e^{\tau}\hat{x}) = t^2 n(t, x). \tag{2.8}$$

Observe that this rescaling preserves *both* total number and mass, that is,

$$\int_0^\infty \hat{n}(\tau, \hat{x}) d\hat{x} = \int_0^\infty \hat{x}\hat{n}(\tau, \hat{x}) d\hat{x} = 1, \quad \tau \geq 0. \tag{2.9}$$

We denote the Laplace transform of $\hat{n}(\tau, \hat{x})$ by

$$u(\tau, s) = \int_0^\infty e^{-s\hat{x}} \hat{n}(\tau, \hat{x})\, d\hat{x} = e^{\tau} N(e^{\tau}, se^{-\tau}) = tN(t, z). \tag{2.10}$$

In these variables, the pointwise convergence of (2.6) takes the simple form

$$\lim_{\tau\to\infty} u(\tau, s) = \frac{1}{1+s} =: u_{*,0}(s), \quad s \in \bar{\mathbb{C}}_+, \tag{2.11}$$

where $u_{*,0}(s)$ denotes the Laplace transform of

$$\hat{n}_{*,0}(\hat{x}) = e^{-\hat{x}}, \quad \hat{x} \geq 0, \tag{2.12}$$

the profile for the self-similar solution in (1.2). Now, (2.11) is equivalent to

$$\hat{n}(\tau, \hat{x})\, d\hat{x} \to \hat{n}_{*,0}(\hat{x})\, d\hat{x}$$

as $\tau \to \infty$, in the sense of weak convergence of measures.

Our goal is to strengthen this to uniform convergence in both continuous and discrete cases, under appropriate hypotheses on initial data. For the continuous Smoluchowski equation (1.1) we prove the following theorem.

THEOREM 2.1. *Let $n(1, x) \geq 0$, $\int_0^\infty n(1, x)\, dx = \int_0^\infty xn(1, x)\, dx = 1$. Assume that the Fourier transform of $xn(1, x)$ is integrable. Then in terms of the rescaling in* (2.7)–(2.8) *we have*

$$\lim_{\tau\to\infty} \sup_{\hat{x}>0} \hat{x}|\hat{n}(\tau, \hat{x}) - \hat{n}_{*,0}(\hat{x})| = 0, \tag{2.13}$$

where $\hat{n}_{*,0}(\hat{x}) = e^{-\hat{x}}$ is the similarity profile in (2.12).

The proof of this theorem extends to the treatment of uniform convergence of coefficients for solutions of the discrete equations (1.6) under only the hypothesis that the zeroth and first moments are finite; see Theorem 2.2 below.

Observe that we prove uniform convergence of the weighted densities $\hat{x}\hat{n}(\tau,\hat{x})$. The reason can be ascribed to use of the Fourier–Laplace inversion formula. We cannot apply the inversion formula directly to $u_{*,0}$ as it is not integrable on the imaginary axis ($|u_{*,0}(ik)| \sim |k|^{-1}$ as $|k| \to \infty$). The slow decay of the Fourier transform is caused by the jump discontinuity at $x = 0$, since $\hat{n}_{*,0}(x) = 0$ for $x < 0$. In order to gain a uniform convergence result, we smooth this discontinuity and consider the mass density $\hat{x}\hat{n}$. Its Laplace transform we denote by

$$v(\tau, s) = -\partial_s u(\tau, s) = \int_0^\infty e^{-s\hat{x}} \hat{x}\hat{n}(\tau, \hat{x})\, d\hat{x}. \tag{2.14}$$

Differentiating (2.11), we obtain a corresponding self-similar profile, with

$$v_{*,0}(s) := \frac{1}{(1+s)^2}, \qquad |v_{*,0}(ik)| = \frac{1}{1+k^2},\ k \in \mathbb{R}. \tag{2.15}$$

2.3. Evolution on Characteristics. The explicit solution for $u(\tau, s)$ and $v(\tau, s)$ can be obtained directly by substituting (2.10) into (2.4). But we rederive the solution to make explicit the geometric idea underlying the proof of Theorem 2.1. The same ideas underlie the proof of Theorem 3.1 for the additive kernel and are more easily understood here. We use the change of variables (2.7) and (2.10) in (2.1), and the conservation of moments in (2.9), to obtain the equation of evolution for u:

$$\partial_\tau u + s\partial_s u = -u(1-u). \tag{2.16}$$

The solution of (2.16) may be described by the method of characteristics. A characteristic curve $s(\tau; \tau_0, s_0)$ is the solution to

$$\frac{ds}{d\tau} = s, \quad s(\tau; \tau_0, s_0) = s_0 \in \bar{\mathbb{C}}_+. \tag{2.17}$$

Explicitly,

$$s(\tau; \tau_0, s_0) = e^{\tau-\tau_0} s_0. \tag{2.18}$$

Equation (2.17) is an autonomous differential equation in $\bar{\mathbb{C}}_+$ and may be thought of geometrically. For fixed $s_0 \in \bar{\mathbb{C}}_+$ the trajectory of the characteristic curve $s(\tau; \tau_0, s_0)$, $\tau \in \mathbb{R}$, is a ray in $\bar{\mathbb{C}}_+$ emanating from the origin. In particular, the imaginary axis is invariant under the flow of (2.17). Equation (2.18) shows that the characteristics expand outward uniformly at the rate e^τ. Among characteristics we have

$$\frac{du}{d\tau} = -u(1-u), \tag{2.19}$$

which may be integrated to obtain the solution

$$u(\tau, s) = \frac{u(\tau_0, s_0)e^{-(\tau-\tau_0)}}{1 - u(\tau_0, s_0)(1 - e^{-(\tau-\tau_0)})}. \tag{2.20}$$

We need to estimate the decay of the derivative $v = -\partial_s u$. Differentiating (2.16), we see that on characteristics the derivative solves

$$\frac{dv}{d\tau} = -2(1-u)v. \tag{2.21}$$

We integrate (2.21) using (2.20) to find

$$v(\tau,s)=\frac{v(\tau_0,s_0)e^{-2(\tau-\tau_0)}}{\left(1-u(\tau_0,s_0)(1-e^{-(\tau-\tau_0)})\right)^2}. \tag{2.22}$$

For $\tau\geq\tau_0$ we may take absolute values in (2.20) and (2.22) to obtain the decay estimates

$$|u(\tau,s)|\leq\frac{|u(\tau_0,s_0)|e^{-(\tau-\tau_0)}}{1-|u(\tau_0,s_0)|(1-e^{-(\tau-\tau_0)})} \tag{2.23}$$

and

$$|v(\tau,s)|\leq\frac{|v(\tau_0,s_0)|e^{-2(\tau-\tau_0)}}{\left(1-|u(\tau_0,s_0)|(1-e^{-(\tau-\tau_0)})\right)^2}\leq\frac{|v(\tau_0,s_0)|e^{-2(\tau-\tau_0)}}{\left(1-|u(\tau_0,s_0)|\right)^2}. \tag{2.24}$$

2.4. Proof of Theorem 2.1. 1. We use the Fourier–Laplace inversion formula

$$\hat{x}(\hat{n}(\tau,\hat{x})-\hat{n}_{*,0}(\hat{x}))=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ik\hat{x}}\left(v(\tau,ik)-v_{*,0}(ik)\right)\,dk. \tag{2.25}$$

Thus, in order to prove (2.13) it suffices to show

$$\lim_{\tau\to\infty}\int_{\mathbb{R}}|v(\tau,ik)-v_{*,0}(ik)|\,dk=0. \tag{2.26}$$

2. Let $\varepsilon\in(0,\frac{1}{2})$ and put $R=\varepsilon^{-1}$. We will prove (2.26) by estimating the integral separately in three regions: $|k|\leq R$, $R\leq|k|\leq Re^{\tau-T}$, and $Re^{\tau-T}\leq|k|$ for $\tau\geq T$, where $T>0$ will be chosen sufficiently large, depending on ε and the initial data v_0. This is essentially the same decomposition used in the proof of uniform convergence in the central limit theorem by Feller [8, Theorem XV.5.2]. The main new idea here is the use of the decay estimates (2.24) and the method of characteristics in the regions where $R\leq|k|$.

