Dynamic Monetary Risk Measures for Bounded Discrete-Time Processes

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October 15, 2004

Abstract

We study time-consistency questions for processes of monetary risk measures that depend on bounded discrete-time processes describing the evolution of financial values. The time horizon can be finite or infinite. We call a process of monetary risk measures time-consistent if it assigns to a process of financial values the same risk irrespective of whether it is calculated directly or in two steps backwards in time, and we show how this property manifests itself in the corresponding process of acceptance sets. For processes of coherent and convex monetary risk measures admitting a robust representation with sigma-additive linear functionals, we give necessary and sufficient conditions for time-consistency in terms of the representing functionals.

Key words: Monetary risk measure processes, convex monetary risk measure processes, coherent risk measure processes, acceptance set processes, time-consistency, concatenation.

1 Introduction

The notion of coherent risk measure was introduced in Artzner et al. (1997, 1999) and further developed in Delbaen (2001, 2002). In Föllmer and Schied (2002a, b, c) and Frittelli and Rosazza Gianin (2002) the more general concepts of monetary and convex monetary risk measures were introduced. All these works discuss one-period risk measurement, that is, the risky objects are real-valued random variables describing future financial values and the risk of such financial values is only measured at the beginning of the time-period considered. Some typical examples of financial values in the context of risk measurement are:

- the market value of a firm's equity

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- the accounting value of a firm's equity
- the market value of a portfolio of financial securities
- the surplus of an insurance company

Cvitanić and Karatzas study the dynamics of a risk associated with hedging a given liability in a continuous-time setup. In Artzner et al. (2002) the evolution of financial values over time is modelled with discrete-time stochastic processes and two special classes of time-consistent processes of coherent risk measures related to m-stable sets of probability measures are introduced. A treatment of the same two classes of time-consistent processes of coherent risk measures in continuous time and more on m-stable sets can be found in Delbaen (2004). Roorda et al. (2003) is similar to Artzner et al. (2002) but also discusses the effects of hedging and the applicability of dynamic programming algorithms. Cheridito et al. (2004a, b) contain representation results for coherent and convex monetary risk measures that depend on processes of financial values evolving in continuous time. Rosazza Gianin (2003) studies the relation between risk measures and g-expectations. Frittelli and Rosazza Gianin (2004) contains a summary of earlier results on convex monetary risk measures and connections to indifference pricing and g-expectations. Riedel (2004), Detlefsen (2003), Scandolo (2003) and Weber (2003) study dynamic coherent or convex monetary risk measures for cash-flow streams in discrete time.

In this paper we follow Artzner et al. and measure the risk of discrete-time processes of financial values. We simply call them value processes. Of course, in discrete-time, value processes can be turned into cash-flow streams by passing to increment processes. But this transformation does not preserve the order of almost sure dominance, and because this order plays a crucial role in our definition of monetary risk measures, it makes a difference whether we take the risky objects to be value processes or cash-flow streams. Since in most practical applications it can be assumed that money can be lent at a risk-free rate, we find it more natural to order value processes than cash-flow streams by almost sure dominance.

The structure of the paper is as follows. In Section 2 we introduce the basic setup and some notation. In Section 3 we introduce monetary risk measures conditional on the information available at stopping times, study the relation between such risk measures and their acceptance sets and prove conditional representation results for coherent and convex monetary risk measures. In Section 4 we define what we mean by time-consistency for processes of monetary risk measures and show how the time-consistency property of processes of monetary risk measures translates into a condition on processes of acceptance sets. For processes of coherent and convex monetary risk measures that can be represented with sigma-additive linear functionals we give necessary and sufficient conditions for time-consistency in terms of the representing sigma-additive linear functionals. In order to do this we define a concatenation operation for adapted increasing processes of integrable variation. The concept of m-stability for probability measures can be viewed as a special case of stability under concatenation. In Section 5 we discuss special cases and examples of time-consistent processes of monetary risk measures for discrete-time value processes.

2 The setup and notation

Throughout the paper $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, P)$ is a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. All equalities and inequalities between random variables or stochastic processes are understood in the P-almost sure sense. For instance, if $(X_t)_{t\in\mathbb{N}}$ and $(Y_t)_{t\in\mathbb{N}}$ are two stochastic processes, we mean by $X \geq Y$ that for P-almost all $\omega \in \Omega$, $X_t(\omega) \geq Y_t(\omega)$ for all $t \in \mathbb{N}$. Also, equalities and inclusions between sets in $\mathcal{F}$ are understood in the P-almost sure sense. By $\mathcal{R}^0$ we denote the space of all adapted stochastic processes $(X_t)_{t\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, P)$, where we identify two processes X and Y if $X = Y$. The two subspaces $\mathcal{R}^\infty$ and $\mathcal{A}^1$ of $\mathcal{R}^0$ are given by

$$\mathcal{R}^\infty := \left\{X \in \mathcal{R}^0 \mid ||X||_{\mathcal{R}^\infty} < \infty\right\} ,$$

where

$$||X||_{\mathcal{R}^\infty} := \inf\left\{m \in \mathbb{R} \mid \sup_{t\in\mathbb{N}} |X_t| \leq m\right\}$$

and

$$\mathcal{A}^1 := \left\{a \in \mathcal{R}^0 \mid ||a||_{\mathcal{A}^1} < \infty\right\} ,$$

where

$$a_{-1} := 0\,, \quad \Delta a_t := a_t - a_{t-1}\,, \text{ for } t \in \mathbb{N}\,, \quad \text{and} \quad ||a||_{\mathcal{A}^1} := \mathrm{E}\left[\sum_{t\in\mathbb{N}} |\Delta a_t|\right] .$$

The set $\mathcal{A}^1_+$ is given by

$$\mathcal{A}^1_+ := \left\{a \in \mathcal{A}^1 \mid \Delta a_t \geq 0 \text{ for all } t \in \mathbb{N}\right\} ,$$

and the bilinear form $\langle ., . \rangle$ on $\mathcal{R}^\infty \times \mathcal{A}^1$ by

$$\langle X, a\rangle := \mathrm{E}\left[\sum_{t\in\mathbb{N}} X_t \Delta a_t\right] .$$

$\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$ denotes the coarsest topology on $\mathcal{R}^\infty$ such that for all $a \in \mathcal{A}^1$, $X \mapsto \langle X, a\rangle$ is a continuous linear functional on $\mathcal{R}^\infty$. $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$ denotes the coarsest topology on $\mathcal{A}^1$ such that for all $X \in \mathcal{R}^\infty$, $a \mapsto \langle X, a\rangle$ is a continuous linear functional on $\mathcal{A}^1$.

For two $(\mathcal{F}_t)$-stopping times τ and θ such that $0 \leq \tau < \infty$ and $\tau \leq \theta \leq \infty$, we define the projection $\pi_{\tau,\theta} : \mathcal{R}^0 \to \mathcal{R}^0$ by

$$\pi_{\tau,\theta}(X)_t := 1_{\{\tau\leq t\}} X_{t\wedge\theta}\,, \quad t \in \mathbb{N}\,.$$

For all $X \in \mathcal{R}^\infty$ and $a \in \mathcal{A}^1$, we define

$$||X||_{\tau,\theta} := \operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid \sup_{t\in\mathbb{N}} |\pi_{\tau,\theta}(X)_t| \leq f\right\} ,$$

where ess inf denotes the essential infimum of a family of random variables (see for instance, Proposition VI.1.1 of Neveu, 1972), and

$$\langle X, a \rangle_{\tau,\theta} := \mathrm{E}\left[\sum_{t \in [\tau,\theta] \cap \mathbb{N}} X_t \Delta a_t \mid \mathcal{F}_\tau \right] .$$

The risky objects considered in this paper are elements of vector spaces of the form

$$\mathcal{R}^\infty_{\tau,\theta} := \pi_{\tau,\theta} \mathcal{R}^\infty .$$

A process $X \in \mathcal{R}^\infty_{\tau,\theta}$ is meant to describe the evolution of a financial value on the time interval $[\tau, \theta] \cap \mathbb{N}$. We assume that there exists a cash account where money can be lent to and borrowed from at the same risk-free rate and use it as numéraire, that is, all prices are expressed in multiples of one dollar put into the cash account at time 0. A monetary risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ is a mapping

$$\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau) ,$$

assigning to a value process $X \in \mathcal{R}^\infty_{\tau,\theta}$ a real number that can depend on the information available at the stopping time τ and specifies the minimal amount of money that has to be held in the cash account to make X acceptable at time τ. By our choice of the numéraire, the infusion of an amount of money m at time τ transforms a value process $X \in \mathcal{R}^\infty_{\tau,\theta}$ into $X + m$ and reduces the risk of X to $\rho(X) - m$. We find it more convenient to work with the negatives of monetary risk measures. If ρ is a monetary risk measure on $\mathcal{R}^\infty_{\tau,\theta}$, we call $\phi = -\rho$ the monetary utility functional corresponding to ρ. $\phi(X)$ can then be viewed as a risk adjusted value of a process $X \in \mathcal{R}^\infty_{\tau,\theta}$ at time τ.

For the representation of conditional convex monetary and coherent risk measures we will need the following subsets of $\mathcal{A}^1$:

$$\mathcal{A}^1_{\tau,\theta} := \pi_{\tau,\theta} \mathcal{A}^1 , \quad (\mathcal{A}^1_{\tau,\theta})_+ := \pi_{\tau,\theta} \mathcal{A}^1_+ \quad \text{and} \quad \mathcal{D}_{\tau,\theta} := \left\{ a \in (\mathcal{A}^1_{\tau,\theta})_+ \mid \langle 1, a \rangle_{\tau,\theta} = 1 \right\} .$$

3 Conditional monetary utility functionals

In all of Section 3, τ and θ are two fixed $(\mathcal{F}_t)$-stopping times such that $0 \leq \tau < \infty$ and $\tau \leq \theta \leq \infty$.

3.1 Basic definitions and easy properties

Definition 3.1 *We call a mapping $\phi : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ a monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ if it has the following three properties:*

(0) $\phi(1_A X) = 1_A \phi(X)$ *for all* $X \in \mathcal{R}^\infty_{\tau,\theta}$ *and* $A \in \mathcal{F}_\tau$

(1) $\phi(X) \leq \phi(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ *such that* $X \leq Y$

(2) $\phi(X+m1_{[\tau,\infty)})=\phi(X)+m$ *for all* $X\in\mathcal{R}^\infty_{\tau,\theta}$ *and* $m\in L^\infty(\mathcal{F}_\tau)$

We call a monetary utility functional ϕ *on* $\mathcal{R}^\infty_{\tau,\theta}$ *a concave monetary utility functional if*
(3) $\phi(\lambda X+(1-\lambda)Y)\geq\lambda\phi(X)+(1-\lambda)\phi(Y)$ *for all* $X,Y\in\mathcal{R}^\infty_{\tau,\theta}$ *and* $\lambda\in L^\infty(\mathcal{F}_\tau)$

We call a concave monetary utility functional ϕ *on* $\mathcal{R}^\infty_{\tau,\theta}$ *a coherent utility functional if*
(4) $\phi(\lambda X)=\lambda\phi(X)$ *for all* $X\in\mathcal{R}^\infty_{\tau,\theta}$ *and* $\lambda\in L^\infty(\mathcal{F}_\tau)$ *such that* $\lambda\geq 0$.

For a monetary utility functional ϕ *on* $\mathcal{R}^\infty_{\tau,\theta}$ *and* $X\in\mathcal{R}^\infty_{\tau,\theta}$, *we define* $\phi(X):=\phi\circ\pi_{\tau,\theta}(X)$.

A monetary risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ *is a mapping* $\rho:\mathcal{R}^\infty_{\tau,\theta}\to L^\infty(\mathcal{F}_\tau)$ *such that* $-\rho$ *is a monetary utility functional on* $\mathcal{R}^\infty_{\tau,\theta}$. ρ *is a convex monetary risk measure if* $-\rho$ *is a concave monetary utility functional and a coherent risk measure if* $-\rho$ *is a coherent utility functional.*

Remarks 3.2
1. It follows from condition (0) of Definition 3.1 that $\phi(0)=0$ for every monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$. This normalization is convenient for the purposes of this paper. Differently normalized monetary utility functionals on $\mathcal{R}^\infty_{\tau,\theta}$ can be obtained by the addition of an $\mathcal{F}_\tau$-measurable random variable.
2. It follows from (1) and (2) of Definition 3.1 that a monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ satisfies the following continuity condition:

(c) $|\phi(X)-\phi(Y)|\leq||X-Y||_{\tau,\theta}$, for all $X,Y\in\mathcal{R}^\infty_{\tau,\theta}$.

3. We call the property (3) of Definition 3.1 $\mathcal{F}_\tau$-concavity.
4. A mapping $\phi:\mathcal{R}^\infty_{\tau,\theta}\to L^\infty(\mathcal{F}_\tau)$ is a coherent utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ if and only if it satisfies (1), (2) and (4) of Definition 3.1 and

(3') $\phi(X+Y)\geq\phi(X)+\phi(Y)$ for all $X,Y\in\mathcal{R}^\infty_{\tau,\theta}$.

Definition 3.3 *The acceptance set* $\mathcal{C}_\phi$ *of a monetary utility functional* ϕ *on* $\mathcal{R}^\infty_{\tau,\theta}$ *is given by*

$$\mathcal{C}_\phi:=\left\{X\in\mathcal{R}^\infty_{\tau,\theta}\mid\phi(X)\geq 0\right\}.$$

Proposition 3.4 *The acceptance set* $\mathcal{C}_\phi$ *of a monetary utility functional* ϕ *on* $\mathcal{R}^\infty_{\tau,\theta}$ *satisfies the following properties:*

(i) $\operatorname{ess\,inf}\left\{f\in L^\infty(\mathcal{F}_\tau)\mid f1_{[\tau,\infty)}\in\mathcal{C}_\phi\right\}=0$.

(ii) $1_AX+1_{A^c}Y\in\mathcal{C}_\phi$ *for all* $X,Y\in\mathcal{C}_\phi$ *and* $A\in\mathcal{F}_\tau$.

(I) $X\in\mathcal{C}_\phi$, $Y\in\mathcal{R}^\infty_{\tau,\theta}$, $X\leq Y\Rightarrow Y\in\mathcal{C}_\phi$

(C) $(X^n)_{n\in\mathbb{N}}\subset\mathcal{C}_\phi$, $X\in\mathcal{R}^\infty_{\tau,\theta}$, $||X^n-X||_{\tau,\theta}\overset{\text{a.s.}}{\to}0\Rightarrow X\in\mathcal{C}_\phi$.

If ϕ *is a concave monetary utility functional, then*

(II) $\lambda X+(1-\lambda)Y\in\mathcal{C}_\phi$ *for all* $X,Y\in\mathcal{C}_\phi$ *and* $\lambda\in L^\infty(\mathcal{F}_\tau)$ *such that* $0\leq\lambda\leq 1$.

If ϕ *is a coherent utility functional, then*

(II') $X + Y \in \mathcal{C}_\phi$ *for all* $X, Y \in \mathcal{C}_\phi$ *and*

(III) λX *for all* $X \in \mathcal{C}_\phi$ *and* $\lambda \in L^\infty_+(\mathcal{F}_\tau)$.

Proof. (i): It follows from the definition of $\mathcal{C}_\phi$ and (0) and (2) of Definition 3.1 that

$$\begin{aligned} & \operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid f 1_{[\tau,\infty)} \in \mathcal{C}_\phi\right\} = \operatorname{ess\,inf}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(f 1_{[\tau,\infty)}) \geq 0\right\} \\ = \quad & \operatorname{ess\,inf}\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(0) + f \geq 0\} = \operatorname{ess\,inf}\{f \in L^\infty(\mathcal{F}_\tau) \mid f \geq 0\} = 0\,. \end{aligned}$$

(ii) follows directly from (0) of Definition 3.1.
(I) follows from (1) of Definition 3.1.
(C): Let $(X^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_\phi$ and $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that $||X^n - X||_{\tau,\theta} \overset{\text{a.s.}}{\to} 0$. It follows from (c) that

$$\phi(X) \geq \phi(X^n) - ||X^n - X||_{\tau,\theta}\,,$$

for all $n \in \mathbb{N}$. Hence, $\phi(X) \geq 0$. The remaining statements of the proposition are obvious. □

Remarks 3.5
1. We call a subset of $\mathcal{R}^\infty_{\tau,\theta}$ that satisfies condition (II) of Proposition 3.4 $\mathcal{F}_\tau$-convex.
2. Let $\mathcal{C}_\phi$ be the acceptance set of a monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$. It can be deduced from (0) of Definition 3.1 or, alternatively, from (ii) and (C) of Proposition 3.4 that $\sum_{n\in\mathbb{N}} 1_{A_n} X^n \in \mathcal{C}_\phi$ for every sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{C}_\phi$ and each sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint events in $\mathcal{F}_\tau$.

Definition 3.6 *If* $\mathcal{C}$ *is a subset of* $\mathcal{R}^\infty_{\tau,\theta}$, *we define for all* $X \in \mathcal{R}^\infty_{\tau,\theta}$,

$$\phi_{\mathcal{C}}(X) := \operatorname{ess\,sup}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid X - f 1_{[\tau,\infty)} \in \mathcal{C}\right\}\,,$$

with the convention

$$\operatorname{ess\,sup} \emptyset := -\infty\,.$$

Remark 3.7 Note that if $\mathcal{C}$ satisfies (ii) of Proposition 3.4 and, for a given $X \in \mathcal{R}^\infty_{\tau,\theta}$, the set

$$\left\{f \in L^\infty(\mathcal{F}_\tau) \mid X - f 1_{[\tau,\infty)} \in \mathcal{C}\right\}$$

is non-empty, then it is directed upwards, and hence, contains an increasing sequence $(f^n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f^n = \phi_{\mathcal{C}}(X)$ almost surely.