3. $|k|\leq R$: Recall that the pointwise convergence of Laplace transforms (2.11) is equivalent to $\hat{n}(\tau,\hat{x})\,d\hat{x}\to\hat{n}_{*,0}(\hat{x})\,d\hat{x}$ in the sense of weak convergence of measures. Combined with (2.9) this also implies that the mass measures $\hat{x}\hat{n}(\tau,\hat{x})\,d\hat{x}$ converge weakly to $\hat{x}\hat{n}_{*,0}(\hat{x})\,d\hat{x}$ as $\tau\to\infty$. But this implies $v(\tau,ik)$ converges to $v_{*,0}(ik)$ uniformly for $|k|\leq R$ [8, Theorem XV.3.2]. Therefore,

$$\lim_{\tau\to\infty}\int_{-R}^{R}|v(\tau,ik)-v_{*,0}(ik)|\,dk=0. \tag{2.27}$$

4. It remains to consider $|k|\geq R$. It is sufficient to consider only $k\geq R$, since $|v(\tau,ik)|=|v(\tau,-ik)|$. We will control $v(\tau,ik)$ and $v_{*,0}$ separately:

$$\int_R^\infty|v(\tau,ik)-v_{*,0}(ik)|\,dk\leq\int_R^\infty|v(\tau,ik)|\,dk+\int_R^\infty|v_{*,0}(ik)|\,dk.$$

But $|v_{*,0}(ik)|=(1+|k|^2)^{-1}$ by (2.15), so that

$$\int_R^\infty|v_{*,0}(ik)|\,dk\leq R^{-1}=\varepsilon.$$

In the rest of the proof we estimate $\int_R^\infty|v(\tau,ik)|dk$.

5. Since $u(\tau, ik) \to u_{*,0}(ik)$ and $v(\tau, ik) \to v_{*,0}$ as $\tau \to \infty$ for each real k, using (2.11) and (2.15) we may choose $T > 0$ such that

$$\sup_{\tau \geq T} |u(\tau, iR)| \leq R^{-1} = \varepsilon, \quad \sup_{\tau \geq T} |v(\tau, iR)| \leq R^{-2}. \tag{2.28}$$

6. $R \leq k \leq Re^{\tau - T}$: The control obtained from (2.28) propagates outward along characteristics as τ increases. Precisely, whenever $\tau \geq T$, for any k such that $R \leq k \leq Re^{\tau-T}$ we have $ik = e^{\tau-\tau_0} iR$, where $\tau_0 \geq T$. By (2.18) this means that $ik = s(\tau; \tau_0, s_0)$, with $s_0 = iR$. Then the decay estimate (2.24) and the boundary control (2.28) imply

$$|v(\tau, ik)| \leq \frac{|v(\tau_0, iR)| e^{-2(\tau-\tau_0)}}{(1 - |u(\tau_0, iR)|)^2} \leq \frac{1}{(1-\varepsilon)^2} R^{-2} \left(\frac{R}{k}\right)^2 \leq 4k^{-2}. \tag{2.29}$$

Integrating this estimate we obtain

$$\int_R^{Re^{\tau-T}} |v(\tau, ik)|\, dk \leq \int_R^\infty 4k^{-2}\, dk = 4R^{-1} = 4\varepsilon.$$

7. $Re^{\tau-T} \leq k$: For brevity, let $\tilde{R} = Re^{-T}$. With $u_0(s) := u(0, s)$, $v_0(s) := v(0, s)$, we use (2.24) and (2.18) with $\tau_0 = 0$ to obtain

$$\begin{aligned} \int_{\tilde{R}e^\tau}^\infty |v(\tau, ik)|\, dk &\leq e^{-2\tau} \int_{\tilde{R}e^\tau}^\infty \frac{|v_0(ike^{-\tau})|}{(1 - |u_0(ike^{-\tau})|)^2} dk \\ &= e^{-\tau} \int_{\tilde{R}}^\infty \frac{|v_0(ik')|}{(1 - |u_0(ik')|)^2} dk' \leq \left(\sup_{|k'| \geq \tilde{R}} \frac{1}{(1 - |u_0(ik')|)^2} \right) e^{-\tau} \|v_0\|_{L^1}, \end{aligned}$$

where $k' = ke^{-\tau}$. Since $|u_0(ik')| < 1$ for $k' \neq 0$ and $u_0(ik') \to 0$ as $k \to \infty$ by the Riemann–Lebesgue lemma, we have $\sup_{|k'| \geq \tilde{R}} (1 - |u_0(ik')|)^{-2} < \infty$.

8. Putting together the estimates we have obtained, it follows that for τ sufficiently large, the integral in (2.26) is less than 12ε. This completes the proof.

2.5. The Discrete Smoluchowski Equations. We consider measure solutions of the form $\nu_t = \sum_{l=1}^\infty n_l(t) \delta_{hl}(x)$, where $\delta_{hl}(x)$ denotes a Dirac mass at hl. To avoid redundancy, we always assume that h is the *span* of the lattice, that is, the maximal $h > 0$ so that all initial clusters, and thus clusters at any time $t > 0$, are concentrated on $h\mathbb{N}$. We will call ν_t a lattice measure with span h. Notice that if the initial number of clusters and the mass are finite, by rescaling n_l and h we may assume that $\int_0^\infty \nu_1(dx) = \int_0^\infty x\nu_1(dx) = 1$. Under these conditions, the weak convergence theorem of [20] asserts that $\lim_{t\to\infty} tN(t, s/t) = u_{*,0}(s)$. We show that this theorem may be strengthened by use of Fourier series. The Fourier transform of ν_t is the Fourier series

$$N(t, ik) = \sum_{l \in \mathbb{N}} n_l(t) e^{-ilhk}, \qquad k \in \mathbb{R},$$

which has minimal period $2\pi/h$. Thus $n_l(t) = (h/2\pi) \int_{-\pi/h}^{\pi/h} e^{ilhk} N(t, ik)\, dk$, or

$$t^2 n_l(t) = \frac{h}{2\pi} \int_{-\pi e^\tau/h}^{\pi e^\tau/h} \exp(ilhke^{-\tau}) u(\tau, ik)\, dk \tag{2.30}$$

in similarity variables from (2.10). We integrate by parts and let

$$\hat{l} = lhe^{-\tau} = lht^{-1}, \quad \hat{n}_l(t) = h^{-1}t^2 n_l(t) \tag{2.31}$$

to obtain

$$\hat{l}\hat{n}_l(t) = tln_l(t) = \frac{1}{2\pi}\int_{-\pi e^\tau/h}^{\pi e^\tau/h} e^{i\hat{l}k} v(\tau, ik)\, dk. \tag{2.32}$$

As in Theorem 2.1, we expect the right-hand side to converge to $\hat{l}\hat{n}_{*,0}(\hat{l})$ as $\tau \to \infty$, indeed uniformly for $\hat{l} \in ht^{-1}\mathbb{N}$.

THEOREM 2.2. *Let $\nu_1 \geq 0$ be a lattice measure with span h such that $\int_0^\infty \nu_1(dx) = \int_0^\infty x\nu_1(dx) = 1$. Then with the scaling* (2.31) *we have*

$$\lim_{t\to\infty} \sup_{l\in\mathbb{N}} \hat{l}\left|\hat{n}_l(t) - \hat{n}_{*,0}(\hat{l})\right| = 0. \tag{2.33}$$

Proof. By (2.32) and the continuous Fourier inversion formulas, it suffices to show that

$$\lim_{\tau\to\infty} \sup_{\hat{l}\geq 0} \left| \int_{-\pi e^\tau/h}^{\pi e^\tau/h} e^{i\hat{l}k} v(\tau, ik)\, dk - \int_{\mathbb{R}} e^{i\hat{l}k} v_{*,0}(ik)\, dk \right| = 0.$$

As earlier, it suffices to consider $k > 0$. The integrals

$$\int_{-R}^{R} |v(\tau, ik) - v_{*,0}(ik)|\, dk, \qquad \int_{R}^{\tilde{R}e^\tau} |v(\tau, ik)|\, dk, \qquad \int_{R}^{\infty} |v_{*,0}(ik)|\, dk,$$

with $\tilde{R} = Re^{-T}$, are controlled exactly as in the proof of Theorem 2.1. It remains only to estimate the integral of $|v(\tau, ik)|$ over the region $\tilde{R}e^\tau < k < \pi e^\tau/h$. We assume that $\pi/h > \tilde{R}$, for otherwise there is nothing to prove. But then by (2.18), the uniform decay estimate (2.24), and the change of variables $k' = ke^{-\tau}$, we have

$$\int_{\tilde{R}e^\tau}^{\pi e^\tau/h} |v(\tau, ik)|\, dk \leq e^{-\tau} \int_{\tilde{R}}^{\pi/h} \frac{|v_0(ik')|}{|1 - u_0(ik')(1 - e^{-\tau})|^2}\, dk'.$$

Since the domain of integration is finite, it suffices to show that the integrand is uniformly bounded in time. Since $|v_0(ik)| \leq 1$, it is necessary only to control the denominator. But $u_0(ik) = \sum_{l\in\mathbb{N}} n_l(0)e^{-ilkh}$ with $n_l(0) \geq 0$. Therefore, $|u_0(ik)| \leq 1$, and [8, Lemma XV.1.4] yields that

$$u_0(ik) = 1 \quad \text{if and only if} \quad k = \frac{2\pi m}{h}, \qquad m \in \mathbb{Z}.$$

In particular, we have the strict inequality

$$\min_{k\in[\tilde{R}, \frac{\pi}{h}]} |1 - u_0(ik)| \geq \delta > 0.$$

Therefore,

$$\left|1 - u_0(ik)(1 - e^{-\tau})\right| \geq |1 - u_0(ik)| - |u_0(ik)|e^{-\tau} \geq \delta - e^{-\tau} \geq \frac{\delta}{2}$$

for sufficiently large τ. Thus,

$$\int_{\tilde{R}e^\tau}^{\pi e^\tau/h} |v(\tau, ik)|\, dk \leq \frac{2\pi}{\delta h} e^{-\tau}. \qquad \square$$

3. Uniform Convergence of Densities for the Additive Kernel.

3.1. Rescaling and Approach to Self-Similarity. In this section we prove the analogues of Theorems 2.1 and 2.2 for the additive kernel. The essential geometric ideas of the proof are similar to the previous section. However, the trajectories of the characteristic curves $s(t; t_0, s_0)$ in the complex plane are no longer rays, and the proofs require more careful analysis. As before, we will work with the explicit solution formula for an appropriate Laplace transform. For $z \in \bar{\mathbb{C}}_+$ we define

$$\Phi(t,z) = \int_0^\infty \left(1 - e^{-zx}\right) n(t,x)dx. \tag{3.1}$$

We observe that $1 - e^{-zx} = zx + O(z^2x^2)$ as $x \to 0$. We use Φ instead of the standard Laplace transform of n because the latter may not be well defined: for example, the similarity profile $\hat{n}_{*,1}$ in (1.3) satisfies $\hat{n}_{*,1}(x) \sim Cx^{-3/2}$ as $x \to 0$. More generally, one needs the initial data to have only a finite first moment for existence and uniqueness of a solution to (1.1) in the case of the additive kernel [20]. A deeper reason for this choice of variables (and notation) is probabilistic: (3.1) is the Lévy–Khintchine formula for the Laplace exponent of a subordinator with no drift [2]. We will always assume that the initial data n_0 satisfies the moment conditions

$$\int_0^\infty xn_0(x)dx = 1, \quad \int_0^\infty x^2 n_0(x)dx = 1. \tag{3.2}$$

We substitute (3.1) in (1.1) and use (3.2) to see that $\Phi(t,z)$ solves the equation

$$\partial_t \Phi - \Phi\partial_z\Phi = -\Phi, \quad \Phi(0,z) = \int_0^\infty (1-e^{-zx})n_0(x)dx. \tag{3.3}$$

As shown in [20] by the method of characteristics, (3.3) has a unique solution for $z > 0$, $t > 0$ which is analytic with derivative $\partial_z\Phi$ completely monotone in z and satisfying $\partial_z\Phi(t,0) = 1$ for all t. For each $t > 0$, then, $\partial_z\Phi(t,\cdot)$ is the Laplace transform of a probability measure, so its domain contains $\bar{\mathbb{C}}_+$ and (3.3) holds by analytic continuation for $z \in \mathbb{C}_+$, $t > 0$.

In contrast with (2.4), it is not obvious that a suitable rescaling will lead to convergence to self-similar form. This point is discussed in [20, section 7], and we refer the reader to that article for motivation for the following change of variables. We define the similarity variables

$$\hat{x} = xe^{-2t}, \quad s = ze^{2t} \tag{3.4}$$

and the rescaled number density

$$\hat{n}(t,\hat{x}) = e^{4t}n(t,\hat{x}e^{2t}) = e^{4t}n(t,x). \tag{3.5}$$

We also define the rescaled Laplace transforms

$$\varphi(t,s) = e^{2t}\Phi(t,e^{-2t}s) = \int_0^\infty (1-e^{-s\hat{x}})\hat{n}(t,\hat{x})d\hat{x}. \tag{3.6}$$

Part of the motivation for the rescaling (3.4) and (3.5) is that this choice preserves *both* moment conditions in (3.2). That is, we have

$$\int_0^\infty \hat{x}\hat{n}(t,\hat{x})d\hat{x} = \int_0^\infty \hat{x}^2\hat{n}(t,\hat{x})d\hat{x} = 1, \quad t \geq 0. \tag{3.7}$$

This should be compared with (2.9) for the constant kernel. The mass measure plays the same role here as the number measure did for $K = 2$. Thus, we denote its Laplace transform by the same letter, and let

$$u(t,s) = \partial_s \varphi(t,s) = \int_0^\infty e^{-s\hat{x}} \hat{x} \hat{n}(t,\hat{x}) d\hat{x}. \tag{3.8}$$

By Theorem 7.1 in [20] (see also [18, Appendix G]), the assumptions in (3.2) imply that the rescaled mass measures converge to the similarity profile, with

$$\hat{x}\hat{n}(t,\hat{x})d\hat{x} \to \hat{x}\hat{n}_{*,1}(\hat{x})d\hat{x} = \frac{1}{\sqrt{2\pi}} \hat{x}^{-1/2} e^{-\hat{x}/2} d\hat{x}, \quad t \to \infty, \tag{3.9}$$

in the sense of weak convergence of measures. It then follows from [8, Theorem XIII.1.2] that (3.9) is equivalent to

$$\lim_{t\to\infty} u(t,s) = \frac{1}{\sqrt{1+2s}} =: u_{*,1}(s), \quad s \in \bar{\mathbb{C}}_+. \tag{3.10}$$

Our goal is to strengthen (3.9) to uniform convergence of densities for (1.1) and uniform convergence of coefficients for (1.6). For the continuous Smoluchowski equations we prove the following theorem.

THEOREM 3.1. *Suppose* $n_0(x) \geq 0$, $\int_0^\infty x n_0(x) dx = \int_0^\infty x^2 n_0(x) dx = 1$. *Suppose also that the Fourier transform of* $x^2 n_0$ *is integrable. Then in terms of the rescaling* (3.4)–(3.5) *we have*

$$\lim_{t\to\infty} \sup_{\hat{x}>0} \hat{x}^2 |\hat{n}(t,\hat{x}) - \hat{n}_{*,1}(\hat{x})| = 0, \tag{3.11}$$

where $\hat{n}_{*,1}(\hat{x})$ *is the similarity profile defined in* (1.3).

Once Theorem 3.1 is established, it is relatively straightforward to obtain the analogous result for the discrete Smoluchowski equations; see Theorem 3.6 below. Thus, most of our effort is devoted to Theorem 3.1.