Proposition 3.8 *Let* ϕ *be a monetary utility functional on* $\mathcal{R}^\infty_{\tau,\theta}$. *Then* $\phi_{\mathcal{C}_\phi} = \phi$.

Proof. For all $X \in \mathcal{R}^\infty_{\tau,\theta}$,

$$\begin{aligned} \phi_{\mathcal{C}_\phi}(X) &= \operatorname{ess\,sup}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid X - f 1_{[\tau,\infty)} \in \mathcal{C}_\phi\right\} \\ &= \operatorname{ess\,sup}\left\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(X - f 1_{[\tau,\infty)}) \geq 0\right\} \\ &= \operatorname{ess\,sup}\{f \in L^\infty(\mathcal{F}_\tau) \mid \phi(X) \geq f\} = \phi(X)\,. \end{aligned}$$

□

Proposition 3.9 *If $\mathcal{C}$ is a subset of $\mathcal{R}^\infty_{\tau,\theta}$ that satisfies* (i), (ii) *and* (I) *of Proposition 3.4, then $\phi_{\mathcal{C}}$ is a monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ and $\mathcal{C}_{\phi_{\mathcal{C}}}$ is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ that contains $\mathcal{C}$ and satisfies condition* (C) *of Proposition 3.4.*
If $\mathcal{C}$ satisfies (i), (I) *and* (II) *of Proposition 3.4, then $\phi_{\mathcal{C}}$ is a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$.*
If $\mathcal{C}$ satisfies (i), (I), (II) *and* (III) *or* (i), (I), (II') *and* (III) *of Proposition 3.4, then $\phi_{\mathcal{C}}$ is a coherent utility functional on $\mathcal{R}^\infty_{\tau,\theta}$.*

Proof. Let $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ such that $X = Y$ on A for some $A \in \mathcal{F}_\tau$. Assume that $X - f 1_{[\tau,\infty)} \in \mathcal{C}$ for some $f \in L^\infty(\mathcal{F}_\tau)$. If $\mathcal{C}$ satisfies (i) and (ii) of Proposition 3.4, there exists an $m \in L^\infty(\mathcal{F}_\tau)$ such that $m 1_{[\tau,\infty)} \in \mathcal{C}$, and

$$1_A(Y - f 1_{[\tau,\infty)}) + 1_{A^c} m 1_{[\tau,\infty)} = 1_A(X - f 1_{[\tau,\infty)}) + 1_{A^c} m 1_{[\tau,\infty)} \in \mathcal{C}\,.$$

If $\mathcal{C}$ also satisfies (I) of Proposition 3.4, then

$$Y - 1_A f 1_{[\tau,\infty)} + 1_{A^c}(m + ||Y||_{\tau,\theta}) 1_{[\tau,\infty)} \in \mathcal{C}\,.$$

Hence, $1_A \phi_{\mathcal{C}}(Y) \geq 1_A \phi_{\mathcal{C}}(X)$, and by symmetry, $1_A \phi_{\mathcal{C}}(Y) = 1_A \phi_{\mathcal{C}}(X)$. It follows that $\phi_{\mathcal{C}}(1_A X) = 1_A \phi_{\mathcal{C}}(X)$ for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $A \in \mathcal{F}_\tau$. (1) of Definition 3.1 follows from (I) of Proposition 3.4. (2) of Definition 3.1 follows directly from the construction of $\phi_{\mathcal{C}}$. By Proposition 3.4, $\mathcal{C}_{\phi_{\mathcal{C}}}$ satisfies condition (C) of Proposition 3.4, and it obviously contains $\mathcal{C}$. On the other hand, if $X \in \mathcal{C}_{\phi_{\mathcal{C}}}$, then there exists an increasing sequence $(f^n)_{n\in\mathbb{N}}$ in $L^\infty(\mathcal{F}_\tau)$ such that $X - f^n 1_{[\tau,\infty)} \in \mathcal{C}$ and $f^n \stackrel{\text{a.s.}}{\to} \phi_{\mathcal{C}}(X) \geq 0$. Set $g^n := f^n \wedge 0$. Then, by (I) of Proposition 3.4, $X - g^n 1_{[\tau,\infty)} \in \mathcal{C}$, and $g^n \to 0$ almost surely. Hence, $\mathcal{C}_{\phi_{\mathcal{C}}}$ is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ that satisfies condition (C) of Proposition 3.4 and contains $\mathcal{C}$. The rest of the statements are obvious. □

3.2 Representations for conditional concave monetary and coherent utility functionals

Definition 3.10 *We say a concave monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ is continuous for bounded decreasing sequences if*

$$\lim_{n\to\infty} \phi(X^n) = \phi(X) \quad \textit{almost surely}$$

for every decreasing sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{R}^\infty_{\tau,\theta}$ and $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that

$$X^n_t \stackrel{\text{a.s.}}{\to} X_t \quad \textit{for all } t \in \mathbb{N}\,.$$

Lemma 3.11 *Let ϕ be a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences. Then the corresponding acceptance set $\mathcal{C}_\phi$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed.*

Proof. Let $(X^\lambda)_{\lambda \in \Lambda}$ be a net in $\mathcal{C}_\phi$ and $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that $X^\lambda \to X$ in $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$, and assume that

$$\phi(X) < 0 \quad \text{on } A \tag{3.1}$$

for some $A \in \mathcal{F}_\tau$ with $P[A] > 0$. Then the map $\tilde{\phi} : \mathcal{R}^\infty \to \mathbb{R}$ given by

$$\tilde{\phi}(X) = \frac{1}{P[A]} \mathrm{E}\left[1_A \, \phi \circ \pi_{\tau,\theta}(X)\right] , \quad X \in \mathcal{R}^\infty ,$$

is a concave monetary utility functional on $\mathcal{R}^\infty$ that is continuous for bounded decreasing sequences. Denote by $\mathcal{G}$ the sigma-algebra on $\Omega \times \mathbb{N}$ generated by all the sets $A \times \{t\}$, $t \in \mathbb{N}$, $A \in \mathcal{F}_t$, and by ν the measure on $(\Omega, \mathcal{G})$ given by

$$\nu(A \times \{t\}) = 2^{-(t+1)} P[A] \, , \, t \in \mathbb{N} \, , \, A \in \mathcal{F}_t \, .$$

Then $\mathcal{R}^\infty = L^\infty(\Omega \times \mathbb{N}, \mathcal{G}, \nu)$ and $\mathcal{A}^1$ can be identified with $L^1(\Omega \times \mathbb{N}, \mathcal{G}, \nu)$. Hence, it can be deduced from the Krein–Šmulian theorem that $\mathcal{C}_{\tilde{\phi}}$ is a $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed subset of $\mathcal{R}^\infty$ (see the proof of Theorem 3.2 in Delbaen (2002) or Remark 4.3 in Cheridito et al. (2004)). Since $(X^\lambda)_{\lambda \in \Lambda} \subset \mathcal{C}_{\tilde{\phi}}$, it follows that

$$\frac{1}{P[A]} \mathrm{E}\left[1_A \, \phi(X)\right] \geq 0 \, ,$$

which contradicts (3.1). □

Definition 3.12 *For a concave monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ and $a \in \mathcal{A}^1_{\tau,\theta}$, we define*

$$\phi^*(a) := \operatorname{ess\,inf}_{X \in \mathcal{R}^\infty_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi(X) \right\}$$

and

$$\phi^\#(a) := \operatorname{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, a \rangle_{\tau,\theta} \, .$$

Remarks 3.13
Let ϕ be a concave monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$.
1. Obviously, for all $a \in \mathcal{A}^1_{\tau,\theta}$, $\phi^*(a)$ and $\phi^\#(a)$ are measurable functions from $(\Omega, \mathcal{F}_\tau)$ to $[-\infty, 0]$ and

$$\phi^*(a) \leq \phi^\#(a) \quad \text{for all } a \in \mathcal{A}^1_{\tau,\theta} \, .$$

Moreover,

$$\phi^*(a) = \phi^\#(a) \quad \text{for all } a \in \mathcal{D}_{\tau,\theta} \tag{3.2}$$

because

$$\langle X, a \rangle_{\tau,\theta} - \phi(X) = \left\langle X - \phi(X) 1_{[\tau,\infty)}, a \right\rangle_{\tau,\theta} \, , \quad \text{and} \quad X - \phi(X) 1_{[\tau,\infty)} \in \mathcal{C}_\phi \, ,$$

for each $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $a \in \mathcal{D}_{\tau,\theta}$.
2. It can easily be checked that

$$\phi^\#(\lambda a + (1-\lambda) b) \geq \lambda \phi^\#(a) + (1-\lambda) \phi^\#(b) \, ,$$

for all $a, b \in \mathcal{A}^1_{\tau,\theta}$ and $\lambda \in L^\infty(\mathcal{F}_\tau)$ such that $0 \le \lambda \le 1$, and

$$\phi^\#(\lambda a) = \lambda \phi^\#(a) \quad \text{for all } a \in \mathcal{A}^1_{\tau,\theta} \text{ and } \lambda \in L^\infty_+(\mathcal{F}_\tau)\,. \tag{3.3}$$

Note that it follows from (3.3) that

$$\phi^\#(1_A a + 1_{A^c} b) = 1_A \phi^\#(a) + 1_{A^c} \phi^\#(b)$$

for all $a, b \in \mathcal{A}^1_{\tau,\theta}$ and $A \in \mathcal{F}_\tau$.

3. For every measurable function $m : (\Omega, \mathcal{F}_\tau) \to [-\infty, 0]$, the set

$$\left\{ a \in \mathcal{A}^1_{\tau,\theta} \mid \phi^\#(a) \ge m \right\}$$

is $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed. Indeed, let $(a^\mu)_{\mu \in M}$ be a net in $\left\{ a \in \mathcal{A}^1_{\tau,\theta} \mid \phi^\#(a) \ge m \right\}$ and $a \in \mathcal{A}^1$ such that $a^\mu \to a$ in $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$. Then, for all $X \in \mathcal{C}_\phi$, $\mu \in M$ and $A \in \mathcal{F}_\tau$ such that $A \subset \{m > -\infty\}$,

$$\langle 1_A X, a^\mu \rangle = \mathrm{E}\left[1_A \langle X, a^\mu \rangle_{\tau,\theta} \right] \ge \mathrm{E}\left[1_A m \right]\,.$$

Hence,

$$\mathrm{E}\left[1_A \langle X, a \rangle_{\tau,\theta} \right] = \langle 1_A X, a \rangle \ge \mathrm{E}\left[1_A m \right]\,,$$

which shows that

$$\langle X, a \rangle_{\tau,\theta} \ge m\,, \quad \text{for all } X \in \mathcal{C}_\phi\,,$$

and therefore $\phi^\#(a) \ge m$.

Definition 3.14 *A penalty function γ on $\mathcal{D}_{\tau,\theta}$ is a mapping from $\mathcal{D}_{\tau,\theta}$ to the space of measurable functions $f : (\Omega, \mathcal{F}_\tau) \to [-\infty, 0]$ with the following property:*

$$\operatorname{ess\,sup}_{a \in \mathcal{D}_{\tau,\theta}} \gamma(a) = 0\,.$$

We call a penalty function γ on $\mathcal{D}_{\tau,\theta}$ special if

$$\gamma(1_A a + 1_{A^c} b) = 1_A \gamma(a) + 1_{A^c} \gamma(b)\,,$$

for all $a, b \in \mathcal{D}_{\tau,\theta}$ and $A \in \mathcal{F}_\tau$.

Theorem 3.15 *The following are equivalent:*

(1) *ϕ is a mapping defined on $\mathcal{R}^\infty_{\tau,\theta}$ that can be represented as*

$$\phi(X) = \operatorname{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \gamma(a) \right\}\,, \quad X \in \mathcal{R}^\infty_{\tau,\theta}\,, \tag{3.4}$$

for a penalty function γ on $\mathcal{D}_{\tau,\theta}$.

(2) *ϕ is a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ whose acceptance set $\mathcal{C}_\phi$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed.*

(3) *ϕ is a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences.*

Moreover, if (1)–(3) are satisfied, then $\phi^\#$ is a special penalty function on $\mathcal{D}_{\tau,\theta}$, $\phi^\#(a) \geq \gamma(a)$ for all $a \in \mathcal{D}_{\tau,\theta}$, and the representation (3.4) also holds with $\phi^\#$ instead of γ.

Proof.
(1) $\Rightarrow$ (3): If ϕ has a representation of the form (3.4), then it is obviously a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$. To show that it is continuous for bounded decreasing sequences, let $(X^n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{R}^\infty_{\tau,\theta}$ and $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that

$$\lim_{n\to\infty} X^n_t = X_t \quad \text{almost surely, for all } t \in \mathbb{N}\,.$$

Note that this implies that

$$\lim_{n\to\infty} \langle X^n, a\rangle_{\tau,\theta} = \langle X, a\rangle_{\tau,\theta} \quad \text{almost surely, for all } a \in \mathcal{D}_{\tau,\theta}\,.$$

By property (1) of Definition 3.1, $\phi_{\tau,\theta}(X^n)$ is decreasing in n. Hence, almost surely, $\lim_{n\to\infty}\phi(X^n)$ exists and $\lim_{n\to\infty}\phi(X^n) \geq \phi(X)$. On the other hand, there exists a sequence $(a^k)_{k\in\mathbb{N}}$ in $\mathcal{D}_{\tau,\theta}$ such that

$$\phi(X) = \inf_{k\in\mathbb{N}} \left\{ \left\langle X, a^k \right\rangle - \gamma(a^k) \right\}\,.$$

Since

$$\left\langle X^n, a^k \right\rangle - \gamma(a^k) \geq \phi(X^n)$$

for all $k, n \in \mathbb{N}$, we have that

$$\left\langle X, a^k \right\rangle - \gamma(a^k) = \lim_{n\to\infty} \left\{ \left\langle X^n, a^k \right\rangle - \gamma(a^k) \right\} \geq \lim_{n\to\infty} \phi(X^n)$$

for all $k \in \mathbb{N}$, and therefore also,

$$\phi(X) \geq \lim_{n\to\infty} \phi(X^n)\,.$$

(3) $\Rightarrow$ (2): follows from Lemma 3.11.
(2) $\Rightarrow$ (1): By (3.2) and the definition of ϕ^*,

$$\phi^\#(a) = \phi^*(a) \leq \langle X, a\rangle_{\tau,\theta} - \phi(X)$$

for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $a \in \mathcal{D}_{\tau,\theta}$. Hence,

$$\phi(X) \leq \operatorname{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}} \left\{ \langle X, a\rangle_{\tau,\theta} - \phi^\#(a) \right\} \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta}\,. \tag{3.5}$$

To show the reverse inequality, let $m \in L^\infty(\mathcal{F}_\tau)$ with

$$m \leq \operatorname{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}} \left\{ \langle X, a\rangle_{\tau,\theta} - \phi^\#(a) \right\}\,, \tag{3.6}$$

and assume that $Y = X - m1_{[\tau,\infty)} \notin \mathcal{C}_\phi$. Since $\mathcal{C}_\phi$ is a convex, $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed subset of $\mathcal{R}^\infty$, there exists an $a \in (\mathcal{A}^1_{\tau,\theta})_+$ such that

$$\mathrm{E}\left[\langle Y, a\rangle_{\tau,\theta}\right] = \langle Y, a\rangle_{0,\infty} < \inf_{Z\in\mathcal{C}_\phi} \langle Z, a\rangle_{0,\infty} = \mathrm{E}\left[\operatorname{ess\,inf}_{Z\in\mathcal{C}_\phi} \langle Z, a\rangle_{\tau,\theta}\right] .$$

Therefore, there exists a $B \in \mathcal{F}_\tau$ with $P[B] > 0$ such that

$$\langle Y, a\rangle_{\tau,\theta} < \operatorname{ess\,inf}_{Z\in\mathcal{C}_\phi} \langle Z, a\rangle_{\tau,\theta} \quad \text{on} \quad B . \tag{3.7}$$

Note that for $A = \left\{\langle 1, a\rangle_{\tau,\theta} = 0\right\}$,

$$1_A \left|\langle Z, a\rangle_{\tau,\theta}\right| \le 1_A \langle |Z|, a\rangle_{\tau,\theta} \le 1_A ||Z||_{\tau,\theta} \langle 1, a\rangle_{\tau,\theta} = 0 \quad \text{for all} \quad Z \in \mathcal{R}^\infty_{\tau,\theta} .$$

Hence, $B \subset \left\{\langle 1, a\rangle_{\tau,\theta} > 0\right\}$. Define the process $b \in \mathcal{D}_{\tau,\theta}$ as follows:

$$b := 1_B \frac{a}{\langle 1, a\rangle_{\tau,\theta}} + 1_{B^c} 1_{[\tau,\infty)} .$$

It follows from (3.7) that

$$\langle X, b\rangle_{\tau,\theta} - m = \langle Y, b\rangle_{\tau,\theta} < \operatorname{ess\,inf}_{Z\in\mathcal{C}_\phi} \langle Z, b\rangle_{\tau,\theta} = \phi^\#(b) \quad \text{on} \quad B .$$

This contradicts (3.6). Hence, $X - m1_{[\tau,\infty)} \in \mathcal{C}_\phi$, and therefore, $\phi(X) \ge m$ for all m satisfying (3.6), which shows that

$$\phi(X) \ge \operatorname{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}} \left\{\langle X, a\rangle_{\tau,\theta} - \phi^\#(a)\right\} .$$

This together with (3.5) proves that (2) implies (1) and also that $\phi^\#$ is a penalty function on $\mathcal{D}_{\tau,\theta}$. By Remark 3.13.2, $\phi^\#$ is special. If ϕ is a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ with a representation of the form (3.4), then

$$\langle X, a\rangle_{\tau,\theta} - \phi(X) \ge \gamma(a)$$

for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $a \in \mathcal{D}_{\tau,\theta}$, which implies that $\phi^\# = \phi^* \ge \gamma$ on $\mathcal{D}_{\tau,\theta}$. □

Corollary 3.16 *The following are equivalent:*

(1) *ϕ is a mapping defined on $\mathcal{R}^\infty_{\tau,\theta}$ that can be represented as*

$$\phi(X) = \operatorname{ess\,inf}_{a\in\mathcal{Q}} \langle X, a\rangle_{\tau,\theta} , \quad X \in \mathcal{R}^\infty_{\tau,\theta} , \tag{3.8}$$

for a non-empty subset $\mathcal{Q}$ of $\mathcal{D}_{\tau,\theta}$.