Observe that we prove uniform convergence of the weighted density $\hat{x}^2\hat{n}(t,\hat{x})$. As in the previous section, this is because Theorem 3.1 is proved using the Fourier–Laplace inversion formula. Since $|u_{*,1}(ik)| \sim |k|^{-1/2}$ as $|k| \to \infty$, $u_{*,1}$ is not integrable on the imaginary axis. This divergence is due to the fact that $\hat{n}_{*,1}(\hat{x}) = 0$ for $\hat{x} < 0$ and $\hat{x}\hat{n}_{*,1}(\hat{x}) \sim C\hat{x}^{-1/2}$ as $\hat{x} \to 0^+$. As before, we resolve the situation by considering the transform of the next moment. Let

$$v(t,s) = -\partial_s u(t,s) = \int_0^\infty e^{-s\hat{x}} \hat{x}^2 \hat{n}(t,\hat{x}) d\hat{x}, \quad s \in \bar{\mathbb{C}}_+. \tag{3.12}$$

We integrate and differentiate (3.10) to obtain

$$\varphi_{*,1}(s) = \sqrt{1+2s} - 1, \quad v_{*,1}(s) = (1+2s)^{-3/2}, \quad s \in \bar{\mathbb{C}}_+. \tag{3.13}$$

3.2. Characteristics and Estimates. The equations of evolution for φ and u are

$$\partial_t \varphi + (2s - \varphi)\partial_s \varphi = \varphi, \tag{3.14}$$

$$\partial_t u + (2s - \varphi)\partial_s u = -u(1-u). \tag{3.15}$$

In what follows, we first derive solution formulas to (3.14) by the method of characteristics. We then show that the solution map for the characteristic equation is never degenerate and that characteristics flow out of the right half into the left half of the complex plane as t increases. For most parts of our analysis, it will suffice to study characteristics in the right half plane only. But for one part, we need to study characteristics that start in the right half plane but move into the left half plane.

We use the notation $s(t; t_0, s_0)$ to denote the solution to

$$\frac{ds}{dt} = 2s - \varphi, \quad s(t_0; t_0, s_0) = s_0. \tag{3.16}$$

Along the characteristic curve $s(t; t_0, s_0)$, we have

$$\frac{d\varphi}{dt} = \varphi \quad \text{and} \quad \frac{du}{dt} = -u(1-u). \tag{3.17}$$

We integrate (3.17) to obtain

$$\varphi(t,s) = e^{t-t_0}\varphi(t_0,s_0), \quad u(t,s) = \frac{u(t_0,s_0)e^{-(t-t_0)}}{1 - u(t_0,s_0)(1-e^{-(t-t_0)})}. \tag{3.18}$$

We now substitute for $\varphi(t,s)$ from (3.18) in (3.16) and integrate to obtain the explicit solution

$$e^{-2(t-t_0)}s(t;t_0,s_0) = s_0 - \varphi(t_0,s_0)(1-e^{-(t-t_0)}). \tag{3.19}$$

This equation can also be rewritten in two other useful forms, namely,

$$e^{-2(t-t_0)}\left(s - \varphi(t,s)\right) = \left(s_0 - \varphi(t_0,s_0)\right) \tag{3.20}$$

and

$$\frac{\varphi(t,s)}{s} = \frac{(\varphi(t_0,s_0)/s_0)e^{-(t-t_0)}}{1-(\varphi(t_0,s_0)/s_0)(1-e^{-(t-t_0)})}. \tag{3.21}$$

The method of characteristics also yields an explicit solution for $v(t,s)$. We differentiate (3.15) to obtain

$$\frac{dv}{dt} = -3(1-u)v. \tag{3.22}$$

We substitute for u from (3.18) and integrate (3.22) to obtain

$$v(t,s) = \frac{v(t_0,s_0)e^{-3(t-t_0)}}{\left(1-u(t_0,s_0)(1-e^{-(t-t_0)})\right)^3}. \tag{3.23}$$

Let $\varphi_0(s) := \varphi(0,s)$, and similarly $u_0(s) := u(0,s)$, $v_0(s) := v(0,s)$. Since $u = \partial_s\varphi$ and $\varphi(t,0) = 0$, the moment conditions (3.2) and the identity $\varphi_0(s)/s = \int_0^1 u_0(\tau s)\,d\tau$ imply

$$|u_0(s)| \le 1, \quad |v_0(s)| \le 1, \quad |\varphi_0(s)| \le |s|, \quad s \in \bar{\mathbb{C}}_+. \tag{3.24}$$

These inequalities are strict for $s \neq 0$ because $xn_0(x)\,dx$ is not a lattice measure [8, Lemma XV.1.4]. Taking $t_0 = 0$ at first, for $t \ge t_0$ we take absolute values in (3.18) and (3.23) to see that $|u|$ and $|v|$ decay along characteristics according to

$$|u(t,s)| \le \frac{|u(t_0,s_0)|e^{-(t-t_0)}}{1-|u(t_0,s_0)|(1-e^{-(t-t_0)})}, \tag{3.25}$$

$$|v(t,s)| \le \frac{|v(t_0,s_0)|e^{-3(t-t_0)}}{(1-|u(t_0,s_0)|)^3}. \tag{3.26}$$

From (3.25) and the fact that $|u_0(s_0)| < 1$ for $s_0 \neq 0$, and a similar estimate using (3.21) and $|\varphi_0(s_0)/s_0| < 1$, it follows that

$$|u(t,s)| < 1, \quad |\varphi(t,s)/s| < 1, \quad t \ge 0, \; s \neq 0. \tag{3.27}$$

Then (3.25) and (3.26) hold also for any $t_0 \ge 0$ if $t \ge t_0$.

Let us also note the uniform outward growth of characteristics implied by (3.27). Using (3.27) together with (3.16) we obtain

$$|s| \le \frac{d|s|}{dt} \le 3|s|. \tag{3.28}$$

Thus, $|s_0|e^{(t-t_0)} \le |s| \le e^{3(t-t_0)}|s_0|$. We will refine this crude estimate in the proof of Theorem 3.1, but we note here that $|s(t;t_0,s_0)|$ is a strictly increasing function of t.

In addition to the decay along characteristics, we will need the following uniform Riemann–Lebesgue lemma. Let $C_R = \{s \in \bar{\mathbb{C}}_+ \mid |s| = R\}$ denote the semicircle of radius R in the right half plane.

LEMMA 3.2. *Let $g(x) \in L^1(0,\infty)$ and $G(s) = \int_0^\infty e^{-sx} g(x)dx$. Then*

$$\lim_{R\to\infty} \sup_{s\in C_R} |G(s)| = 0. \tag{3.29}$$

Proof. Let $\varepsilon > 0$. We choose a step function $g_\varepsilon = \sum_{k=1}^K c_k 1_{[a_k,b_k]}$ so that $\|g - g_\varepsilon\|_{L^1} < \varepsilon$. But then $\|e^{-sx}(g-g_\varepsilon)\|_{L^1} < \varepsilon$. Therefore, for $s \in \bar{\mathbb{C}}_+$,

$$|G(s)| \le \varepsilon + \left|\int_0^\infty e^{-sx} g_\varepsilon(x)dx\right| = \varepsilon + \left|\sum_{k=1}^K c_k \int_{a_k}^{b_k} e^{-sx}dx\right| \le \varepsilon + \frac{C_\varepsilon}{|s|}. \qquad \square$$

We apply this lemma and (3.7) to $g(\hat{x}) = \hat{x}^j \hat{n}(t,\hat{x})$ for $j = 1, 2$ to infer that for every $t \ge 0$, as $|s| \to \infty$ with $\operatorname{Re} s \ge 0$, we have

$$|u(t,s)| \to 0, \quad |v(t,s)| \to 0, \quad \left|\frac{\varphi(t,s)}{s}\right| \to 0. \tag{3.30}$$

3.3. Geometry of the Characteristic Map in the Complex Plane. In this subsection, we study the solution formula (3.19). Our goal is to delineate some key properties of the map $s_0 \mapsto s(t;t_0,s_0)$ for $t, t_0 \ge 0$.

Let $\mathbb{C}_+$ denote the open right half plane. We let Ω_t denote the image of $\mathbb{C}_+$ under the map $s_0 \mapsto s(t;0,s_0)$, and let Γ_t denote the image of the imaginary axis under the same map. We aim to prove the following lemma.