(2) *ϕ is a coherent utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ whose acceptance set $\mathcal{C}_\phi$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed.*

(3) *ϕ is a coherent utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences.*

Moreover, if (1)–(3) *are satisfied, then the set*

$$\mathcal{Q}^0_\phi := \left\{ a \in \mathcal{D}_{\tau,\theta} \mid \phi^\#(a) = 0 \right\}$$

is equal to the smallest $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, $\mathcal{F}_\tau$-convex subset of $\mathcal{D}_{\tau,\theta}$ that contains $\mathcal{Q}$, and the representation (3.4) *also holds with $\mathcal{Q}^0_\phi$ instead of $\mathcal{Q}$.*

Proof. If (1) holds, then it follows from Theorem 3.15 that ϕ is a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences, and it is clear that ϕ is coherent. This shows that (1) implies (3). The implication (3) $\Rightarrow$ (2) follows directly from Theorem 3.15. If (2) holds, then Theorem 3.15 implies that $\phi^\#$ is a special penalty function on $\mathcal{D}_{\tau,\theta}$, and

$$\phi(X) = \inf_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^\#(a) \right\} \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} \, .$$

Since $\phi^\#$ is special, the set $\left\{ \phi^\#(a) \mid a \in \mathcal{D}_{\tau,\theta} \right\}$ is directed upwards. Therefore, there exists a sequence $(a^k)_{k \in \mathbb{N}}$ in $\mathcal{D}_{\tau,\theta}$ such that almost surely,

$$\phi^\#(a^k) \nearrow \operatorname{ess\,sup}_{a \in \mathcal{D}_{\tau,\theta}} \phi^\#(a) = 0 \, , \quad \text{as } k \to \infty \, .$$

It can easily be deduced from the fact that ϕ is coherent, that

$$\left\{ \phi^\#(a) = 0 \right\} \cup \left\{ \phi^\#(a) = -\infty \right\} = \Omega \quad \text{for all } a \in \mathcal{D}_{\tau,\theta} \, .$$

Hence, the sets $A_k := \left\{ \phi^\#(a^k) = 0 \right\}$ are increasing in k, and $\bigcup_{k \in \mathbb{N}} A_k = \Omega$. Therefore,

$$a^* := 1_{A_0} a^0 + \sum_{k \geq 1} 1_{A_k \setminus A_{k-1}} a^k \in \mathcal{D}_{\tau,\theta} \, ,$$

and it follows from Remark 3.13.2 that $\phi^\#(a^*) = 0$. Note that for all $a \in \mathcal{D}_{\tau,\theta}$,

$$1_{\left\{ \phi^\#(a) = 0 \right\}} a + 1_{\left\{ \phi^\#(a) = -\infty \right\}} a^* \in \mathcal{Q}^0_\phi \, .$$

This shows that

$$\phi(X) = \operatorname{ess\,inf}_{a \in \mathcal{Q}^0_\phi} \langle X, a \rangle_{\tau,\theta} \, , \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} \, . \tag{3.9}$$

It remains to show that $\mathcal{Q}^0_\phi$ is equal to the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, $\mathcal{F}_\tau$-convex hull $\langle \mathcal{Q} \rangle_\tau$ of $\mathcal{Q}$. It follows from Theorem 3.15 that $\phi^\#$ is the largest among all penalty functions on $\mathcal{D}_{\tau,\theta}$ that induce ϕ. This implies $\mathcal{Q} \subset \mathcal{Q}^0_\phi$. By Remarks 3.13.2 and 3.13.3, $\mathcal{Q}^0_\phi$ is $\mathcal{F}_\tau$-convex and $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed. Hence, $\langle \mathcal{Q} \rangle_\tau \subset \mathcal{Q}^0_\phi$. Now, assume that there exists a $b \in \mathcal{Q}^0_\phi \setminus \langle \mathcal{Q} \rangle_\tau$.

Then, it follows from the separating hyperplane theorem that there exists an $X \in \mathcal{R}^\infty_{\tau,\theta}$ such that

$$\langle X, b \rangle < \inf_{a \in \langle \mathcal{Q} \rangle_\tau} \langle X, a \rangle = \mathrm{E}\left[\operatorname{ess\,inf}_{a \in \langle \mathcal{Q} \rangle_\tau} \langle X, a \rangle_{\tau,\theta}\right] = \mathrm{E}\left[\operatorname{ess\,inf}_{a \in \mathcal{Q}} \langle X, a \rangle_{\tau,\theta}\right] = \mathrm{E}\left[\phi(X)\right] . \tag{3.10}$$

But, by (3.9),

$$\langle X, b \rangle - \mathrm{E}\left[\phi(X)\right] = \mathrm{E}\left[\langle X, b \rangle_{\tau,\theta} - \phi(X)\right] \geq 0$$

for all $b \in \mathcal{Q}^0_\phi$, which contradicts (3.10). Hence, $\mathcal{Q}^0_\phi \setminus \langle \mathcal{Q} \rangle_\tau$ is empty, that is, $\mathcal{Q}^0_\phi \subset \langle \mathcal{Q} \rangle_\tau$.□

Remark 3.17 Detlefsen (2003) and Scandolo (2003) give representation results for conditional concave monetary utility functionals that depend on random variables. Since monetary utility functionals that depend on random variables can be seen as special cases of monetary utility functionals for stochastic processes, Theorem 3.15 generalizes the representation results in Detlefsen (2003) and Scandolo (2003).

3.3 Relevance

Definition 3.18 *Let ϕ be a monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$. We call ϕ θ-relevant if*

$$A \subset \left\{\phi(-\varepsilon 1_A 1_{[t \wedge \theta, \infty)}) < 0\right\}$$

for all $\varepsilon > 0$, $t \in \mathbb{N}$ and $A \in \mathcal{F}_{t \wedge \theta}$.

Definition 3.19

$$\mathcal{D}^e_{\tau,\theta} := \left\{ a \in \mathcal{D}_{\tau,\theta} \mid P\left[\sum_{j \geq t \wedge \theta} \Delta a_j > 0\right] = 1 \quad \textit{for all } t \in \mathbb{N} \right\} .$$

Remarks 3.20

1. If ϕ is a θ-relevant monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ and ξ is an $(\mathcal{F}_t)$-stopping time such that $\tau \leq \xi \leq \theta$, then, obviously, the restriction of ϕ to $\mathcal{R}^\infty_{\tau,\xi}$ is ξ-relevant.

2. Assume that θ is finite. Then it can easily be checked that a monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ is θ-relevant if and only if

$$A \subset \left\{\phi(-\varepsilon 1_A 1_{[\theta, \infty)}) < 0\right\}$$

for all $\varepsilon > 0$ and $A \in \mathcal{F}_\theta$. Also, in this case,

$$\mathcal{D}^e_{\tau,\theta} = \left\{ a \in \mathcal{D}_{\tau,\theta} \mid P\left[\Delta a_\theta > 0\right] = 1 \right\} .$$

Definition 3.21 *For a concave monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ and a constant $K \geq 0$, we define*

$$\mathcal{Q}^K_\phi := \left\{ a \in \mathcal{D}_{\tau,\theta} \mid \phi^\#(a) \geq -K \right\} .$$

By the Remarks 3.13.2 and 3.13.3, $\mathcal{Q}_\phi^K$ is $\mathcal{F}_\tau$-convex and $\sigma(\mathcal{A}^1,\mathcal{R}^\infty)$-closed for every concave monetary utility functional ϕ on $\mathcal{R}^\infty_{\tau,\theta}$ and each constant $K \geq 0$.

Proposition 3.22 *Let ϕ be a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences and θ-relevant. Then*

$$\mathcal{Q}_\phi^K \cap \mathcal{D}^e_{\tau,\theta} \neq \emptyset \quad \textit{for all} \quad K > 0\,.$$

Proof. Fix $K > 0$ and $t \in \mathbb{N}$. For $a \in \mathcal{D}_{\tau,\theta}$, we denote

$$e_t(a) := \sum_{j \geq t\wedge\theta} \Delta a_j\,,$$

and we define

$$\alpha_t := \sup_{a\in\mathcal{Q}_\phi^K} P\left[e_t(a) > 0\right]\,. \tag{3.11}$$

Let $(a^{t,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{Q}_\phi^K$ with

$$\lim_{n\to\infty} P\left[e_t(a^{t,n}) > 0\right] = \alpha_t\,.$$

Since $\mathcal{Q}_\phi^K$ is convex and $\sigma(\mathcal{A}^1,\mathcal{R}^\infty)$-closed,

$$a^t := \sum_{n\geq 1} 2^{-n} a^{t,n} \in \mathcal{Q}_\phi^K\,,$$

and, obviously,

$$P\left[e_t(a^t) > 0\right] = \alpha_t\,.$$

In the next step we show that $\alpha_t = 1$. Assume to the contrary that $\alpha_t < 1$ and denote $A_t := \left\{e_t(a^t) = 0\right\}$. Since ϕ is θ-relevant,

$$A_t \subset \left\{\phi(-K1_{A_t}1_{[t\wedge\theta,\infty)}) < 0\right\}\,,$$

and therefore also,

$$\hat{A}_t := \bigcap_{B\in\mathcal{F}_\tau,\,A_t\subset B} B \;\subset\; \left\{\phi(-K1_{A_t}1_{[t\wedge\theta,\infty)}) < 0\right\}\,.$$

By Theorem 3.15,

$$\phi(-K1_{A_t}1_{[t\wedge\theta,\infty)}) = \operatorname{ess\,inf}_{a\in\mathcal{D}_{\tau,\theta}}\left\{\left\langle -K1_{A_t}1_{[t\wedge\theta,\infty)}, a\right\rangle_{\tau,\theta} - \phi^{\#}(a)\right\}\,.$$

Hence, there must exist an $a \in \mathcal{D}_{\tau,\theta}$ with $P\left[A_t \cap \{e_t(a) > 0\}\right] > 0$ and $\phi^{\#}(a) \geq -K$ on $\hat{A}_t$. Then,

$$b^t := 1_{\hat{A}_t}\, a + 1_{\hat{A}^c}\, a^t \in \mathcal{Q}_\phi^K\,, \quad c^t := \frac{1}{2} b^t + \frac{1}{2} a^t \in \mathcal{Q}_\phi^K\,,$$

and $P\left[e_t(c^t) > 0\right] > P\left[e_t(a^t) > 0\right] = \alpha$. This contradicts (3.11). Therefore, we must have $\alpha_t = 1$ for all $t \in \mathbb{N}$. Finally, set

$$a^* = \sum_{t \geq 1} 2^{-t} a^t ,$$

and note that $a^* \in \mathcal{Q}^K_\phi \cap \mathcal{D}^e_{\tau,\theta}$. □

Corollary 3.23 *Let ϕ be a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences and θ-relevant. Then*

$$\phi(X) = \operatorname{ess\,inf}_{a \in \mathcal{D}^e_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^\#(a) \right\} , \quad \textit{for all } X \in \mathcal{R}^\infty_{\tau,\theta} .$$

Proof. By Theorem 3.15,

$$\phi(X) = \operatorname{ess\,inf}_{a \in \mathcal{D}_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^\#(a) \right\} , \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} ,$$

which immediately shows that

$$\phi(X) \leq \operatorname{ess\,inf}_{a \in \mathcal{D}^e_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^\#(a) \right\} , \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} .$$

To show the reverse inequality, we choose a $b \in \mathcal{D}_{\tau,\theta}$. It follows from Proposition 3.22 that there exists a process $c \in \mathcal{Q}^1_\phi \cap \mathcal{D}^e_{\tau,\theta}$. Then, for all $n \geq 1$,

$$b^n := (1 - \frac{1}{n}) b + \frac{1}{n} c \in \mathcal{D}^e_{\tau,\theta} ,$$

$$\lim_{n \to \infty} \langle X, b^n \rangle_{\tau,\theta} = \lim_{n \to \infty} \left\{ (1 - \frac{1}{n}) \langle X, b \rangle_{\tau,\theta} + \frac{1}{n} \langle X, c \rangle_{\tau,\theta} \right\} = \langle X, b \rangle_{\tau,\theta} \quad \text{almost surely,}$$

and

$$\begin{aligned} \phi^\#(b^n) &= \operatorname{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, b^n \rangle_{\tau,\theta} \geq (1 - \frac{1}{n}) \operatorname{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, b \rangle_{\tau,\theta} + \frac{1}{n} \operatorname{ess\,inf}_{X \in \mathcal{C}_\phi} \langle X, c \rangle_{\tau,\theta} \\ &= (1 - \frac{1}{n}) \phi^\#(b) + \frac{1}{n} \phi^\#(c) \quad \to \quad \phi^\#(b) \quad \text{almost surely} . \end{aligned}$$

This shows that

$$\langle X, b \rangle_{\tau,\theta} - \phi^\#(b) \geq \operatorname{ess\,inf}_{a \in \mathcal{D}^e_{\tau,\theta}} \left\{ \langle X, a \rangle_{\tau,\theta} - \phi^\#(a) \right\} ,$$

and therefore,

$$\phi(X) \geq \operatorname{ess\,inf}_{a \in \mathcal{D}^e_{\tau,\theta}} \left\{ \langle X, a \rangle_\tau - \phi^\#(a) \right\} ,$$

which completes the proof. □

Corollary 3.24 *Let ϕ be a coherent utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ that is continuous for bounded decreasing sequences and θ-relevant. Then*

$$\phi(X) = \operatorname{ess\,inf}_{a \in \mathcal{Q}^e_\phi} \langle X, a \rangle_{\tau,\theta} \,, \quad X \in \mathcal{R}^\infty_{\tau,\theta} \,,$$

where $\mathcal{Q}^e_\phi := \left\{ a \in \mathcal{D}^e_{\tau,\theta} \mid \phi^{\#}(a) = 0 \right\}$.

Proof. This corollary can either be deduced from Corollary 3.16 and Proposition 3.22 like Corollary 3.23 from Theorem 3.15 and Proposition 3.22 or from Corollary 3.23 with the arguments used in the proof of the implication (2) $\Rightarrow$ (1) of Corollary 3.16. □

4 Processes of monetary utility functionals and acceptance sets

Definition 4.1 *Let $S \in \mathbb{N}$ and $T \in \mathbb{N} \cup \{\infty\}$ such that $S \le T$. Assume that for all $t \in [S,T] \cap \mathbb{N}$, $\phi_{t,T}$ is a monetary utility functional on $\mathcal{R}^\infty_{t,T}$ with acceptance set $\mathcal{C}_{t,T}$. Then we call $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ a monetary utility functional process and $(\mathcal{C}_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ an acceptance set process. We call $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ a relevant monetary utility functional process if all $\phi_{t,T}$ are T-relevant. We call $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ a concave monetary utility process if $\phi_{t,T}$ is a concave monetary utility functional on $\mathcal{R}^\infty_{t,T}$ for all $t \in [S,T] \cap \mathbb{N}$. If every $\phi_{t,T}$ is coherent, we call $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ a coherent utility functional process.*

Definition 4.2 *Let $S \in \mathbb{N}$ and $T \in \mathbb{N}\cup\{\infty\}$ such that $S \le T$. Let $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ be a monetary utility functional process with corresponding acceptance set process $(\mathcal{C}_{t,T})_{t\in[S,T]\cap\mathbb{N}}$. Let τ and θ be two $(\mathcal{F}_t)$-stopping times such that τ is finite (i.e. $\tau < \infty$) and $S \le \tau \le \theta \le T$. Then we define the mapping $\phi_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ by*

$$\phi_{\tau,\theta}(X) := \sum_{t\in[S,T]\cap\mathbb{N}} \phi_{t,T}(1_{\{\tau=t\}}X) \,, \tag{4.12}$$

and the set $\mathcal{C}_{\tau,\theta} \subset \mathcal{R}^\infty_{\tau,\theta}$ by

$$\mathcal{C}_{\tau,\theta} := \left\{ X \in \mathcal{R}^\infty_{\tau,\theta} \mid 1_{\{\tau=t\}}X \in \mathcal{C}_{t,T} \text{ for all } t \in [S,T] \cap \mathbb{N} \right\} . \tag{4.13}$$

It can easily be checked that $\phi_{\tau,\theta}$ defined by (4.12) is a monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$ and that the set $\mathcal{C}_{\tau,\theta}$ given in (4.13) is the acceptance set of $\phi_{\tau,\theta}$. Moreover, if $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ is a concave monetary utility functional process, then $\phi_{\tau,\theta}$ is a concave monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$. If $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ is coherent, then so is $\phi_{\tau,\theta}$.

4.1 Time-consistency

Definition 4.3 *Let $S \in \mathbb{N}$ and $T \in \mathbb{N}\cup\{\infty\}$ such that $S \le T$. We call a monetary utility functional process $(\phi_{t,T})_{t\in[S,T]\cap\mathbb{N}}$ time-consistent if*

$$\phi_{t,T}(X) = \phi_{t,T}(X1_{[t,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)})$$

for each $t \in [S,T] \cap \mathbb{N}$, every finite $(\mathcal{F}_t)$-stopping time θ such that $t \leq \theta \leq T$ and all processes $X \in \mathcal{R}^\infty_{t,T}$.

Remarks 4.4
1. Let $S \in \mathbb{N}$ and $T \in \mathbb{N} \cup \{\infty\}$ such that $S \leq T$. It is easy to see that a monetary utility functional process $(\phi_{t,T})_{t \in [S,T]\cap\mathbb{N}}$ is time-consistent, if and only if

$$\phi_{t,T}(X) \leq \phi_{t,T}(Y)\,,$$

for each $t \in [S,T] \cap \mathbb{N}$ and all processes $X, Y \in \mathcal{R}^\infty_{t,T}$ such that

$$X 1_{[t,\theta)} \leq Y 1_{[t,\theta)} \quad \text{and} \quad \phi_{\theta,T}(X) \leq \phi_{\theta,T}(Y)\,,$$

for some finite $(\mathcal{F}_t)$-stopping time θ with $t \leq \theta \leq T$.
2. Let $S \in \mathbb{N}$ and $T \in \mathbb{N} \cup \{\infty\}$ such that $S \leq T$ and $(\phi_{t,T})_{t \in [S,T]\cap\mathbb{N}}$ a time-consistent monetary utility functional process. Then it can easily be seen from Definition 4.2 that

$$\phi_{\tau,T}(X) = \phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)})$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ and θ such that $S \leq \tau \leq \theta \leq T$ and all processes $X \in \mathcal{R}^\infty_{\tau,T}$.

Proposition 4.5 *Let $S, T \in \mathbb{N}$ such that $S \leq T$ and $(\phi_{t,T})_{t=S}^T$ a monetary utility functional process that satisfies*

$$\phi_{t,T}(X) = \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}) \tag{4.14}$$

for all $t = S, \dots, T-1$ and $X \in \mathcal{R}^\infty_{t,T}$. Then $(\phi_{t,T})_{t=S}^T$ is time-consistent.

Proof. For $t \in [S,T] \cap \mathbb{N}$, an $(\mathcal{F}_t)$-stopping time θ such that $t \leq \theta \leq T$ and a process $X \in \mathcal{R}^\infty_{t,T}$, we denote $Y = X 1_{[t,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}$ and show

$$\phi_{t,T}(X) = \phi_{t,T}(Y) \tag{4.15}$$

by induction. For $t = T$, (4.15) is obvious. If $t \leq T-1$, we assume that

$$\phi_{t+1,T}(Z) = \phi_{t+1,T}(Z 1_{[t+1,\xi)} + \phi_{\xi,T}(Z) 1_{[\xi,\infty)})\,,$$

for every $(\mathcal{F}_t)$-stopping time ξ such that $t+1 \leq \xi \leq T$ and all $Z \in \mathcal{R}^\infty_{t+1,T}$. Then

$$1_{\{\theta \geq t+1\}} \phi_{t+1}(X) = \phi_{t+1}(1_{\{\theta \geq t+1\}} X) = \phi_{t+1}(1_{\{\theta \geq t+1\}} Y) = 1_{\{\theta \geq t+1\}} \phi_{t+1}(Y)\,.$$

Hence, it follows from the assumption (4.14) that

$$\begin{aligned}
\phi_{t,T}(Y) &= \phi_{t,T}\left(1_{\{\theta = t\}} \phi_{t,T}(X) 1_{[t,\infty)} + 1_{\{\theta \geq t+1\}} Y\right) \\
&= 1_{\{\theta = t\}} \phi_{t,T}(X) + 1_{\{\theta \geq t+1\}} \phi_{t,T}(Y) \\
&= 1_{\{\theta = t\}} \phi_{t,T}(X) + 1_{\{\theta \geq t+1\}} \phi_{t,T}(Y 1_{\{t\}} + \phi_{t+1}(Y) 1_{[t+1,\infty)}) \\
&= 1_{\{\theta = t\}} \phi_{t,T}(X) + 1_{\{\theta \geq t+1\}} \phi_{t,T}(X 1_{\{t\}} + \phi_{t+1}(X) 1_{[t+1,\infty)}) \\
&= 1_{\{\theta = t\}} \phi_{t,T}(X) + 1_{\{\theta \geq t+1\}} \phi_{t,T}(X) \\
&= \phi_{t,T}(X)\,.
\end{aligned}$$

□

Proposition 4.6 *Let $S \in \mathbb{N}$, $T \in \mathbb{N} \cup \{\infty\}$ and τ, θ finite $(\mathcal{F}_t)$-stopping times such that $S \le \tau \le \theta \le T$. For a monetary utility functional process $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ with corresponding acceptance set process $(\mathcal{C}_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ the following two conditions are equivalent:*

(1) $\phi_{\tau,T}(X) = \phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)})$ *for all* $X \in \mathcal{R}^{\infty}_{\tau,T}$.

(2) $\mathcal{C}_{\tau,T} = \mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T}$

Proof.
(1) $\Rightarrow$ (2):
Assume $Y \in \mathcal{C}_{\tau,\theta}$ and $Z \in \mathcal{C}_{\theta,T}$. Then $X = Y + Z \in \mathcal{R}^{\infty}_{\tau,T}$, $X 1_{[\tau,\theta)} = Y 1_{[\tau,\theta)}$ and $\phi_{\theta,T}(X) = Y_\theta + \phi_{\theta,T}(Z) \ge Y_\theta$. Therefore,

$$\phi_{\tau,T}(X) = \phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}) \ge \phi_{\tau,T}(Y) \ge 0 \, .$$

This shows that $\mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T} \subset \mathcal{C}_{\tau,T}$. To show $\mathcal{C}_{\tau,T} \subset \mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T}$, let $X \in \mathcal{C}_{\tau,T}$ and set $Z := (X - \phi_{\theta,T}(X)) 1_{[\theta,\infty)}$ and $Y := X - Z = X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}$. It follows directly from the translation invariance of $\phi_{\theta,T}$ that $Z \in \mathcal{C}_{\theta,T}$. Moreover, $\phi_{\tau,T}(Y) = \phi_{\tau,T}(X) \ge 0$, which shows that $Y \in \mathcal{C}_{\tau,\theta}$.
(2) $\Rightarrow$ (1):
Let $X \in \mathcal{R}^{\infty}_{\tau,T}$ and $f \in L^\infty(\mathcal{F}_\tau)$ such that $X - f 1_{[\tau,\infty)} \in \mathcal{C}_{\tau,T}$. Since

$$\phi_{\theta,T}(X) = \operatorname{ess\,sup} \left\{ g \in L^\infty(\mathcal{F}_\theta) \mid (X - g) 1_{[\theta,\infty)} \in \mathcal{C}_{\theta,T} \right\}$$

and

$$\mathcal{C}_{\tau,T} \subset \mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T} \, ,$$

the process

$$X - f 1_{[\tau,\infty)} - (X - \phi_{\theta,T}(X)) 1_{[\theta,\infty)} = X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)} - f 1_{[\tau,\infty)}$$

has to be in $\mathcal{C}_{\tau,\theta}$. This shows that

$$\phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}) \ge \phi_{\tau,T}(X) \, .$$

On the other hand, if $X \in \mathcal{R}^{\infty}_{\tau,T}$ and $f \in L^\infty(\mathcal{F}_\tau)$ such that

$$X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)} - f 1_{[\tau,\infty)} \in \mathcal{C}_{\tau,T} \, ,$$

then also $X - f 1_{[\tau,\infty)} \in \mathcal{C}_{\tau,T}$ because $(X - \phi_{\theta,T}(X)) 1_{[\theta,\infty)} \in \mathcal{C}_{\theta,T}$ and $\mathcal{C}_{\tau,\theta} + \mathcal{C}_{\theta,T} \subset \mathcal{C}_{\tau,T}$. It follows that

$$\phi_{\tau,T}(X) \ge \phi_{\tau,T}(X 1_{[\tau,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}) \, .$$

□

Proposition 4.7 *Let $S \in \mathbb{N}$ and $T \in \mathbb{N} \cup \{\infty\}$ such that $S \leq T$. Let $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ be a time-consistent monetary utility functional process with corresponding acceptance set process $(\mathcal{C}_{t,T})_{t \in [S,T] \cap \mathbb{N}}$, and let τ and θ be two finite $(\mathcal{F}_t)$-stopping times such that $S \leq \tau \leq \theta \leq T$. Then*

1. *$1_A X \in \mathcal{C}_{\tau,T}$ for all $X \in \mathcal{C}_{\theta,T}$ and $A \in \mathcal{F}_\theta$.*

2. *If $\phi_{\tau,\theta}$ is θ-relevant, and X is a process in $\mathcal{R}^\infty_{\theta,T}$ such that $1_A X \in \mathcal{C}_{\tau,T}$ for all $A \in \mathcal{F}_\theta$, then $X \in \mathcal{C}_{\theta,T}$.*

3. *If ξ is an $(\mathcal{F}_t)$-stopping time such that $\theta \leq \xi \leq T$ and $\phi_{\tau,\xi}$ is ξ-relevant, then $\phi_{\theta,\xi}$ is ξ-relevant too. In particular, the monetary utility functional process $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is relevant if and only if $\phi_{S,T}$ is T-relevant.*

Proof.
1. If $X \in \mathcal{C}_{\theta,T}$ and $A \in \mathcal{F}_\theta$, then also $1_A X \in \mathcal{C}_{\theta,T}$. Obviously, $0 \in \mathcal{C}_{\tau,\theta}$. Hence, it follows from Proposition 4.6 that $1_A X = 0 + 1_A X \in \mathcal{C}_{\tau,T}$.
2. Assume $1_A X \in \mathcal{C}_{\tau,T}$ for all $A \in \mathcal{F}_\theta$ but $X \notin \mathcal{C}_{\theta,T}$. Then there exists an $\varepsilon > 0$ such that $P[A] > 0$, where $A = \{\phi_{\theta,T}(X) \leq -\varepsilon\}$. By Proposition 4.6, there exist $Y \in \mathcal{C}_{\tau,\theta}$ and $Z \in \mathcal{C}_{\theta,T}$ such that $1_A X = Y + Z$. Since $\phi_{\theta,T}(1_A X) \leq -\varepsilon 1_A$, $Z_\theta \geq 1_A(X_\theta + \varepsilon)$ and therefore, $Y_\theta \leq -\varepsilon 1_A$. But then, since $\phi_{\tau,\theta}$ is θ-relevant, $Y \notin \mathcal{C}_{\tau,\theta}$, which is a contradiction.
3. Let $\varepsilon > 0$, $t \in \mathbb{N}$ and $A \in \mathcal{F}_{t \wedge \xi}$. Set

$$B := A \cap \left\{\phi_{\theta,\xi}(-\varepsilon 1_A 1_{[t \wedge \xi, \infty)}) = 0\right\}$$

and note that

$$\phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t \wedge \xi, \infty)}) = 0 \quad \text{on } B\,.$$

Therefore, also

$$\phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t \wedge \xi, \infty)}) = 0 \quad \text{on } \hat{B} := \bigcap_{C \in \mathcal{F}_\theta\,;\, B \subset C} C\,.$$

Since

$$1_{\hat{B}^c}\, \phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t \wedge \xi, \infty)}) = \phi_{\theta,\xi}(-\varepsilon 1_{\hat{B}^c} 1_B 1_{[t \wedge \xi, \infty)}) = 0\,,$$

it follows that $\phi_{\theta,\xi}(-\varepsilon 1_B 1_{[t \wedge \xi, \infty)}) = 0$. Hence, $-\varepsilon 1_B 1_{[t \wedge \xi, \infty)} \in \mathcal{C}_{\theta,\xi}$, and therefore, by statement 1, $-\varepsilon 1_B 1_{[t \wedge \xi, \infty)} \in \mathcal{C}_{\tau,\xi}$. If $\phi_{\tau,\xi}$ is ξ-relevant, then $P[B] = 0$, which shows that $\phi_{\theta,\xi}$ is ξ-relevant. □

Corollary 4.8 *Let $S \in \mathbb{N}$ and $T \in \mathbb{N} \cup \{\infty\}$ such that $S \leq T$. Let ϕ be a T-relevant monetary utility functional on $\mathcal{R}^\infty_{S,T}$. Then there exists at most one time-consistent monetary utility process $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ with $\phi_{S,T} = \phi$.*

Proof. Let $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ be a time-consistent monetary utility process with $\phi_{S,T} = \phi$ and $(\mathcal{C}_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ the corresponding acceptance set process. By Proposition 4.7.3, $\phi_{t,T}$ is T-relevant for all $t \in [S,T] \cap \mathbb{N}$. Therefore it follows from 1. and 2. of Proposition 4.7 that for all $t \in [S,T] \cap \mathbb{N}$, a process $X \in \mathcal{R}^\infty_{t,T}$ is in $\mathcal{C}_{t,T}$ if and only if $1_A X \in \mathcal{C}_{S,T}$ for all $A \in \mathcal{F}_t$. This shows that $\mathcal{C}_{t,T}$ is uniquely determined by the acceptance set $\mathcal{C}_{S,T}$ of ϕ. Hence, $\phi_{t,T}$ is uniquely determined by ϕ. □

4.2 Consistent extension of the time horizon.

Proposition 4.9 *Let $S \in \mathbb{N}$ and $T \in \mathbb{N} \cup \{\infty\}$ such that $S \le T$. Let $(\phi_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ be two time-consistent monetary utility functional processes with corresponding acceptance set processes $(\mathcal{C}_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\mathcal{C}_{t,T})_{t \in [S,T] \cap \mathbb{N}}$, respectively. For $t \in [0,S)$, define*

$$\phi_{t,T}(X) := \phi_{t,S}\left(X 1_{[t,S)} + \phi_{S,T}(X) 1_{[S,\infty)}\right) , \quad X \in \mathcal{R}^{\infty}_{t,T} , \tag{4.16}$$

and

$$\mathcal{C}_{t,T} := \mathcal{C}_{t,S} + \mathcal{C}_{S,T} . \tag{4.17}$$

Then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is a time-consistent monetary utility functional process with corresponding acceptance set process $(\mathcal{C}_{t,T})_{t \in [0,T] \cap \mathbb{N}}$. If $(\phi_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ are concave monetary utility functional processes, then so is $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$. If $(\phi_{t,S})_{t \in [0,S]} \cap \mathbb{N}$ and $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ are coherent, then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is coherent too.

Proof. It can easily be checked that for all $t \in [0,S)$, the mapping $\phi_{t,T}$ defined in (4.16) is a monetary utility functional on $\mathcal{R}^{\infty}_{t,T}$ with acceptance set $\mathcal{C}_{t,T}$ given by (4.17). Also, it is obvious that $\phi_{t,T}$ is a concave monetary utility functional on $\mathcal{R}^{\infty}_{t,T}$ if $\phi_{t,S}$ and $\phi_{S,T}$ are concave monetary utility functionals, and $\phi_{t,T}$ is coherent if $\phi_{t,S}$ and $\phi_{S,T}$ are coherent. To prove that $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time-consistent, it is by Proposition 4.6 enough to show that

$$\mathcal{C}_{t,T} = \mathcal{C}_{t,\theta} + \mathcal{C}_{\theta,T} ,$$

for all $t \in [0,S) \cap \mathbb{N}$ and every finite $(\mathcal{F}_t)$-stopping time θ such that $t \le \theta \le T$.