LEMMA 3.3.

(i) *For any $t > 0$, Γ_t is a C^2 curve that passes through the origin but otherwise lies in the open left half plane. On Γ_t, $\operatorname{Re} s$ is a C^2 function of $\operatorname{Im} s$.*

(ii) *Ω_t is the component of the complex plane to the right of Γ_t. Consequently $\Gamma_t = \partial\Omega_t$ and $\Omega_t \supset \bar{\mathbb{C}}_+ \setminus \{0\}$.*

(iii) *Whenever $t_1 \ge t_0 \ge 0$, the map $s_0 \mapsto s_1 = s(t_1;t_0,s_0)$ is one to one from $\bar{\Omega}_{t_0}$ onto $\bar{\Omega}_{t_1}$. It is C^2 on $\bar{\Omega}_{t_0}$ and analytic in Ω_{t_0}. The inverse map is given by $s_1 \mapsto s_0 = s(t_0;t_1,s_1)$ and is C^2 on $\bar{\Omega}_{t_1}$ and analytic in Ω_{t_1}.*

(iv) *Whenever* $t_1 \geq 0$ *and* $s_1 \in \bar{\mathbb{C}}_+$, *the backward characteristic curve* $s(t_0; t_1, s_1)$, $t_0 \in [0, t_1]$, *lies in* $\bar{\mathbb{C}}_+$.

Proof. We first establish part (iii), taking $t_0 = 0$ at first. Since $x^2 n_0$ is integrable, $v_0(s)$ is continuous in $\bar{\mathbb{C}}_+$ and analytic for $\operatorname{Re} s > 0$. It follows by a standard dominated convergence argument that u_0 is C^1 and φ_0 is C^2 in $\bar{\mathbb{C}}_+$, and these functions are analytic in $\mathbb{C}_+$. From (3.19) we see that the map $s_0 \mapsto s(t; 0, s_0)$ is analytic in $\mathbb{C}_+$ and C^2 on $\bar{\mathbb{C}}_+$ (meaning derivatives up to second order extend continuously to $\bar{\mathbb{C}}_+$).

We next claim that this map is one to one. The proof relies on the fact that φ_0 is contractive, with

$$|\varphi_0(\tilde{s}_0) - \varphi_0(s_0)| \leq |\tilde{s}_0 - s_0|, \quad \tilde{s}_0, s_0, \in \bar{\mathbb{C}}_+. \tag{3.31}$$

This holds because $|\partial_s \varphi_0(s)| \leq 1$ for $s \in \bar{\mathbb{C}}_+$ as an immediate consequence of (3.7) and (3.8). Now suppose $s(t; 0, \tilde{s}_0) = s(t; 0, s_0)$, where $\tilde{s}_0, s_0 \in \bar{\mathbb{C}}_+$. Then (3.19) implies

$$\tilde{s}_0 - s_0 = \left(1 - e^{-t}\right)\left(\varphi_0(\tilde{s}_0) - \varphi_0(s_0)\right).$$

From this and (3.31) we infer $|\tilde{s}_0 - s_0| \leq (1 - e^{-t})|\tilde{s}_0 - s_0|$, whence $\tilde{s}_0 = s_0$. So $s_0 \mapsto s(t; 0, s_0)$ is one to one.

We observe that the derivative of this map is uniformly bounded away from zero. Indeed, (3.19) and (3.24) yield

$$\left|\frac{ds}{ds_0}\right| \geq e^{2t}\left(1 - |u_0(s_0)|(1 - e^{-t})\right) \geq e^t.$$

It follows by the inverse function theorem that Ω_t is an open set, and by continuity the image of $\bar{\mathbb{C}}_+$ is $\bar{\Omega}_t$. The inverse map from $\bar{\Omega}_t$ to $\bar{\mathbb{C}}_+$ is analytic in Ω_t, and C^2 on $\bar{\Omega}_t$.

For $t_1 > 0$, the inverse of the map $s_0 \mapsto s_1 = s(t_1; 0, s_0)$ may be obtained by solving the characteristic equation in (3.16) backward from time t_1 to $t_0 = 0$, so that we have $s_0 = s(0; t_1, s_1)$. Now whenever $t_1 \geq t_0 \geq 0$ in general, we may follow any characteristic curve back from a point in $\bar{\Omega}_{t_1}$ at time t_1 to a point in $\bar{\mathbb{C}}_+$ at time 0 and then forward to a point in $\bar{\Omega}_{t_0}$ at time t_0. This means that $s(t_1; t_0, s_0) = s(t_1; 0, s(0; t_0, s_0))$. Part (iii) of the lemma now follows from the properties established in the case $t_0 = 0$.

Next we prove part (i). For $t > 0$, Γ_t is the image of the map $k \mapsto s(t; 0, ik) = e^{2t}(ik - \varphi_0(ik)(1 - e^{-t}))$, $k \in \mathbb{R}$, and this is a C^2 function of k. We have $s(t; 0, 0) = 0$, but $\operatorname{Re} s < 0$ for $k \neq 0$. This is so because $\operatorname{Re} s$ and $\operatorname{Re} \varphi_0(ik)$ have opposite signs, and

$$\operatorname{Re} \varphi_0(ik) = \int_0^\infty (1 - \cos kx) n_0(x) dx > 0, \quad k \neq 0,$$

since n_0 is continuous. Finally, we find that

$$\operatorname{Im} \frac{d}{dk} s(t; 0, ik) \geq e^{2t}(1 - |u_0(ik)|(1 - e^{-t})) > 0$$

using (3.24). Hence $\operatorname{Re} s$ is a function of $\operatorname{Im} s$ on Γ_t.

Now we establish part (ii). By (3.30) we have that as $|s_0| \to \infty$ with $s_0 \in \bar{\mathbb{C}}_+$, $|\varphi_0(s_0)/s_0| \to 0$, so $s = s_0 e^{2t}(1 + o(1))$ by (3.19). Let $s_1 \in \mathbb{C}$ lie to the right of Γ_t, and put $f(s_0) = s(t; 0, s_0) - s_1$. It follows by applying the argument principle to large semicircles that the analytic function f has a single zero at some point $s_0 \in \mathbb{C}_+$. Indeed, $\arg f(re^{i\theta}) \to \theta$ as $r \to \infty$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and as k goes from ∞ to $-\infty$, $f(ik)$ does not cross the positive real axis so $\arg f(ik)$ changes from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. Thus, f maps a large semicircle to a curve that winds exactly once about 0. Hence $s_1 \in \Omega_t$.

Finally, part (iv) follows by a change of variables, replacing $t - t_0$ by t, and applying parts (i)–(iii). □

3.4. Proof of Theorem 3.1. 1. By the Fourier–Laplace inversion formula, it suffices to prove

$$\lim_{t\to\infty} \sup_{x>0} \left| \int_{\mathbb{R}} e^{ikx} \left[v(t, ik) - v_{*,1}(ik) \right] dk \right| = 0. \tag{3.32}$$

2. Let $\varepsilon \in (0, \frac{1}{8})$, and put $R = \frac{1}{2}\varepsilon^{-2}$. We will prove (3.32) by estimating the integral for $t \ge T$ separately in three regions: $|k| \le R$, $R \le |k| \le \tilde{R}e^{2t}$, and $\tilde{R}e^{2t} \le |k|$, where $\tilde{R} = Re^{-2T}$ and T depends only on ε and the initial data v_0. This is the same decomposition used in the proof of Theorem 2.1, and convergence in the region $|k| \le R$ will follow as before. However, estimates for $|k| \ge R$ are more subtle and use the analyticity and geometry of the characteristic map.