We first show $\mathcal{C}_{t,\theta} + \mathcal{C}_{\theta,T} \subset \mathcal{C}_{t,T}$. Let $Y \in \mathcal{C}_{t,\theta}$ and $Z \in \mathcal{C}_{\theta,T}$. By definition of $\mathcal{C}_{t,T}$, Y can be decomposed into $Y = Y' + Y''$, where $Y' \in \mathcal{C}_{t,S}$ and $Y'' \in \mathcal{C}_{S,T}$. It is easy to see that Y'' can be chosen such that $Y'' = 0$ on $\{\theta \le S\}$. Then

$$Y' \in \mathcal{C}_{t,\theta \wedge S} \quad \text{and} \quad Y'' \in \mathcal{C}_{S,\theta \vee S} .$$

Similarly, $Z = Z' + Z''$, where $Z' \in \mathcal{C}_{t,S}$ and $Z'' \in \mathcal{C}_{S,T}$ can be chosen such that

$$Z' \in \mathcal{C}_{\theta \wedge S,S} \quad \text{and} \quad Z'' \in \mathcal{C}_{\theta \vee S,T} .$$

Hence,

$$Y' + Z' \in \mathcal{C}_{t,S} \quad \text{and} \quad Y'' + Z'' \in \mathcal{C}_{S,T} ,$$

and therefore,

$$Y + Z = Y' + Z' + Y'' + Z'' \in \mathcal{C}_{t,T} .$$

To show $\mathcal{C}_{t,T} \subset \mathcal{C}_{t,\theta} + \mathcal{C}_{\theta,T}$, we let $X \in \mathcal{C}_{t,T}$. By definition of $\mathcal{C}_{t,T}$, $X = X' + X''$, where $X' \in \mathcal{C}_{t,S}$ and $X'' \in \mathcal{C}_{S,T}$. Since $(\phi_{t,S})_{t \in [0,S] \cap \mathbb{N}}$ and $(\phi_{t,T})_{t \in [S,T] \cap \mathbb{N}}$ are time-consistent, we get from Proposition 4.6 that

$$X' = Y' + Z' \quad \text{and} \quad X'' = Y'' + Z'' ,$$

where $Y' \in \mathcal{C}_{t,\theta\wedge S}$, $Z' \in \mathcal{C}_{\theta\wedge S,S}$, $Y'' \in \mathcal{C}_{S,\theta\vee S}$ and $Z'' \in \mathcal{C}_{\theta\vee S,T}$. Note that $1_{\{\theta>S\}}Z' \in \mathcal{C}_{S,S}$. Hence,

$$1_{\{\theta>S\}}Z' = f 1_{\{\theta>S\}} 1_{[S,\infty)} \quad \text{for some } f \in L^\infty_+(\mathcal{F}_S)\,.$$

It follows that

$$1_{\{\theta>S\}}Y'' + 1_{\{\theta>S\}}Z' \in \mathcal{C}_{S,\theta\vee S} \cap \mathcal{R}^\infty_{t,\theta}\,,$$

and therefore,

$$Y' + 1_{\{\theta>S\}}Y'' + 1_{\{\theta>S\}}Z' \in \mathcal{C}_{t,\theta}\,.$$

On the other hand,

$$1_{\{\theta\le S\}}Y'' \in \mathcal{C}_{S,S}\,,$$

and therefore,

$$1_{\{\theta\le S\}}Y'' = g 1_{\{\theta\le S\}} 1_{[S,\infty)} \quad \text{for some } g \in L^\infty_+(\mathcal{F}_S)\,.$$

Hence,

$$1_{\{\theta\le S\}}Y'' + 1_{\{\theta\le S\}}Z' \in \mathcal{C}_{\theta\wedge S,S} \cap \mathcal{R}^\infty_{\theta,T}\,,$$

and

$$1_{\{\theta\le S\}}Y'' + 1_{\{\theta\le S\}}Z' + Z'' \in \mathcal{C}_{\theta,T}\,.$$

□

Remark 4.10 Let $T \in \mathbb{N}$. Note that for all $t = 0,\dots,T$, there exists only one monetary utility functional $\phi_{t,t}$ on $\mathcal{R}^\infty_{t,t}$. It is given by

$$\phi_{t,t}(m 1_{[t,\infty)}) = m\,, \quad \text{for } m \in L^\infty(\mathcal{F}_t)\,,$$

and its acceptance set is

$$C_{t,t} = \left\{ m 1_{[t,\infty)} \mid m \in L^\infty_+(\Omega,\mathcal{F}_t,P) \right\}\,.$$

Now, for every $t = 0,\dots T-1$, let $\phi_{t,t+1}$ be an arbitrary monetary utility functional on $\mathcal{R}^\infty_{t,t+1}$ with acceptance set $\mathcal{C}_{t,t+1}$. It can easily be checked that for all $t = 0,\dots T-1$, the monetary utility functional process $(\phi_{s,t+1})_{s=t}^{t+1}$ is time-consistent. Therefore, it follows from Proposition 4.9 that an acceptance set process $(\mathcal{C}_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ corresponding to a time-consistent monetary utility functional process $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ can be obtained by defining

$$\mathcal{C}_{t,T} := \mathcal{C}_{t,t+1} + \mathcal{C}_{t+1,t+2} + \cdots + \mathcal{C}_{T-1,T}\,, \quad \text{for all } t = 0,1,\dots,T-1\,.$$

4.3 Concatenation of elements in $\mathcal{A}^1_+$

Definition 4.11 *Let $a, b \in \mathcal{A}^1_+$, θ a finite $(\mathcal{F}_t)$-stopping time and $A \in \mathcal{F}_\theta$. Then the concatenation $a \oplus^\theta_A b$ is defined by*

$$(a \oplus^\theta_A b)_t := \begin{cases} a_t & \text{on } \{t<\theta\} \cup A^c \cup \left\{\langle 1,b\rangle_{\theta,\infty} = 0\right\} \\ a_{\theta-1} + \frac{\langle 1,a\rangle_{\theta,\infty}}{\langle 1,b\rangle_{\theta,\infty}}(b_t - b_{\theta-1}) & \text{on } \{t\ge\theta\} \cap A \cap \left\{\langle 1,b\rangle_{\theta,\infty} > 0\right\}\,, \end{cases}$$

where we set $a_{-1} = b_{-1} = 0$.

We say a subset $\mathcal{Q}$ of $\mathcal{A}^1_+$ is stable under concatenation if $a \oplus^\theta_A b \in \mathcal{Q}$ for all $a, b \in \mathcal{Q}$, every finite $(\mathcal{F}_t)$-stopping time θ and all $A \in \mathcal{F}_\theta$.

Remarks 4.12
1. Let $\mathcal{Q}$ be a subset of $\mathcal{A}^1_+$ such that

$$a \oplus^s_A b \in \mathcal{Q} \quad \text{for all } a, b \in \mathcal{Q}\,,\ s \in \mathbb{N} \text{ and } A \in \mathcal{F}_s\,. \tag{4.18}$$

Then

$$a \oplus^\theta_A b \in \mathcal{Q} \quad \text{for all } a, b \in \mathcal{Q}, \text{ each bounded } (\mathcal{F}_t)\text{-stopping time } \theta, \text{ and } A \in \mathcal{F}_\theta\,,$$

and

$$a \oplus^\theta_A b \quad \text{is in the } ||.||_{\mathcal{A}^1}\text{-closure of } \mathcal{Q}$$

for all $a, b \in \mathcal{Q}$, each finite $(\mathcal{F}_t)$-stopping time θ and $A \in \mathcal{F}_\theta$.

Indeed, if $\mathcal{Q}$ has the property (4.18), set for each $(\mathcal{F}_t)$-stopping time θ and $A \in \mathcal{F}_\theta$, $A_n := A \cap \{\theta = n\}$, $n \in \mathbb{N}$. Then all the following processes are in $\mathcal{Q}$:

$$a^0 := a \oplus^0_{A_0} b\,, \quad a^n := a^{n-1} \oplus^n_{A_n} b\,, \quad n \geq 1\,.$$

If θ is bounded, then $a^n = a \oplus^\theta_A b$ for all n such that $n \geq \theta$. If θ is finite, then $a^n \to a \oplus^\theta_A b$ in $||.||^1_{\mathcal{A}}$, as $n \to \infty$.
2. Let θ be a finite $(\mathcal{F}_t)$-stopping time and $A \in \mathcal{F}_\theta$. It can easily be checked that the concatenation $\oplus^\theta_A$ has the following properties:
(i) Let $a^1, a^2, b \in \mathcal{A}^1_+$ and $\lambda \in (0, 1)$. Then

$$(\lambda a^1 + (1-\lambda) a^2) \oplus^\theta_A b = \lambda(a^1 \oplus^\theta_A b) + (1-\lambda)(a^2 \oplus^\theta_A b)\,.$$

(ii) Let $a, b \in \mathcal{A}^1_+$ and $(a^\mu)_{\mu \in M}$ a net in $\mathcal{A}^1_+$ with

$$a^\mu \to a \quad \text{in } \sigma(\mathcal{A}^1, \mathcal{R}^\infty)\,.$$

Then

$$a^\mu \oplus^\theta_A b \to a \oplus^\theta_A b \quad \text{in } \sigma(\mathcal{A}^1, \mathcal{R}^\infty)\,.$$

(iii) Let $a, b \in \mathcal{A}^1_+$, $B = \{\langle 1, b\rangle_\theta > 0\}$ and $(b^\mu)_{\mu \in M}$ a net in $\mathcal{A}^1_+$ with

$$b^\mu \to b \quad \text{in } \sigma(\mathcal{A}^1, \mathcal{R}^\infty)\,.$$

Then

$$a \oplus^\theta_{A \cap B} b^\mu \to a \oplus^\theta_A b \quad \text{in } \sigma(\mathcal{A}^1, \mathcal{R}^\infty)\,.$$

Proposition 4.13 *Let $\mathcal{Q}$ be a non-empty subset of $\mathcal{A}^1_+$ and denote by $\langle \mathcal{Q} \rangle$ the smallest $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, convex subset of $\mathcal{A}^1_+$ that contains $\mathcal{Q}$. Assume that*

$$a \oplus^s_A b \in \langle \mathcal{Q} \rangle\,,$$

for all $a, b \in \mathcal{Q}$, $s \in \mathbb{N}$ and $A \in \mathcal{F}_s$. Then $\langle \mathcal{Q} \rangle$ is stable under concatenation.

Proof. It follows from the properties (i) and (ii) of Remark 4.12.2 that

$$a \oplus_A^s b \in \langle \mathcal{Q} \rangle$$

for all $a \in \langle \mathcal{Q} \rangle$, $b \in \mathcal{Q}$, $s \in \mathbb{N}$ and $A \in \mathcal{F}_s$. Then, it can be shown as in Remark 4.12.1 that

$$a \oplus_A^\theta b \in \langle \mathcal{Q} \rangle$$

for all $a \in \langle \mathcal{Q} \rangle$, $b \in \mathcal{Q}$, every finite $(\mathcal{F}_t)$-stopping time θ and $A \in \mathcal{F}_\theta$. Now, let $a \in \langle \mathcal{Q} \rangle, b^1, b^2 \in \mathcal{Q}$ and $\lambda \in (0,1)$. Set

$$B_1 := \left\{ \langle 1, b^1 \rangle_\theta > 0 \right\} \quad \text{and} \quad B_2 := \left\{ \langle 1, b^2 \rangle_\theta > 0 \right\} .$$

Let $C_1, \dots, C_N$ be finitely many disjoint sets in $\mathcal{F}_\theta$ such that $\bigcup_{n=1}^N C_n = B_1 \cap B_2$ and $\lambda_1, \dots, \lambda_N$ numbers in $[0,1]$. Then the following processes are all in $\langle \mathcal{Q} \rangle$:

$$c^1 = \lambda_1 (a \oplus_{C_1}^\theta b^1) + (1 - \lambda_1)(a \oplus_{C_1}^\theta b^2) ,$$

$$c^2 = \lambda_2 (c^1 \oplus_{C_2}^\theta b^1) + (1 - \lambda_2)(c^1 \oplus_{C_2}^\theta b^2) , \dots$$

$$\dots , c^N = \lambda_N (c^{N-1} \oplus_{C_N}^\theta b^1) + (1 - \lambda_N)(c^{N-1} \oplus_{C_N}^\theta b^2) .$$

Note that

$$c_t^N = \begin{cases} a_t & \text{on } \{t < \theta\} \cup B_1^c \cup B_2^c \\ a_{\theta-1} + \lambda_n \frac{\langle 1,a \rangle_\theta}{\langle 1,b^1 \rangle_\theta} (b_t^1 - b_{\theta-1}^1) + (1-\lambda_n) \frac{\langle 1,a \rangle_\theta}{\langle 1,b^2 \rangle_\theta} (b_t^2 - b_{\theta-1}^2) & \text{on } \{t \geq \theta\} \cap C_n \end{cases} .$$

Hence, since $\langle \mathcal{Q} \rangle$ is $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, it also contains the process c given by

$$c_t = \begin{cases} a_t & \text{on } \{t < \theta\} \cup B_1^c \cup B_2^c \\ a_{\theta-1} + \frac{\langle 1,a \rangle_\theta}{\langle 1, \lambda b^1 + (1-\lambda) b^2 \rangle_\theta} \left[\lambda (b_t^1 - b_{\theta-1}^1) + (1-\lambda)(b_t^2 - b_\theta^2) \right] & \text{on } \{t \geq \theta\} \cap B^1 \cap B^2 \end{cases} .$$

Next, notice that the processes

$$d^1 = c \oplus_{B^1 \setminus B^2}^\theta b^1 \quad \text{and} \quad d^2 = d^1 \oplus_{B^2 \setminus B^1}^\theta b^2$$

are in $\langle \mathcal{Q} \rangle$, and

$$d_2 = a \oplus_A^\theta (\lambda b^1 + (1-\lambda) b^2) .$$

Together with property (iii) of Remark 4.12.2, this implies that

$$a \oplus_A^\theta b \in \langle \mathcal{Q} \rangle$$

for all $a, b \in \langle \mathcal{Q} \rangle$, every finite $(\mathcal{F}_t)$-stopping time θ and $A \in \mathcal{F}_\theta$. □

4.4 Time-consistent coherent utility functional processes

Theorem 4.14 *Let $T \in \mathbb{N} \cup \{\infty\}$ and $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ a relevant time-consistent coherent utility process such that $\phi_{0,T}$ can be represented as*

$$\phi_{0,T}(X) = \inf_{a\in\mathcal{Q}} \langle X,a\rangle_{0,T} \ , \quad X \in \mathcal{R}^\infty_{0,T} \ ,$$

for some $\sigma(\mathcal{R}^\infty,\mathcal{A}^1)$-closed, convex subset $\mathcal{Q}$ of $\mathcal{D}_{0,T}$. Then,
1. *For every finite $(\mathcal{F}_t)$-stopping time $\tau \le T$,*

$$\phi_{\tau,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{Q}} \frac{\langle X,a\rangle_{\tau,T}}{\langle 1,a\rangle_{\tau,T}} = \operatorname{ess\,inf}_{a\in\mathcal{Q}^e} \frac{\langle X,a\rangle_{\tau,T}}{\langle 1,a\rangle_{\tau,T}} \ , \quad X \in \mathcal{R}^\infty_{\tau,T} \ ,$$

where

$$\frac{\langle X,a\rangle_{\tau,T}}{\langle 1,a\rangle_{\tau,T}} \quad \text{is understood to be } \infty \text{ on } \left\{\langle 1,a\rangle_{\tau,T} = 0\right\} \ ,$$

and

$$\mathcal{Q}^e := \mathcal{Q} \cap \mathcal{D}^e_{0,T} \ .$$

2. *$\mathcal{Q}$ and $\mathcal{Q}^e$ are stable under concatenation.*

Proof.
1. Let $(\mathcal{C}_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ be the acceptance set process corresponding to $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$. Parts 1 and 2 of Proposition 4.7 imply that for every finite $(\mathcal{F}_t)$-stopping time $\tau \le T$ and $X \in \mathcal{R}^\infty_{\tau,T}$,

$$X \in \mathcal{C}_{\tau,T} \ \Leftrightarrow \ 1_A X \in \mathcal{C}_{0,T} \quad \text{for all } A \in \mathcal{F}_\tau \ .$$

It follows from Corollary 3.16 that

$$\mathcal{Q} = \left\{ a \in \mathcal{D}_{0,T} \mid \phi^{\#}_{0,T}(a) = 0 \right\} \ ,$$

and from Corollary 3.24 that

$$\phi_{0,T}(X) = \inf_{a\in\mathcal{Q}^e} \langle X,a\rangle_{0,T} \ ,$$

where $\mathcal{Q}^e = \mathcal{Q} \cap \mathcal{D}^e_{0,T}$. Hence, for all $X \in \mathcal{R}^\infty_{\tau,T}$,

$$\begin{aligned} X \in \mathcal{C}_{\tau,T} \quad &\Leftrightarrow \quad \langle 1_A X, a\rangle_{0,T} \ge 0 \quad \text{for all } A \in \mathcal{F}_\tau \text{ and } a \in \mathcal{Q}^e \\ &\Leftrightarrow \quad \langle X,a\rangle_{\tau,T} \ge 0 \quad \text{for all } a \in \mathcal{Q}^e \ . \end{aligned}$$

This shows that $\phi_{\tau,T}$ and the coherent utility functional

$$\operatorname{ess\,inf}_{a\in\mathcal{Q}^e} \frac{\langle X,a\rangle_{\tau,T}}{\langle 1,a\rangle_{\tau,T}} \ , \quad X \in \mathcal{R}^\infty_{\tau,T}$$

have the same acceptance set. Hence, they must be equal. It is clear that

$$\operatorname{ess\,inf}_{a \in \mathcal{Q}} \frac{\langle X, a \rangle_{\tau,T}}{\langle 1, a \rangle_{\tau,T}} \leq \operatorname{ess\,inf}_{a \in \mathcal{Q}^e} \frac{\langle X, a \rangle_{\tau,T}}{\langle 1, a \rangle_{\tau,T}}, \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,T}.$$

On the other hand, since $X - \phi_{\tau,T}(X) 1_{[\tau,\infty)} \in \mathcal{C}_{\tau,T}$, it follows that

$$\left\langle 1_A \left(X - \phi_{\tau,T}(X) 1_{[\tau,\infty)} \right), a \right\rangle_{0,T} \geq 0, \quad \text{for all } A \in \mathcal{F}_\tau \text{ and } a \in \mathcal{Q},$$

and therefore,

$$\frac{\left\langle \left(X - \phi_{\tau,T}(X) 1_{[\tau,\infty)} \right), a \right\rangle_{\tau,T}}{\langle 1, a \rangle_{\tau,T}} \geq 0, \quad \text{for all } a \in \mathcal{Q},$$

which shows that

$$\operatorname{ess\,inf}_{a \in \mathcal{Q}} \frac{\langle X, a \rangle_{\tau,T}}{\langle 1, a \rangle_{\tau,T}} \geq \phi_{\tau,T}(X), \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,T}.$$

2. To show that $\mathcal{Q}$ is stable under concatenation, we assume by way of contradiction that there exist $a, b \in \mathcal{Q}$, a finite $(\mathcal{F}_t)$-stopping time $\theta \leq T$ and $A \in \mathcal{F}_\theta$ such that $c := a \oplus^\theta_A b \notin \mathcal{Q}$. Since $\mathcal{Q}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed and convex, it follows from the separating hyperplane theorem that there exists an $X \in \mathcal{R}^\infty_{0,T}$ such that

$$\langle X, c \rangle_{0,T} < \inf_{d \in \mathcal{Q}} \langle X, d \rangle_{0,T} = \phi_{0,T}(X).$$

Note that

$$\begin{aligned}
& \mathrm{E}\left[\sum_{j \in [\theta,T] \cap \mathbb{N}} X_j \Delta c_j \right] \\
= \; & \mathrm{E}\left[1_{\{A^c \cup \{\langle 1,b \rangle_{\theta,T} = 0\}\}} \sum_{j \in [\theta,T] \cap \mathbb{N}} X_j \Delta a_j + 1_{\{A \cap \{\langle 1,b \rangle_{\theta,T} > 0\}\}} \frac{\langle 1, a \rangle_{\theta,T}}{\langle 1, b \rangle_{\theta,T}} \sum_{j \in [\theta,T] \cap \mathbb{N}} X_j \Delta b_j \right] \\
= \; & E\Bigg[\Bigg(1_{\{A^c \cup \{\langle 1,b \rangle_{\theta,T} = 0\}\}} \frac{\langle X, a \rangle_{\theta,T}}{\langle 1, a \rangle_{\theta,T}} 1_{\{\langle 1,a \rangle_{\theta,T} > 0\}} \\
& \qquad + 1_{\{A \cap \{\langle 1,b \rangle_{\theta,T} > 0\}\}} \frac{\langle X, b \rangle_{\theta,T}}{\langle 1, b \rangle_{\theta,T}} \Bigg) \sum_{j \in [\theta,T] \cap \mathbb{N}} \Delta a_j \Bigg] \\
\geq \; & \mathrm{E}\left[\phi_{\theta,T}(X) \sum_{j \in [\theta,T] \cap \mathbb{N}} \Delta a_j \right]
\end{aligned}$$

Hence,

$$\begin{aligned}
\phi_{0,T}(X) &> \langle X, c \rangle_{0,T} = \mathrm{E}\left[\sum_{j \in [0,T] \cap \mathbb{N}} X_j \Delta c_j \right] \geq \mathrm{E}\left[\sum_{j \in [0,\theta)} X_j \Delta a_j + \sum_{j \in [\theta,T] \cap \mathbb{N}} \phi_{\theta,T}(X) \Delta a_j \right] \\
&\geq \phi_{0,T}\left(X 1_{[0,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)} \right),
\end{aligned}$$

and therefore, $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is not time-consistent. But this contradicts the assumptions, and therefore, $\mathcal{Q}$ has to be stable under concatenation. It follows immediately that $\mathcal{Q}^e$ is stable under concatenation. □

Remark 4.15 Let $T \in \mathbb{N}\cup\{\infty\}$ and $\mathcal{Q}^e$ a non-empty subset of $\mathcal{D}^e_{0,T}$. Define for all $t \in [0,T]\cap\mathbb{N}$,

$$\phi_{t,T}(X) := \operatorname{ess\,inf}_{a\in\mathcal{Q}^e} \frac{\langle X,a\rangle_{t,T}}{\langle 1,a\rangle_{t,T}}\,, \quad X \in \mathcal{R}^\infty_{t,T}\,.$$

Then, obviously, $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is a relevant coherent utility functional process, and it is easy to see that for every finite $(\mathcal{F}_t)$-stopping time $\tau \le T$ and all $X \in \mathcal{R}^\infty_{\tau,T}$,

$$\phi_{\tau,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{Q}^e} \frac{\langle X,a\rangle_{\tau,T}}{\langle 1,a\rangle_{\tau,T}}\,.$$

Theorem 4.16 *Let $T \in \mathbb{N}\cup\{\infty\}$ and $\mathcal{Q}^e$ a non-empty subset of $\mathcal{D}^e_{0,T}$ that is stable under concatenation. Define for all $t \in [0,T]\cap\mathbb{N}$,*

$$\phi_{t,T}(X) := \operatorname{ess\,inf}_{a\in\mathcal{Q}^e} \frac{\langle X,a\rangle_{t,T}}{\langle 1,a\rangle_{t,T}}\,, \quad X \in \mathcal{R}^\infty_{t,T}\,.$$

Then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is a relevant time-consistent coherent utility functional process.

Proof. By Remark 4.15, $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is a relevant coherent utility functional process such that for every finite $(\mathcal{F}_t)$-stopping time $\tau \le T$ and all $X \in \mathcal{R}^\infty_{\tau,T}$,

$$\phi_{\tau,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{Q}^e} \frac{\langle X,a\rangle_{\tau,T}}{\langle 1,a\rangle_{\tau,T}}\,.$$

To show time-consistency, we denote by $(\mathcal{C}_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ the acceptance set process corresponding to $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ and prove that

$$\mathcal{C}_{t,T} = \mathcal{C}_{t,\theta} + \mathcal{C}_{\theta,T}\,,$$

for all $t \in [0,T]\cap\mathbb{N}$ and every finite $(\mathcal{F}_t)$-stopping time θ such that $t \le \theta \le T$. If $Y \in \mathcal{C}_{t,\theta} \subset \mathcal{C}_{t,T}$ and $Z \in \mathcal{C}_{\theta,T}$, then

$$\langle Z,a\rangle_{\theta,T} \ge 0 \quad \text{for all } a \in \mathcal{Q}^e\,,$$

which implies

$$\langle Z,a\rangle_{t,T} \ge 0 \quad \text{for all } a \in \mathcal{Q}^e\,.$$

Therefore, $Z \in \mathcal{C}_{t,T}$ and $Y + Z \in \mathcal{C}_{t,T}$. This shows that $\mathcal{C}_{t,\theta} + \mathcal{C}_{\theta,T} \subset \mathcal{C}_{t,T}$. To prove $\mathcal{C}_{t,T} \subset \mathcal{C}_{t,\theta} + \mathcal{C}_{\theta,T}$, we choose a process $X \in \mathcal{C}_{t,T}$. Obviously, $Z := (X - \phi_{\theta,T}(X))1_{[\theta,\infty)} \in \mathcal{C}_{\theta,T}$, and it remains to show that

$$Y := X - Z = X1_{[t,\theta)} + \phi_{\theta,T}(X)1_{[\theta,\infty)} \in \mathcal{C}_{\tau,T}\,. \tag{4.19}$$

Since $\mathcal{Q}^e$ is stable under concatenation, the set

$$\left\{ \frac{\langle X, a \rangle_{\theta,T}}{\langle 1, a \rangle_{\theta,T}} \mid a \in \mathcal{Q}^e \right\}$$

is directed downwards. Therefore, there exists a sequence $(b^n)_{n \in \mathbb{N}}$ in $\mathcal{Q}^e$ such that

$$\frac{\left\langle X, b^0 \right\rangle_{\theta,T}}{\langle 1, b^0 \rangle_{\theta,T}} \leq ||X||_{\theta,\infty} \leq ||X||_\infty$$

and

$$\frac{\langle X, b^n \rangle_{\theta,T}}{\langle 1, b^n \rangle_{\theta,T}} \searrow \phi_{\theta,T}(X) \quad \text{almost surely} \, .$$

Moreover, $a \oplus_\Omega^\theta b^n \in \mathcal{Q}^e$ for all $a \in \mathcal{Q}^e$ and $n \in \mathbb{N}$. Hence,

$$0 \leq \left\langle X, a \oplus_\Omega^\theta b^n \right\rangle_{t,T} \searrow \langle Y, a \rangle_{t,T} \quad \text{almost surely} \, ,$$

and therefore,

$$\langle Y, a \rangle_{t,T} \geq 0 \quad \text{for all } a \in \mathcal{Q}^e \, ,$$

which completes the proof. □

4.5 Time-consistent concave monetary utility functional processes

Definition 4.17 *Let $f \in L^0(\mathcal{F})$ and τ a finite $(\mathcal{F}_t)$-stopping time. If there exists a $g \in L^1(\mathcal{F})$ such that $f \geq g$, we define*

$$\mathrm{E}\left[f \mid \mathcal{F}_\tau\right] := \lim_{n \to \infty} \mathrm{E}\left[f \wedge n \mid \mathcal{F}_\tau\right] \, .$$

If there exists a $g \in L^1(\mathcal{F})$ such that $f \leq g$, we define

$$\mathrm{E}\left[f \mid \mathcal{F}_\tau\right] := \lim_{n \to -\infty} \mathrm{E}\left[f \vee n \mid \mathcal{F}_\tau\right] \, .$$

If X is an adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}}, P)$ taking values in the interval $[m, \infty]$ for some $m \in \mathbb{R}$, we define for all $a \in \mathcal{A}^1_+$,

$$\langle X, a \rangle_{\tau,\theta} := \lim_{n \to \infty} \langle X \wedge n, a \rangle_{\tau,\theta} \, .$$

If X takes values in $[-\infty, m]$, we define for all $a \in \mathcal{A}^1_+$,

$$\langle X, a \rangle_{\tau,\theta} := \lim_{n \to -\infty} \langle X \vee n, a \rangle_{\tau,\theta} \, .$$

Remark 4.18 Let $T \in \mathbb{N} \cup \{\infty\}$ and $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ a concave monetary utility functional process such that for each $t \in [0,T] \cap \mathbb{N}$, $\phi_{t,T}$ is given by

$$\phi_{t,T}(X) = \operatorname{ess\,inf}_{a \in \mathcal{D}_{t,T}} \left\{ \langle X, a \rangle_{t,T} - \gamma_{t,T}(a) \right\} , \quad X \in \mathcal{R}^\infty_{t,T} ,$$

for a a special penalty function $\gamma_{t,T}$ on $\mathcal{D}_{t,T}$. Then it can easily be checked that for all finite $(\mathcal{F}_t)$-stopping times $\tau \leq T$,

$$\phi_{\tau,T}(X) = \operatorname{ess\,inf}_{a \in \mathcal{D}_{\tau,T}} \left\{ \langle X, a \rangle_{\tau,T} - \gamma_{\tau,T}(a) \right\} , \quad X \in \mathcal{R}^\infty_{\tau,T} ,$$

where $\gamma_{\tau,T}$ is the special penalty function on $\mathcal{D}_{\tau,T}$ given by

$$\gamma_{\tau,T}(a) := \sum_{t \in [0,T] \cap \mathbb{N}} 1_{\{\tau = t\}} \gamma_t (1_{\{\tau = t\}} a + 1_{\{\tau \neq t\}} 1_{[t,\infty)}) , \quad a \in \mathcal{D}_{\tau,T} . \tag{4.20}$$

Theorem 4.19 *Let $T \in \mathbb{N} \cup \{\infty\}$ and $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ a time-consistent concave monetary utility process such that for all $t \in [0,T] \cap \mathbb{N}$,*

$$\phi_{t,T}(X) = \operatorname{ess\,inf}_{a \in \mathcal{D}_{t,T}} \left\{ \langle X, a \rangle_{t,T} - \phi^{\#}_{t,T}(a) \right\} , \quad X \in \mathcal{R}^\infty_{t,T} .$$

Then

$$\phi^{\#}_{\tau,T}(a) = \operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^{\#}_{\tau,T} \left(a \oplus^{\theta}_{\Omega} b \right) + \mathrm{E} \left[\phi^{\#}_{\theta,T}(a) \mid \mathcal{F}_\tau \right] , \tag{4.21}$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ, θ such that $0 \leq \tau \leq \theta \leq T$ and all $a \in \mathcal{D}_{\tau,T}$.

Proof. Let τ and θ be two finite $(\mathcal{F}_t)$-stopping times such that $0 \leq \tau \leq \theta \leq T$, and $(\mathcal{C}_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ the acceptance set process corresponding to $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$. It follows from Remark 4.4.2 and Proposition 4.6 that for all $a \in \mathcal{D}_{\tau,T}$,

$$\begin{aligned}
\phi^{\#}_{\tau,T}(a) &= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,T}} \langle X, a \rangle_{\tau,T} \\
&= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \operatorname{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \langle X, a \rangle_{\tau,T} \\
&= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E} \left[\operatorname{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \langle X, a \rangle_{\theta,T} \mid \mathcal{F}_\tau \right] \\
&= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E} \left[\phi^{\#}_{\theta,T}(a) \mid \mathcal{F}_\tau \right] .
\end{aligned} \tag{4.22}$$

and for all $a \in \mathcal{D}_{\tau,T}$ and $b \in \mathcal{D}_{\theta,T}$,

$$\begin{aligned}
\phi^{\#}_{\tau,T}(a \oplus^{\theta}_{\Omega} b) &= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,T}} \left\langle X, a \oplus^{\theta}_{\Omega} b \right\rangle_{\tau,T} \\
&= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \left\langle X, a \oplus^{\theta}_{\Omega} b \right\rangle_{\tau,T} + \operatorname{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \left\langle X, a \oplus^{\theta}_{\Omega} b \right\rangle_{\tau,T} \\
&= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E} \left[\operatorname{ess\,inf}_{X \in \mathcal{C}_{\theta,T}} \langle X, b \rangle_{\theta,T} \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau \right] \\
&= \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E} \left[\phi^{\#}_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_\tau \right] .
\end{aligned}$$

By Remark 4.18,

$$\phi_{\theta,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{D}_{\theta,T}} \left\{ \langle X, a\rangle_{\theta,T} - \phi^{\#}_{\theta,T}(a) \right\} , \quad X \in \mathcal{R}^{\infty}_{\theta,T} ,$$

which implies

$$\operatorname{ess\,sup}_{b\in\mathcal{D}_{\theta,T}} \phi^{\#}_{\theta,T}(b) = 0 ,$$

and therefore,

$$\operatorname{ess\,sup}_{b\in\mathcal{D}_{\theta,T}} \mathrm{E}\left[\phi^{\#}_{\theta,T}(b) \langle 1, a\rangle_{\theta,T} \mid \mathcal{F}_\tau \right] = 0 ,$$

Hence,

$$\operatorname{ess\,sup}_{b\in\mathcal{D}_{\theta,T}} \phi^{\#}_{\tau,T}\left(a \oplus^{\theta}_{\Omega} b \right) = \operatorname{ess\,inf}_{X\in\mathcal{C}_{\tau,\theta}} \langle X, a\rangle_{\tau,T} ,$$

which together with (4.22), proves (4.21). □

Corollary 4.20 *Let $T \in \mathbb{N} \cup \{\infty\}$ and $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ a relevant time-consistent concave monetary utility process such that $\phi_{0,T}$ is continuous for bounded decreasing sequences. Then*

$$\phi_{\tau,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{D}_{\tau,T}} \left\{ \langle X, a\rangle_{\tau,T} - \phi^{\#}_{\tau,T}(a) \right\} = \operatorname{ess\,inf}_{a\in\mathcal{D}^e_{\tau,T}} \left\{ \langle X, a\rangle_{\tau,T} - \phi^{\#}_{\tau,T}(a) \right\} ,$$

for every finite $(\mathcal{F}_t)$-stopping time $\tau \le T$, and

$$\begin{aligned} \phi^{\#}_{\tau,T}(a) &= \operatorname{ess\,sup}_{b\in\mathcal{D}_{\theta,T}} \phi^{\#}_{\tau,T}\left(a \oplus^{\theta}_{\Omega} b \right) + \mathrm{E}\left[\phi^{\#}_{\theta,T}(a) \mid \mathcal{F}_\tau \right] \\ &= \operatorname{ess\,sup}_{b\in\mathcal{D}^e_{\theta,T}} \phi^{\#}_{\tau,T}\left(a \oplus^{\theta}_{\Omega} b \right) + \mathrm{E}\left[\phi^{\#}_{\theta,T}(a) \mid \mathcal{F}_\tau \right] , \end{aligned}$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ, θ such that $0 \le \tau \le \theta \le T$ and all $a \in \mathcal{D}_{\tau,\theta}$.