3. $|k| \le R$: Theorem 7.1 in [20] implies that $\hat{x}\hat{n}(\tau, \hat{x})\, d\hat{x} \to \hat{x}\hat{n}_{*,0}(\hat{x})\, d\hat{x}$ in the sense of weak convergence of measures. Combined with (3.7) this also implies that the measures $\hat{x}^2\hat{n}(\tau, \hat{x})\, d\hat{x}$ converge weakly to $\hat{x}^2\hat{n}_{*,1}(\hat{x})\, d\hat{x}$ as $t \to \infty$. But this implies $v(t, ik)$ converges to $v_{*,1}(ik)$ uniformly on compact subsets of $\bar{\mathbb{C}}_+$, and in particular on compact subsets of the imaginary axis [8, Theorem XV.3.2]. Thus,

$$\lim_{t\to\infty} \int_{-R}^{R} |v(t, ik) - v_{*,1}(ik)|\, dk = 0. \tag{3.33}$$

4. $|k| \ge R$: It is sufficient to consider only $k \ge R$, since $|v(t, ik)| = |v(t, -ik)|$. We will control $v(t, ik)$ and $v_{*,1}$ separately:

$$\int_R^\infty |v(t, ik) - v_{*,1}(ik)|\, dk \le \int_R^\infty |v(t, ik)|\, dk + \int_R^\infty |v_{*,1}(ik)|\, dk.$$

But $|v_{*,1}(ik)| \le (2k)^{-3/2}$ by (3.13). Thus,

$$\int_R^\infty |v_{*,1}(ik)|\, dk \le \int_R^\infty (2k)^{-3/2}\, dk = (2R)^{-1/2} = \varepsilon.$$

5. In the rest of the proof we estimate $\int_R^\infty |v(t, ik)| dk$. In order to aid the reader, we state the main estimates as two distinct lemmas.

LEMMA 3.4. *Let $\varepsilon \in (0, \frac{1}{8})$. There exist $T > 0$, depending on ε and the initial data, and a universal constant C such that if $t \ge T$, then*

$$\int_R^{Re^{2(t-T)}} |v(t, ik)|\, dk \le C\varepsilon. \tag{3.34}$$

LEMMA 3.5. *Let $\tilde{R} > 0$. There exists $\tilde{C}$ depending on $\tilde{R}$ and the initial data such that for all $t \ge 0$ we have*

$$\int_{\tilde{R}e^{2t}}^\infty |v(t, ik)|\, dk \le \tilde{C}e^{-t}. \tag{3.35}$$

6. We now prove (3.32). We choose T as in Lemma 3.4, and then $\tilde{R} = Re^{-2T}$ in Lemma 3.5. Choose $T_* \geq T$ such that for $t \geq T_*$,

$$\int_{-R}^{R} |v(t,ik) - v_{*,1}(ik)|\, dk < \varepsilon, \qquad \tilde{C}e^{-t} \leq \tilde{C}e^{-T_*} < \varepsilon.$$

Thus, for $t \geq T_*$ we have

$$\begin{aligned} &\int_{\mathbb{R}} |v(t,ik) - v_{*,1}(ik)|\, dk \leq \int_{-R}^{R} |v(t,ik) - v_{*,1}(ik)|\, dk \\ &+2\left(\int_{R}^{\infty} |v_{*,1}(ik)|\, dk + \int_{R}^{\tilde{R}e^{2t}} |v(t,ik)|\, dk + \int_{\tilde{R}e^{2t}}^{\infty} |v(t,ik)|\, dk\right) \\ &\leq \varepsilon + 2\left(\varepsilon + C\varepsilon + \varepsilon\right). \end{aligned}$$

Since $\varepsilon \in (0, \frac{1}{8})$ may be chosen arbitrarily small, this completes the proof.

3.5. Proof of Lemma 3.4. In this subsection we will always suppose $s \in \bar{\mathbb{C}}_+$. In a manner similar to step 6 of the proof of Theorem 2.1, the idea is to get estimates on the semicircle $C_R := \{s \in \bar{\mathbb{C}}_+ \mid |s| = R\}$ valid for large time and propagate these estimates outward along characteristics. We first use (3.10) and (3.13) to obtain the following estimates for $s \in \bar{\mathbb{C}}_+$:

$$|\varphi_{*,1}(s)| < |2s|^{1/2}, \quad |u_{*,1}(s)| < |2s|^{-1/2}, \quad |v_{*,1}(s)| < |2s|^{-3/2}. \tag{3.36}$$

Next, we use the uniform convergence on compact sets and (3.36) to see that there exists T_0 (depending on ε and the initial data) such that for all $s_0 \in C_R$ and $t_0 \geq T_0$ we have

$$|\varphi(t_0,s_0)/s_0| \leq 2(2R)^{-1/2} = 2\varepsilon \leq 1/4, \tag{3.37}$$

$$|u(t_0,s_0)| \leq (2R)^{-1/2} = \varepsilon, \tag{3.38}$$

$$|v(t_0,s_0)| \leq (2R)^{-3/2} = \varepsilon^3. \tag{3.39}$$

We first extend (3.37) to a larger domain in s.

Claim 1. There exists $T_1 \geq T_0$ such that

$$\left|\frac{\varphi(t,s)}{s}\right| \leq 1/3, \quad t \geq T_1,\ s \in \bar{\mathbb{C}}_+,\ |s| \geq R. \tag{3.40}$$

Proof of Claim 1. Observe that by using (3.27) and (3.30) in (3.21), we have

$$a := \sup\{|\varphi(T_0,s)/s| \mid s \in \bar{\mathbb{C}}_+,\ |s| \geq R\} < 1.$$

Fix $t_1 \geq T_0$, $s_1 \in \bar{\mathbb{C}}_+$ with $|s_1| \geq R$. Either the characteristic curve $s(t;t_1,s_1)$ that passes through s_1 at time t_1 intersects C_R at some time $t_0 \in [T_0,t_1]$ or it does not. If so, then $s_1 = s(t_1;t_0,s_0)$ for some $s_0 \in C_R$, and (3.21) and (3.37) directly yield

$$\left|\frac{\varphi(t_1,s_1)}{s_1}\right| \leq \frac{1/4}{1-1/4} = \frac{1}{3}.$$

If not, then $|s(t;t_1,s_1)| > R$ for all $t \in [T_0,t_1]$, by continuity and the fact that $s(t;t_1,s_1) \in \bar{\mathbb{C}}_+$ for all $t \in [0,t_1]$ by part (iv) of Lemma 3.3. Then taking $t_0 = T_0$, $s_0 = s(T_0;t_1,s_1)$ in (3.21) yields

$$\left|\frac{\varphi(t_1,s_1)}{s_1}\right| \leq \frac{ae^{-(t_1-T_0)}}{1-a} \leq \frac{1}{3},$$

provided $t_1 \geq T_1$ with T_1 sufficiently large. This proves the claim.

Claim 2. Let $T = T_1 + \frac{1}{2}\ln 2$. Suppose $t_1 \geq T$ and $R \leq |s_1| \leq Re^{2(t_1-T)}$. Then the characteristic curve $s(t;t_1,s_1)$ that passes through s_1 at time t_1 intersects C_R at some time $t_0 \in [T_1, t_1]$.

Proof of Claim 2. Suppose the claim were false. Then the continuity of $|s(t;t_1,s_1)|$ and part (iv) of Lemma 3.3 imply $R < |s(t_0;t_1,s_1)|$ for all $t_0 \in [T_1,t_1]$. But now, by (3.20) with $s_0 = s(t_0;t_1,s_1)$ we have

$$s_0\left(1 - \frac{\varphi(t_0,s_0)}{s_0}\right) = e^{-2(t_1-t_0)}s_1\left(1 - \frac{\varphi(t_1,s_1)}{s_1}\right). \tag{3.41}$$

We take $t_0 = T_1$ and apply (3.40) and the hypothesis $|s_1| \leq Re^{2(t_1-T)} = \frac{1}{2}Re^{2(t_1-T_1)}$ to deduce

$$R < |s_0| \leq |s_1| e^{-2(t_1-T_1)} \frac{1+1/3}{1-1/3} \leq R,$$

a contradiction. This proves the claim.