Proof. By Theorem 3.15, $\mathcal{C}_{0,T}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed. Let $\tau \le T$ be a finite $(\mathcal{F}_t)$-stopping time and $(X^\mu)_{\mu\in M}$ a net in $\mathcal{C}_{\tau,T}$ such that $X^\mu \to X$ in $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$ for some $X \in \mathcal{R}^\infty_{\tau,T}$. Then, for each $A \in \mathcal{F}_\tau$, $1_A X^\mu \to 1_A X$ in $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$, and by Proposition 4.7.1, $(1_A X^\mu)_{\mu\in M}$ is a net in $\mathcal{C}_{0,T}$. Hence, $1_A X \in \mathcal{C}_{0,T}$, which by Proposition 4.7.2, implies that $X \in \mathcal{C}_{\tau,T}$. This shows that $\mathcal{C}_{\tau,T}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed. Hence, it follows from Theorem 3.15 that

$$\phi_{\tau,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{D}_{\tau,T}} \left\{ \langle X, a\rangle_{\tau,T} - \phi^{\#}_{\tau,T}(a) \right\} . \tag{4.23}$$

By Proposition 4.7.3, $\phi_{\tau,T}$ is T-relevant, which by Corollary 3.23, implies that

$$\phi_{\tau,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{D}^e_{\tau,T}} \left\{ \langle X, a\rangle_{\tau,T} - \phi^{\#}_{\tau,T}(a) \right\} . \tag{4.24}$$

By Theorem 4.19, it follows from (4.23) that

$$\phi^{\#}_{\tau,T}(a) = \operatorname{ess\,sup}_{b\in\mathcal{D}_{\theta,T}} \phi^{\#}_{\tau,T}\left(a \oplus^{\theta}_{\Omega} b \right) + \mathrm{E}\left[\phi^{\#}_{\theta,T}(a) \mid \mathcal{F}_\tau \right] ,$$

for every pair of finite $(\mathcal{F}_t)$-stopping times τ, θ such that $0 \le \tau \le \theta \le T$ and all $a \in \mathcal{D}_{\tau,\theta}$. In the proof of Theorem 4.19 we showed that for all $a \in \mathcal{D}_{\tau,T}$ and $b \in \mathcal{D}_{\theta,T}$,

$$\phi^{\#}_{\tau,T}(a \oplus^{\theta}_{\Omega} b) = \operatorname{ess\,inf}_{X \in \mathcal{C}_{\tau,\theta}} \langle X, a \rangle_{\tau,T} + \mathrm{E}\left[\phi^{\#}_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_{\tau}\right] ,$$

and it follows from (4.23) and (4.24) that

$$\operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^{\#}_{\theta,T}(b) = \operatorname{ess\,sup}_{b \in \mathcal{D}^e_{\theta,T}} \phi^{\#}_{\theta,T}(b) = 0 \,.$$

Hence,

$$\operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \mathrm{E}\left[\phi^{\#}_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_{\tau}\right] = \operatorname{ess\,sup}_{b \in \mathcal{D}^e_{\theta,T}} \mathrm{E}\left[\phi^{\#}_{\theta,T}(b) \langle 1, a \rangle_{\theta,T} \mid \mathcal{F}_{\tau}\right] = 0 \,,$$

and therefore,

$$\operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \phi^{\#}_{\tau,T}\left(a \oplus^{\theta}_{\Omega} b\right) = \operatorname{ess\,sup}_{b \in \mathcal{D}^e_{\theta,T}} \phi^{\#}_{\tau,T}\left(a \oplus^{\theta}_{\Omega} b\right) .$$

□

Definition 4.21 *Let $T \in \mathbb{N} \cup \infty$ and $\theta \le T$ a finite $(\mathcal{F}_t)$-stopping time.*

For every $a \in \mathcal{D}_{0,T}$, we define the process $\overset{\rightarrow\theta}{a} \in \mathcal{D}_{\theta,T}$ as follows:

$$\overset{\rightarrow\theta}{a} := \begin{cases} \frac{a}{\langle 1, a \rangle_{\theta,T}} 1_{[\theta,\infty)} & \text{on } \left\{\langle 1, a \rangle_{\theta,T} > 0\right\} \\ 1_{[\theta,\infty)} & \text{on } \left\{\langle 1, a \rangle_{\theta,T} = 0\right\} \end{cases}$$

If $\gamma_{\theta,T}$ is a special penalty function on $\mathcal{D}_{\theta,T}$, we extend it to $\mathcal{D}_{0,T}$ by setting

$$\gamma^{\text{ext}}_{\theta,T}(a) := \begin{cases} \langle 1, a \rangle_{\theta,T} \, \gamma_{\theta,T}\left(\overset{\rightarrow\theta}{a}\right) & \text{on } \left\{\langle 1, a \rangle_{\theta,T} > 0\right\} \\ 0 & \text{on } \left\{\langle 1, a \rangle_{\theta,T} = 0\right\} \end{cases} , \; a \in \mathcal{D}_{0,T} \,.$$

Theorem 4.22 *Let $T \in \mathbb{N} \cup \{\infty\}$ and $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ a concave monetary utility process such that for every $t \in [0,T] \cap \mathbb{N}$ and $X \in \mathcal{R}^{\infty}_{t,T}$,*

$$\phi_{t,T}(X) = \operatorname{ess\,inf}_{a \in \mathcal{D}_{t,T}} \left\{\langle X, a \rangle_{t,T} - \gamma_{t,T}(a)\right\} ,$$

for a special penalty function $\gamma_{t,T}$ on $\mathcal{D}_{t,T}$. Assume that at least one of the following two conditions is satisfied:

(1) *For each $t \in [0,T] \cap \mathbb{N}$ and every finite $(\mathcal{F}_t)$-stopping time θ such that $t \le \theta \le T$,*

$$\gamma_{t,T}(a) = \operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \gamma_{t,T}(a \oplus^{\theta}_{\Omega} b) + \mathrm{E}\left[\gamma^{\text{ext}}_{\theta,T}(a) \mid \mathcal{F}_t\right] , \quad \textit{for all } a \in \mathcal{D}_{t,T} \,.$$

(2) *$T \in \mathbb{N}$, and for each $t = 0, \dots, T-1$,*

$$\gamma_{t,T}(a) = \operatorname{ess\,sup}_{b \in \mathcal{D}_{t+1,T}} \gamma_{t,T}(a \oplus^{t+1}_{\Omega} b) + \mathrm{E}\left[\gamma^{\text{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right] , \quad \textit{for all } a \in \mathcal{D}_{t,T} \,.$$

Then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time-consistent.

Proof. Let $t \in [0,T] \cap \mathbb{N}$ and θ a finite $(\mathcal{F}_t)$-stopping time such that $0 \le t \le \theta \le T$. First, note that

$$\gamma_{t,T}(a) \ge \operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \gamma_{t,T}(a \oplus^\theta_\Omega b) + \mathrm{E}\left[\gamma^{\text{ext}}_{\theta,T}(a) \mid \mathcal{F}_t\right], \quad \text{for all } a \in \mathcal{D}_{t,T}, \tag{4.25}$$

implies that for all $a \in \mathcal{D}_{t,T}$ and $b \in \mathcal{D}_{\theta,T}$,

$$\gamma_{t,T}(a \oplus^\theta_\Omega b) \ge \gamma_{t,T}(a) + \mathrm{E}\left[\gamma^{\text{ext}}_{\theta,T}(a \oplus^\theta_\Omega b) \mid \mathcal{F}_t\right],$$

and therefore,

$$\begin{aligned}
&\left\langle X_{[t,\theta)} + \left[\langle X, b\rangle_{\theta,T} - \gamma_{\theta,T}(b)\right] 1_{[\theta,\infty)}, a \right\rangle_{t,T} - \gamma_{t,T}(a)\\
= \;& \left\langle X, a \oplus^\theta_\Omega b \right\rangle_{t,T} - \left\langle \gamma_{\theta,T}(b) 1_{[\theta,\infty)}, a\right\rangle_{t,T} - \gamma_{t,T}(a)\\
= \;& \left\langle X, a \oplus^\theta_\Omega b \right\rangle_{t,T} - \mathrm{E}\left[\gamma^{\text{ext}}_{\theta,T}(a \oplus^\theta_\Omega b) \mid \mathcal{F}_t\right] - \gamma_{t,T}(a)\\
\ge \;& \left\langle X, a \oplus^\theta_\Omega b \right\rangle_{t,T} - \gamma_{t,T}(a \oplus^\theta_\Omega b),
\end{aligned}$$

for all $X \in \mathcal{R}^\infty_{t,T}$. This shows that

$$\phi_{t,T}(X) \le \phi_{t,T}(X 1_{[t,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty]}), \quad \text{for all } X \in \mathcal{R}^\infty_{t,T}.$$

On the other hand, it follows from

$$\gamma_{t,T}(a) \le \operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \gamma_{t,T}(a \oplus^\theta_\Omega b) + \mathrm{E}\left[\gamma^{\text{ext}}_{\theta,T}(a) \mid \mathcal{F}_t\right], \quad \text{for all } a \in \mathcal{D}_{t,T}$$

that

$$\begin{aligned}
&\langle X, a\rangle_{t,T} - \gamma_{t,T}(a)\\
\ge \;& \langle X, a\rangle_{t,T} - \mathrm{E}\left[\gamma^{\text{ext}}_{\theta,T}(a) \mid \mathcal{F}_t\right] - \operatorname{ess\,sup}_{b \in \mathcal{D}_{\theta,T}} \gamma_{t,T}(a \oplus^\theta_\Omega b)\\
= \;& \operatorname{ess\,inf}_{b \in \mathcal{D}_{\theta,T}} \left\{ \left\langle X 1_{[t,\theta)} + \left[\left\langle X, \overset{\rightarrow\theta}{a} \right\rangle_{\theta,T} - \gamma_{\theta,T}\left(\overset{\rightarrow\theta}{a}\right)\right] 1_{[\theta,\infty)}, a \oplus^\theta_\Omega b \right\rangle_{t,T} - \gamma_{t,T}(a \oplus^\theta_\Omega b) \right\}\\
\ge \;& \operatorname{ess\,inf}_{b \in \mathcal{D}_{\theta,T}} \left\{ \left\langle X 1_{[t,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}, a \oplus^\theta_\Omega b \right\rangle_{t,T} - \gamma_{t,T}(a \oplus^\theta_\Omega b) \right\}\\
\ge \;& \phi_{t,T}\left(X 1_{[t,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}\right),
\end{aligned}$$

for all $X \in \mathcal{R}^\infty_{t,T}$ and $a \in \mathcal{D}_{t,T}$, which shows that

$$\phi_{t,T}(X) \ge \phi_{t,T}(X 1_{[t,\theta)} + \phi_{\theta,T}(X) 1_{[\theta,\infty)}), \quad \text{for all } X \in \mathcal{R}^\infty_{t,T}.$$

This shows that it follows directly from condition (1) that $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time-consistent. If condition (2) is satisfied, then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is time consistent because it fulfills the assumption (4.14) of Proposition 4.5. □

Exactly the same arguments as in the proof of Theorem 4.22 yield the following

Corollary 4.23 *Let $T \in \mathbb{N} \cup \{\infty\}$ and $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ a concave monetary utility process such that for all $t \in [0,T] \cap \mathbb{N}$ and $X \in \mathcal{R}^\infty_{t,T}$,*

$$\phi_{t,T}(X) = \operatorname{ess\,inf}_{a\in\mathcal{D}^e_{t,T}} \left\{ \langle X, a\rangle_{t,T} - \gamma_{t,T}(a) \right\} ,$$

for a special penalty function $\gamma_{t,T}$ on $\mathcal{D}_{t,T}$. Assume that at least one of the following two conditions is satisfied:

(1) *For each $t \in [0,T] \cap \mathbb{N}$ and every finite $(\mathcal{F}_t)$-stopping time θ such that $t \le \theta \le T$,*

$$\gamma_{t,T}(a) = \operatorname{ess\,sup}_{b\in\mathcal{D}^e_{\theta,T}} \gamma_{t,T}(a \oplus^\theta_\Omega b) + \mathrm{E}\left[\gamma^{\mathrm{ext}}_{\theta,T}(a) \mid \mathcal{F}_t\right] , \quad \text{for all } a \in \mathcal{D}^e_{t,T} .$$

(2) *$T \in \mathbb{N}$, and for each $t = 0, \dots, T-1$,*

$$\gamma_{t,T}(a) = \operatorname{ess\,sup}_{b\in\mathcal{D}^e_{t+1,T}} \gamma_{t,T}(a \oplus^{t+1}_\Omega b) + \mathrm{E}\left[\gamma^{\mathrm{ext}}_{t+1,T}(a) \mid \mathcal{F}_t\right] , \quad \text{for all } a \in \mathcal{D}^e_{t,T} .$$

Then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is time-consistent.

5 Special cases and examples

In much of this section the concept of m-stability plays an important role. It can be viewed as a special case of the concept of stability under concatenation and appears under various names in Artzner et al. (2002), Roorda et al. (2003), Epstein and Schneider (2003), Wang (2003), Riedel (2004) and Delbaen (2004).

If $T = \infty$, we denote by $\mathcal{F}_\infty$ the sigma-algebra generated by $\bigcup_{t\in\mathbb{N}} \mathcal{F}_t$.

Definition 5.1 *For $T \in \mathbb{N} \cup \{\infty\}$, $f, g \in \left\{h \in L^1(\mathcal{F}_T) \mid h \ge 0 , \mathrm{E}[h] = 1\right\}$, a finite $(\mathcal{F}_t)$-stopping time $\theta \le T$ and $A \in \mathcal{F}_\theta$, we define*

$$f \otimes^\theta_A g := \begin{cases} f & \text{on } A^c \cup \{\mathrm{E}[g \mid \mathcal{F}_\theta] = 0\} \\ \mathrm{E}[f \mid \mathcal{F}_\theta] \frac{g}{\mathrm{E}[g|\mathcal{F}_\theta]} & \text{on } A \cap \{\mathrm{E}[g \mid \mathcal{F}_\theta] > 0\} \end{cases} , \tag{5.26}$$

and we call a subset $\mathcal{P}$ of $\left\{h \in L^1(\mathcal{F}_T) \mid h \ge 0 , \mathrm{E}[h] = 1\right\}$ m-stable if it contains $f \otimes^\theta_A g$ for all $f, g \in \mathcal{P}$, every finite $(\mathcal{F}_t)$-stopping time $\theta \le T$ and $A \in \mathcal{F}_\theta$.

Let $T \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{P}$ a non-empty subset of

$$\left\{h \in L^1(\mathcal{F}_T) \mid h \ge 0 , \mathrm{E}[h] = 1\right\} .$$

If for all $s \in [0,T] \cap \mathbb{N}$, $A \in \mathcal{F}_s$ and $f, g \in \mathcal{P}$, $f \otimes^s_A g$ is in the $\sigma(L^1, L^\infty)$-closed, convex hull $\langle\mathcal{P}\rangle$ of $\mathcal{P}$, then it can be shown as in the proof of Proposition 4.13 that $\langle\mathcal{P}\rangle$ is m-stable.

5.1 Processes of coherent utility functionals that depend on the final value

Let $T \in \mathbb{N}$ and $\mathcal{P}$ a non-empty subset of the set

$$\left\{h \in L^1(\mathcal{F}_T) \mid h \geq 0\,,\, \mathrm{E}\,[h] = 1\right\}\,.$$

Then

$$\mathcal{Q}(\mathcal{P}) := \left\{f 1_{[T,\infty)} \mid f \in \mathcal{P}\right\}$$

is a non-empty subset of $\mathcal{D}_{0,T}$, and the concatenation of two elements

$$a = f 1_{[T,\infty)} \quad \text{and} \quad b = g 1_{[T,\infty)}$$

in $\mathcal{Q}(\mathcal{P})$ at an $(\mathcal{F}_t)$-stopping time $\theta \leq T$ for a set $A \in \mathcal{F}_\theta$ is equal to

$$\left(f \otimes^\theta_A g\right) 1_{[T,\infty)}\,.$$

This shows that $\mathcal{Q}(\mathcal{P})$ is stable under concatenation if and only if $\mathcal{P}$ is m-stable.

If $\mathcal{P}^e$ is a non-empty subset of

$$\left\{h \in L^1(\mathcal{F}_T) \mid h > 0\,,\, \mathrm{E}\,[h] = 1\right\}\,,$$

then $\mathcal{Q}(\mathcal{P}^e)$ is a non-empty subset of $\mathcal{D}^e_{0,T}$, and

$$\phi_{t,T}(X) := \operatorname{ess\,inf}_{a \in \mathcal{Q}(\mathcal{P}^e)} \frac{\langle X, a\rangle_{t,T}}{\langle 1, a\rangle_{t,T}} = \operatorname{ess\,inf}_{f \in \mathcal{P}^e} \frac{\mathrm{E}\,[f X_T \mid \mathcal{F}_t]}{\mathrm{E}\,[f \mid \mathcal{F}_t]}\,,\; t = 0, \dots, T\,,\; X \in \mathcal{R}^\infty_{t,T}\,,$$

defines a relevant coherent utility functional process.

If $\mathcal{P}^e$ is m-stable, it follows from Theorem 4.16 that $(\phi_{t,T})_{t=0}^T$ is time-consistent. On the other hand, if $(\phi_{t,T})_{t=0}^T$ is time consistent, then by Theorem 4.14, the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed convex hull of $\mathcal{Q}(\mathcal{P}^e)$ is stable under concatenation, which implies that the $\sigma(L^1, L^\infty)$-closed, convex hull of $\mathcal{P}^e$ is m-stable.

This class of time-consistent coherent utility functional processes appears in Artzner et al. (2002), Roorda et al. (2003) and in a continuous-time setup in Delbaen (2004). Rosazza Gianin (2003) studies the relation between time-consistent monetary utility functionals that depend on real-valued random variables and g-expectations.

5.2 Processes of coherent utility functionals defined by m-stable sets and worst stopping

Let $T \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{P}^e$ a non-empty m-stable subset of

$$\left\{h \in L^1(\mathcal{F}_T) \mid h > 0\,,\, \mathrm{E}\,[h] = 1\right\}\,.$$

For all $t \in [0, T] \cap \mathbb{N}$, define

$$\psi_t(Y) := \operatorname{ess\,inf}_{f \in \mathcal{P}^e} \frac{\mathrm{E}\,[f\, Y \mid \mathcal{F}_t]}{\mathrm{E}\,[f \mid \mathcal{F}_t]}\,, \quad Y \in L^\infty(\mathcal{F}_T)\,,$$

and for all $X \in \mathcal{R}^\infty_{t,T}$,

$$\phi_{t,T}(X) := \operatorname{ess\,inf} \{\psi_t(X_\xi) \mid \xi \text{ a finite } (\mathcal{F}_t)\text{-stopping time such that } t \le \xi \le T\} \ . \quad (5.27)$$

Then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is a time-consistent relevant coherent utility functional process.