We now apply these claims to propagate the decay estimate (3.39). From Claim 2, for any $t = t_1 \geq T$, $R \leq k \leq Re^{2(t-T)}$, with $s_1 = ik$, we obtain $t_0 \in [T_1, t]$ and $s_0 \in C_R$ and substitute (3.20), (3.39), and (3.40) into the decay estimate (3.26) to obtain

$$\begin{aligned} |v(t,ik)| &\leq \frac{|v(t_0,s_0)|}{(1-|u(t_0,s_0)|)^3}\left|\frac{s_0 - \varphi(t_0,s_0)}{ik - \varphi(t,ik)}\right|^{3/2} \\ &\leq (1-\varepsilon)^{-3}\ |v(t_0,s_0)|\left|\frac{2s_0}{k}\right|^{3/2} \\ &\leq (1-\varepsilon)^{-3}(2R)^{-3/2}\left(\frac{2R}{k}\right)^{3/2} = (1-\varepsilon)^{-3}k^{-3/2}. \end{aligned}$$

Therefore,

$$\int_R^{Re^{2(t-T)}} |v(t,ik)|\,dk \leq (1-\varepsilon)^{-3}\int_R^\infty k^{-3/2}\,dk = \frac{2R^{-1/2}}{(1-\varepsilon)^3} \leq C\varepsilon, \tag{3.42}$$

with $C = 2(8/7)^3 2^{1/2}$. This completes the proof of Lemma 3.4.

3.6. Proof of Lemma 3.5. We consider the initial time $t_0 = 0$ and the following special case of (3.19):

$$s = s(t;0,s_0) = e^{2t}\left[s_0 - \varphi_0(s_0)(1-e^{-t})\right]. \tag{3.43}$$

For any $t \geq 0$, the map $s_0 \mapsto s(t;0,s_0)$ is analytic for $\mathrm{Re}(s_0) > 0$, and

$$\frac{ds}{ds_0} = e^{2t}\left(1 - u_0(s_0)(1-e^{-t})\right), \qquad u_0(s_0) = u(0,s_0). \tag{3.44}$$

Recall that Ω_t denotes the image of $\mathbb{C}_+$ under $s_0 \mapsto s(t;0,s_0)$, and Γ_t denotes the image of the imaginary axis; we let Γ_{-t} denote its preimage. As was observed in Lemma 3.3, Γ_t is a graph over the imaginary axis, contained in the left half plane.

We will use the analyticity of $v(t,s)$ in Ω_t and contour deformation. For large finite $R_2 < \infty$, consider the domain $ABCD$ shown in Figure 3.1. The path AB is

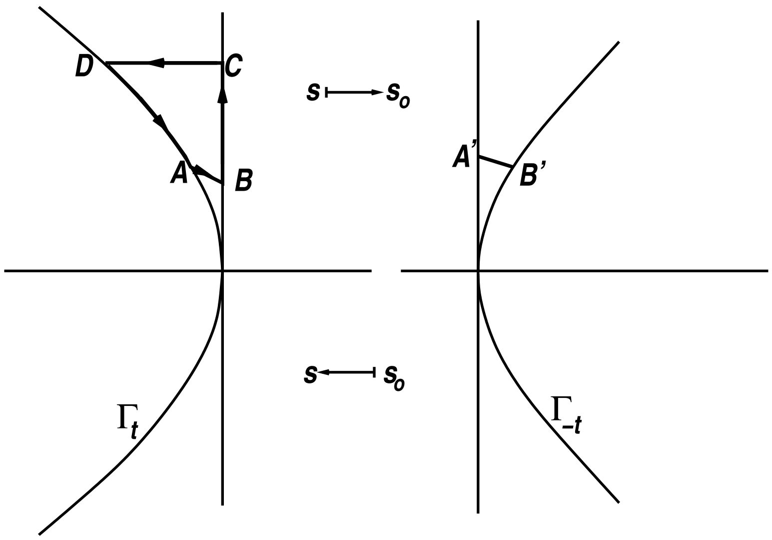

Fig. 3.1 *The s-plane is on the left, the s_0-plane on the right. Ω_t is the region to the right of Γ_t. $A = s(t;0,i\tilde{R})$, $B = i\tilde{R}e^{2t}$, $C = iR_2$, $\mathrm{Im}(D) = R_2$, $A' = i\tilde{R}$, $B' = s(0;t,i\tilde{R}e^{2t})$.*

chosen so that $A'B'$ is a straight line. CD is parallel to the real axis and lies in Ω_t since Γ_t is a graph over the imaginary axis. Then by Cauchy's theorem,

$$\begin{aligned}\int_{\tilde{R}e^{2t}}^{R_2} e^{ikx} v(t,ik)\,dk &= \int_{BC} e^{ikx} v(t,ik)\,dk \\ &= \int_{DA} e^{sx} v(t,s)\,ds + \int_{AB} e^{sx} v(t,s)\,ds + \int_{CD} e^{sx} v(t,s)\,ds.\end{aligned}$$

Let σ denote $\mathrm{Re}\, s$. Since $\sigma < 0$ in Ω_t for $s \in CD$ we see that the last integral is estimated by

$$\left| \int_{CD} e^{sx} v(t,s)\,ds \right| \le \sup_{s\in CD} |v(t,s)| \int_{-\infty}^{0} e^{\sigma x} d\sigma = \frac{\sup_{s\in CD} |v(t,s)|}{x}.$$

By the decay estimate (3.26) we have

$$\sup_{s\in CD} |v(t,s)| \le \sup_{s_1 \in CD} \frac{|v_0(s_0)| e^{-3t}}{(1 - |u_0(s_0)|)^3}, \quad s_0 = s(0;t,s_1).$$

It follows from (3.30) and the fact that $|s_0| = |s_1| e^{-2t}(1 + o(1)) \to \infty$ as $R_2 \to \infty$ that $\sup_{s_1\in CD} |v_0(s_0)| \to 0$. We thus let $R_2 \to \infty$ to conclude that

$$\int_{\tilde{R}e^{2t}}^{\infty} e^{ikx} v(t,ik)\,dk = \int_{\Gamma_{t,A}} e^{sx} v(t,s)\,ds + \int_{AB} e^{sx} v(t,s)\,ds, \tag{3.45}$$

where $\Gamma_{t,A}$ denotes the path from ∞ to A on Γ_t. Notice that (3.45) holds independent of x.

The virtue of deforming the contour is that the integrals may now be estimated by changing variables from s to s_0. We use the solution formula (3.23) together with the change of variables $s = s(t;0,ik)$ and (3.44) to obtain

$$\int_{\Gamma_{t,A}} e^{sx} v(t,s)\,ds = ie^{-t} \int_{\tilde{R}}^{\infty} e^{s(t;0,ik)x} \frac{v_0(ik)}{(1-u_0(ik)(1-e^{-t}))^2}\,dk.$$

Since $\operatorname{Re} s(t;0,ik) \le 0$ and $\sup_{|k|\ge \tilde{R}} |u_0(ik)| < 1$, this yields the estimate

$$\left| \int_{\Gamma_{t,A}} e^{sx} v(t,s)\,ds \right| \le C_1 e^{-t} \|v_0\|_{L^1}. \tag{3.46}$$

Similarly, we have by (3.23) and (3.44)

$$\begin{aligned} \left| \int_{AB} e^{sx} v(t,s)\,ds \right| &= e^{-t} \left| \int_{A'B'} e^{s(t;0,s_0)x} \frac{v_0(s_0)}{(1-u_0(s_0)(1-e^{-t}))^2}\,ds_0 \right| \\ &\le e^{-t} |A'B'| \sup_{s_0 \in A'B'} \left|1-u_0(s_0)(1-e^{-t})\right|^{-2}. \end{aligned}$$

The point $A' = i\tilde{R}$ is independent of t. It also follows from (3.43) that $B' = s(0;t,i\tilde{R}e^{2t})$ converges to the point $s_0 \in \bar{\mathbb{C}}_+$ that solves $i\tilde{R} = s_0 - \varphi_0(s_0)$. Thus, we have the exponential decay estimate

$$\left| \int_{AB} e^{sx} v(t,s)\,ds \right| \le C_2 e^{-t}. \tag{3.47}$$

The constants C_i in (3.46) and (3.47) depend only on $\tilde{R}$ and the initial data u_0. To be explicit, we set $\tilde{C} = C_1\|v_0\|_{L^1} + C_2$. This completes the proof.

3.7. The Discrete Smoluchowski Equations. We now use the proof of Theorem 3.1 to obtain a uniform convergence theorem for the discrete Smoluchowski equations with additive kernel. The proof is simpler and we do not need the contour deformation argument.

Let $\nu_t = \sum_{l=1}^{\infty} n_l(t)\delta_{hl}(x)$ denote a measure-valued solution to (1.1). We first adapt the rescaling (3.4) and (3.5) to similarity variables. Let

$$\hat{l} = lhe^{-2t}, \quad \hat{n}_l(t) = h^{-1}e^{4t}n_l(t). \tag{3.48}$$

Then the discrete Fourier inversion formula analogous to (2.32) is

$$\hat{l}^2 \hat{n}_l(t) = \frac{1}{2\pi} \int_{-\pi e^{2t}/h}^{\pi e^{2t}/h} e^{i\hat{l}k} v(t,ik)\,dk. \tag{3.49}$$

THEOREM 3.6. *Let $\nu_0 \ge 0$ be a lattice measure with span h such that $\int_0^\infty x\nu_0(dx) = \int_0^\infty x^2\nu_0(dx) = 1$. Then with the scaling* (3.48) *we have*

$$\lim_{t\to\infty} \sup_{l\in\mathbb{N}} \hat{l}^2 \left| \hat{n}_l(t) - \hat{n}_{*,1}(\hat{l}) \right| = 0.$$

Proof. By (3.49) and the continuous Fourier inversion formulas it suffices to show that

$$\lim_{t\to\infty} \sup_{\hat{l}\ge 0} \left| \int_{-\pi e^{2t}/h}^{\pi e^{2t}/h} e^{i\hat{l}hk} v(t,ik)\,dk - \int_{\mathbb{R}} e^{i\hat{l}hk} v_{*,1}(ik)\,dk \right| = 0. \tag{3.50}$$

Let $\varepsilon \in (0, \frac{1}{8})$ and choose $R = \frac{1}{2}\varepsilon^{-2}$. The integrals over $[-R, R]$ and $R < |k| < \tilde{R}e^{2t}$ with $\tilde{R} = e^{-2T}$ are controlled as in the proof of Theorem 3.1, and it remains only to control the integral of $|v(t, ik)|$ over $\tilde{R}e^{2t} < k < \pi e^{2t}/h$. This is considerably simpler than in the previous proof. We use the solution formula (3.23) and change variables via $ik = s(t; 0, s_0)$, then use (3.44) to obtain

$$\int_{\tilde{R}e^{2t}}^{\pi e^{2t}/h} e^{ikx} v(t, ik)\, dk = ie^{-t} \int_{\Gamma_{-t}(\tilde{R}, \pi/h)} \frac{e^{xs(t;0,s_0)} v_0(s_0)}{(1 - u_0(s_0)(1 - e^{-t}))^2}\, ds_0.$$

Here $\Gamma_{-t}(\tilde{R}, \pi/h)$ denotes the segment along the curve Γ_{-t} from $s(0; t, i\tilde{R}e^{2t})$ to $s(0; t, i\pi e^{2t}/h)$. The formula (3.19) shows that $\Gamma_{-t}(\tilde{R}, \pi/h)$ converges to a compact C^2 curve defined implicitly by $ik = s_0 - \varphi_0(s_0)$, $\tilde{R} \le k \le \pi/h$. Thus, for $t \ge T$ we have

$$e^{-t} \left| \int_{\Gamma_{-t}(\tilde{R}, \pi/h)} \frac{e^{xs(t;0,s_0)} v_0(s_0)}{(1 - u_0(s_0)(1 - e^{-t}))^2}\, ds_0 \right| \le C(T, \tilde{R}, u_0, v_0) e^{-t}.$$

Thus, this term is less than ε for all t large enough. □

4. Self-Similar Gelation for the Multiplicative Kernel. For $K = xy$, McLeod solved the coagulation equation explicitly for monodisperse initial data and showed that a mass-conserving solution fails to exist for $t > 1$ [19]. The second moment satisfies $m_2(t) = (1 - t)^{-1}$. The divergence of the second moment indicates that breakdown is associated with an explosive flux of mass toward large clusters. A rescaled limit of McLeod's solution is the following self-similar solution for $K = xy$ [1]:

$$n(t, x) = \frac{1}{\sqrt{2\pi}} x^{-5/2} e^{-(1-t)^2 x/2}, \quad x > 0, \quad t < 1. \tag{4.1}$$

Evidently this solution has infinite mass (first moment). This should not be thought unnatural, however, since it was shown in [20] that (1.1) has a unique weak solution for any initial distribution with finite second moment.

The problem of solving Smoluchowski's equation with multiplicative kernel can be reduced to that for the additive kernel by a change of variables [4]. Let us briefly review this. In unscaled variables we define

$$\Psi(t, z) = \int_0^\infty (1 - e^{-zx}) x n(t, x)\, dx. \tag{4.2}$$

Then Ψ solves the inviscid Burgers equation

$$\partial_t \Psi - \Psi \partial_z \Psi = 0, \tag{4.3}$$

with initial data

$$\Psi_0(z) = \int_0^\infty (1 - e^{-zx}) x n_0(x)\, dx. \tag{4.4}$$

The gelation time for initial data with finite second moment is $T_{\text{gel}} = (\int_0^\infty x^2 \nu_0(dx))^{-1}$, and this is exactly the time for the first intersection of characteristics [20]. We presume that the initial data is scaled to ensure

$$\int_0^\infty x^2 n_0(x)\, dx = \int_0^\infty x^3 n_0(x)\, dx = 1. \tag{4.5}$$

Then the gelation time is $T_{\text{gel}} = 1$. The connection between the additive and multiplicative kernels is that Ψ solves (4.3) with initial data Ψ_0 if and only if $\Phi(\tau, z)$ is a solution to (3.3) with the same initial data, where

$$\Psi(t, z) = e^{\tau}\Phi(\tau, z), \quad \text{with } \tau = \log(1-t)^{-1}. \tag{4.6}$$

For solutions $n^{\text{mul}}(t, x)$ and $n^{\text{add}}(\tau, x)$ to Smoluchowski's equation with multiplicative and additive kernels, respectively, this means that

$$xn^{\text{mul}}(t, x) = (1-t)^{-1} n^{\text{add}}(\tau, x) \tag{4.7}$$

for all $t \in (0, 1)$ if and only if the same holds at $t = 0$. We thus obtain a scaling limit as $t \to T_{\text{gel}}$ directly from Theorem 3.1.

THEOREM 4.1. *Suppose* $n_0(x) \geq 0$, $\int_0^\infty x^2 n_0(x)\,dx = \int_0^\infty x^3 n_0(x)\,dx = 1$. *Suppose also that the Fourier transform of* $x^3 n_0$ *is integrable. Then, in terms of the rescaling* (1.8), *we have*

$$\lim_{t\to 1} \sup_{\hat{x}>0} \hat{x}^3 |\hat{n}(t, \hat{x}) - \hat{n}_{*,2}(\hat{x})| = 0, \tag{4.8}$$

where $\hat{n}_{*,2}(\hat{x})$ *is the self-similar density in* (1.9).

Theorem 3.6 may be similarly adapted to $K = xy$. In the discrete case, the correspondence (4.7) between solutions of Smoluchowski's equations with multiplicative and additive kernels becomes

$$hln_l^{\text{mul}}(t) = (1-t)^{-1} n_l^{\text{add}}(\log(1-t)^{-1}). \tag{4.9}$$

We introduce similarity variables via

$$\hat{l} = lh(1-t)^2, \quad \hat{n}_l(t) = h^{-1}(1-t)^{-5} n_l(t). \tag{4.10}$$

Then directly from Theorem 3.6 we obtain the following theorem.

THEOREM 4.2. *Let* $\nu_0 \geq 0$ *be a lattice measure with span* h *such that* $\int_0^\infty x^2 \nu_0(dx) = \int_0^\infty x^3 \nu_0(dx) = 1$. *Then with the rescaling* (4.10) *we have*

$$\lim_{t\to 1} \sup_{l\in\mathbb{N}} \hat{l}^3 \left|\hat{n}_l(t) - \hat{n}_{*,2}(\hat{l})\right| = 0. \tag{4.11}$$

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