To see this, note that $\phi_{0,T}$ is a T-relevant coherent utility functional on $\mathcal{R}^\infty_{0,T}$ that can be represented as

$$\phi_{0,T}(X) = \inf_{a\in\mathcal{Q}(\mathcal{P}^e)} \langle X, a\rangle_{0,T} \ , \quad X \in \mathcal{R}^\infty_{0,T} \, ,$$

where $\mathcal{Q}(\mathcal{P}^e)$ is the non-empty subset of $\mathcal{D}_{0,T}$ given by

$$\mathcal{Q}(\mathcal{P}^e) := \left\{ \mathrm{E}\left[f \mid \mathcal{F}_\xi\right] 1_{[\xi,\infty)} \mid f \in \mathcal{P}^e \, , \, \xi \le T \text{ a finite } (\mathcal{F}_t)\text{-stopping time} \right\} \, .$$

It follows from the Corollaries 3.16 and 3.24 that

$$\phi_{0,T}(X) = \inf_{a\in\mathcal{Q}} \langle X, a\rangle_{0,T} = \inf_{a\in\mathcal{Q}^e} \langle X, a\rangle_{0,T} \ , \quad X \in \mathcal{R}^\infty_{0,T} \, ,$$

where $\mathcal{Q}$ is the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, convex hull of $\mathcal{Q}(\mathcal{P}^e)$ and $\mathcal{Q}^e = \mathcal{Q} \cap \mathcal{D}^e_{0,T}$.

Let $\theta \le T$ be a finite $(\mathcal{F}_t)$-stopping time, $A \in \mathcal{F}_\theta$ and a, b two processes in $\mathcal{Q}(\mathcal{P}^e)$ of the form

$$a = f_a 1_{[\xi_a,\infty)} \quad \text{and} \quad b = f_b 1_{[\xi_b,\infty)} \, ,$$

where $\xi_a \le T$ and $\xi_b \le T$ are finite $(\mathcal{F}_t)$-stopping times, $f_a = \mathrm{E}\left[\hat{f}_a \mid \mathcal{F}_{\xi_a}\right]$ and $f_b = \mathrm{E}\left[\hat{f}_b \mid \mathcal{F}_{\xi_b}\right]$ for $\hat{f}_a, \hat{f}_b \in \mathcal{P}^e$. Then

$$\begin{aligned}
(a \oplus^\theta_A b)_t &= 1_{B^c} \, f_a \, 1_{\{t\ge\xi_a\}} + 1_B \, \frac{\mathrm{E}\left[f_a \mid \mathcal{F}_\theta\right]}{\mathrm{E}\left[f_b \mid \mathcal{F}_\theta\right]} \, f_b \, 1_{\{t\ge\xi_b\}} \\
&= 1_{B^c} \, f_a \, 1_{\{t\ge\xi_a\}} + 1_B \, \frac{\mathrm{E}\left[\hat{f}_a \mid \mathcal{F}_\theta\right]}{\mathrm{E}\left[\hat{f}_b \mid \mathcal{F}_\theta\right]} \, f_b \, 1_{\{t\ge\xi_b\}} \\
&= \mathrm{E}\left[\hat{f} \mid \mathcal{F}_\xi\right] 1_{\{t\ge\xi\}} \, ,
\end{aligned}$$

where

$$B = A \cap \{t \ge \theta\} \cap \{\xi_b \ge \theta\} \cap \{\xi_a \ge \theta\} \in \mathcal{F}_{\theta\wedge\xi_a\wedge\xi_b} \, ,$$
$$\hat{f} = \hat{f}_a \otimes^\theta_B \hat{f}_b \quad \text{and} \quad \xi = 1_{B^c}\xi_a + 1_B \xi_b \, .$$

It follows from the m-stability of $\mathcal{P}^e$ that $\mathcal{Q}(\mathcal{P}^e)$ is stable under concatenation. Proposition 4.13 implies that $\mathcal{Q}$, and therefore also $\mathcal{Q}^e$ are stable under concatenation. Hence, it follows from Theorem 4.16 that the sequence of functions $(\tilde{\phi}_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ given by

$$\tilde{\phi}_{t,T}(X) := \operatorname{ess\,inf}_{a\in\mathcal{Q}^e} \frac{\langle X, a\rangle_{t,T}}{\langle 1, a\rangle_{t,T}} \, , \quad t \in [0,T] \cap \mathbb{N} \, , \, X \in \mathcal{R}^\infty_{t,T} \, ,$$

is a time-consistent relevant coherent utility functional process, and it can easily be checked that $\tilde{\phi}_{t,T} = \phi_{t,T}$ for all $t \in [0,T] \cap \mathbb{N}$.

For finite time horizon T, this class of time-consistent coherent utility functional processes is also discussed in Artzner et al. (2002) and in a continuous-time setup in Delbaen (2004).

5.3 Processes of coherent utility functionals that depend on the infimum over time

Let $T \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{P}^e$ a non-empty subset of the set

$$\left\{h \in L^1(\mathcal{F}_T) \mid h > 0 \,,\, \mathrm{E}\,[h] = 1\right\} \,.$$

For all $t \in [0,T] \cap \mathbb{N}$, define

$$\psi_t(Y) := \operatorname{ess\,inf}_{f \in \mathcal{P}^e} \frac{\mathrm{E}\,[f\,Y \mid \mathcal{F}_t]}{\mathrm{E}\,[f \mid \mathcal{F}_t]} \,, \quad Y \in L^\infty(\mathcal{F}_T) \,,$$

and

$$\phi_{t,T}(X) := \psi_t\left(\inf_{s \in [t,T] \cap \mathbb{N}} X_s\right) \,, \quad X \in \mathcal{R}^\infty_{t,T} \,.$$

Then $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is a relevant coherent utility functional process. But even if $\mathcal{P}^e$ is m-stable, $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ is in general not time-consistent.

For an easy counter-example, consider a probability space of the form $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with $P[\omega_j] = \frac{1}{4}$ for all $j = 1, \dots, 4$. Let $T = 2$ and assume that the filtration $(\mathcal{F}_t)_{t=0}^2$ is given as follows: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1$ is generated by the set $\{\omega_1, \omega_2\}$ and $\mathcal{F}_2$ is generated by the sets $\{\omega_j\}$, $j = 1, \dots, 4$. If $\mathcal{P}^e = \{1\}$. Then, for $t \in \{0,1,2\}$ and $X \in \mathcal{R}^\infty_{t,2}$,

$$\phi_{t,2}(X) = \mathrm{E}\left[\inf_{t \le s \le 2} X_s \mid \mathcal{F}_t\right] \,.$$

If $X_0 = 2$, $X_1(\omega_1) = X_1(\omega_2) = 4$, $X_1(\omega_3) = X_1(\omega_4) = 1$, $X_2(\omega_1) = 5$, $X_2(\omega_2) = 1$, $X_2(\omega_3) = 2$ and $X_2(\omega_4) = -1$, then $\phi_{0,2}(X) = \frac{3}{4}$. On the other hand, $\phi_{1,2}(X) = \frac{5}{2}$ on $\{\omega_1, \omega_2\}$ and $\phi_{1,2}(X) = 0$ on $\{\omega_3, \omega_4\}$. Hence, $\phi_{0,2}(X 1_{\{0\}} + \phi_{1,2}(X) 1_{[1,2]}) = 1$.

5.4 Processes of monetary risk measures that depend on an average over time

Let $T \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{P}^e$ a non-empty subset of the set

$$\left\{h \in L^1(\mathcal{F}_T) \mid h > 0 \,,\, \mathrm{E}\,[h] = 1\right\} \,.$$

For all $t \in [0,T] \cap \mathbb{N}$, define

$$\psi_t(Y) := \operatorname{ess\,inf}_{f \in \mathcal{P}^e} \frac{\mathrm{E}\,[f\,Y \mid \mathcal{F}_t]}{\mathrm{E}\,[f \mid \mathcal{F}_t]} \,, \quad Y \in L^\infty(\mathcal{F}_T) \,,$$

and

$$\phi_{t,T}(X) := \psi_t\left(\frac{\sum_{s \in [t,T] \cap \mathbb{N}} \mu_s X_s}{\sum_{s \in [t,T] \cap \mathbb{N}} \mu_s}\right) \,, \quad X \in \mathcal{R}^\infty_{t,T} \,,$$

where $(\mu_s)_{s \in \mathbb{N}}$ is a sequence of non-negative numbers such that

$$\sum_{s \in [0,T] \cap \mathbb{N}} \mu_s = 1 \,,$$

and

$$\sum_{s\in[t,T]\cap\mathbb{N}} \mu_s > 0 \quad \text{for all } t \in [0,T]\cap\mathbb{N}\,.$$

Then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is obviously a relevant coherent utility functional process, and if $\mathcal{P}^e$ is m-stable, then $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is time-consistent.

To see this we denote for $f \in \mathcal{P}^e$ by $J(f)$ the process $a \in \mathcal{D}^e_{0,T}$ given by

$$\Delta a_t := \mu_t \mathrm{E}\left[f \mid \mathcal{F}_t\right] \quad \text{for all } t \in \mathbb{N}\,.$$

Clearly,

$$\phi_{t,T}(X) = \operatorname{ess\,inf}_{a\in J(\mathcal{P}^e)} \frac{\langle X, a\rangle_{t,T}}{\langle 1, a\rangle_{t,T}} \quad \text{for all } t \in [0,T]\cap\mathbb{N} \quad \text{and} \quad X \in \mathcal{R}^\infty_{t,T}\,,$$

and it can easily be checked that for all $f, g \in \mathcal{P}^e$, every finite $(\mathcal{F}_t)$-stopping time $\theta \le T$ and $A \in \mathcal{F}_\theta$,

$$J(f) \oplus^\theta_A J(g) = J(f \otimes^\theta_A g)\,.$$

Hence, $J(\mathcal{P}^e)$ is stable under concatenation, and it follows from Theorem 4.16 that $(\phi_{t,T})_{t\in[0,T]\cap\mathbb{N}}$ is time-consistent.

5.5 Processes of robust entropic utility functionals

Let $T \in \mathbb{N}$ and $\mathcal{P}^e$ a non-empty subset of

$$\left\{h \in L^1(\mathcal{F}_T) \mid h > 0\,,\, \mathrm{E}\left[h\right] = 1\right\}\,.$$

For $t = 0, \dots, T$ and $X \in \mathcal{R}^\infty_{t,T}$, define

$$\phi_{t,T}(X) := \operatorname{ess\,inf}_{f\in\mathcal{P}^e} \left\{-\log \frac{\mathrm{E}\left[f \exp(-X_T) \mid \mathcal{F}_t\right]}{\mathrm{E}\left[f \mid \mathcal{F}_t\right]}\right\}\,, \quad X \in \mathcal{R}^\infty_{0,T}\,.$$

Then, for all $t = 0, \dots, T$, $\phi_{t,T}$ is a T-relevant concave monetary utility functional on $\mathcal{R}^\infty_{t,T}$ that is continuous for bounded decreasing sequences, and $(\phi_{t,T})_{t=0}^T$ is time-consistent if $\mathcal{P}^e$ is m-stable. Indeed, it is obvious that for all $t = 0, \dots, T$, $\phi_{t,T}$ is a T-relevant monetary utility functional on $\mathcal{R}^\infty_{t,T}$. To show the other assertions, we introduce for all $f \in \mathcal{P}^e$ and $t = 0, \dots, T$, the mappings

$$\psi^f_t(Y) := -\log \frac{\mathrm{E}\left[f \exp(-Y) \mid \mathcal{F}_t\right]}{\mathrm{E}\left[f \mid \mathcal{F}_t\right]}\,, \quad Y \in L^\infty(\mathcal{F}_T)$$

and

$$\psi_t(Y) := \operatorname{ess\,inf}_{f\in\mathcal{P}^e} \psi^f_t(Y)\,, \quad Y \in L^\infty(\mathcal{F}_T)\,.$$

To see that for all $t = 0, \dots, T$, $\phi_{t,T}$ is continuous for bounded decreasing sequences we let $(Y^n)_{n\in\mathbb{N}}$ be a decreasing sequence in $L^\infty(\mathcal{F}_T)$ and $Y \in L^\infty(\mathcal{F}_T)$ such that $\lim_{n\to\infty} Y^n = Y$

almost surely. Then $\lim_{n\to\infty}\psi_t(Y^n)$ exists almost surely and $\lim_{n\to\infty}\psi_t(Y^n)\geq\psi_t(Y)$. On the other hand, there exists a sequence $(f^k)_{k\in\mathbb{N}}$ in $\mathcal{P}^e$ such that $\psi_t(Y)=\inf_{k\in\mathbb{N}}\psi_t^{f^k}(Y)$ almost surely, and for all $k\in\mathbb{N}$,

$$\psi_t^{f^k}(Y)=\lim_{n\to\infty}\psi_t^{f^k}(Y^n)\geq\lim_{n\to\infty}\psi_t(Y^n)\,,$$

Hence, $\psi_t(Y)\geq\lim_{n\to\infty}\psi_t(Y^n)$, which shows that $\phi_{t,T}$ is continuous for bounded decreasing sequences.

To see that $\phi_{t,T}$ satisfies condition (3) of Definition 3.1, fix an $f\in\mathcal{P}^e$ and a $Z\in L^\infty(\mathcal{F}_T)$. Then, define $f_Z\in L^1(\mathcal{F}_T)$ by

$$f_Z:=\frac{f\exp\left(-Z\right)}{\mathrm{E}\left[f\exp\left(-Z\right)\mid\mathcal{F}_t\right]}\,,$$

and note that it follows from Jensen's inequality that for all $Y\in L^\infty(\mathcal{F}_T)$,

$$\begin{aligned}\psi_t^f(Y) &= -\log\frac{\mathrm{E}\left[f_Z\frac{f}{f_Z}\exp\left(-Y\right)\mid\mathcal{F}_t\right]}{\mathrm{E}\left[f\mid\mathcal{F}_t\right]}\\ &= -\log\mathrm{E}\left[f_Z\exp\left(Z-Y\right)\mid\mathcal{F}_t\right]-\log\frac{\mathrm{E}\left[f\exp\left(-Z\right)\mid\mathcal{F}_t\right]}{\mathrm{E}\left[f\mid\mathcal{F}_t\right]}\\ &\leq \mathrm{E}\left[f_Z(Y-Z)\mid\mathcal{F}_t\right]+\psi_t^f(Z)\,.\end{aligned}$$

This shows that for all $Y\in L^\infty(\mathcal{F}_T)$,

$$\psi_t^f(Y)=\operatorname{ess\,inf}_{Z\in L^\infty(\mathcal{F}_T)}\left\{\mathrm{E}\left[f_ZY\mid\mathcal{F}_t\right]-\mathrm{E}\left[f_ZZ\mid\mathcal{F}_t\right]+\psi_t^f(Z)\right\}\,,\tag{5.28}$$

and therefore,

$$\psi_t(Y)=\operatorname{ess\,inf}_{f\in\mathcal{P}^e\,,\,Z\in L^\infty(\mathcal{F}_T)}\left\{\mathrm{E}\left[f_ZY\mid\mathcal{F}_t\right]-\mathrm{E}\left[f_ZZ\mid\mathcal{F}_t\right]+\psi_t^f(Z)\right\}\,,$$

from which it can be seen that $\phi_{t,T}$ satisfies condition (3) of Definition 3.1.

Now, assume that $\mathcal{P}^e$ is m-stable. Then, for all $t=0,\dots,T$ and $Y\in L^\infty(\mathcal{F}_T)$, the set

$$\left\{\psi_t^f(Y)\mid f\in\mathcal{P}^e\right\}$$

is directed downwards because for all $f,g\in\mathcal{P}^e$,

$$\psi_t^f(Y)\wedge\psi_t^g(Y)=\psi_t^h(Y)\,,$$

where

$$h=f\otimes_A^t g\quad\text{for}\quad A=\left\{\psi_t^g(Y)<\psi_t^f(Y)\right\}\,.$$

Hence, there exists a sequence $(f^k)_{k\in\mathbb{N}}$ in $\mathcal{P}^e$ such that almost surely,

$$\psi_t^{f^k}(Y)\searrow\psi_t(T)\,,\quad\text{as }k\to\infty\,.$$

Next, note that for all $t = 0, \ldots, T-1$, $f, g \in \mathcal{P}^e$ and $Y \in L^\infty(\mathcal{F}_T)$,

$$\psi_t^f(\psi_{t+1}^g(Y)) = \psi_t^h(Y)\,,$$

where

$$h = f \otimes_\Omega^{t+1} g\,.$$

It follows that

$$\psi_t(\psi_{t+1}(Y)) = \psi_t(Y)\,,$$

for all $t = 0, \ldots, T-1$ and $Y \in L^\infty(\mathcal{F}_T)$, and therefore,

$$\phi_{t,T}(X 1_{\{t\}} + \phi_{t+1,T}(X) 1_{[t+1,\infty)}) = \phi_{t,T}(X)\,,$$

for all $t = 0, \ldots, T-1$ and $X \in \mathcal{R}^\infty_{t,T}$, which by Proposition 4.5, implies that $(\phi_{t,T})_{t=0}^T$ is time-consistent.

The utility functional $\phi_{0,T}$ is a robust version of the mapping

$$C : L^\infty(\mathcal{F}_T) \to \mathbb{R}\,, \quad Y \mapsto -\log \mathrm{E}\left[\exp(-Y)\right]\,,$$

which assigns a random variable $Y \in L^\infty(\mathcal{F}_T)$ its certainty equivalent under the exponential utility function

$$x \mapsto -\exp(-x)\,.$$

It is well known that C admits the representation

$$C(Y) = \inf_Q \left\{ \mathrm{E}_Q[Y] + H(Q \mid P) \right\}\,, \tag{5.29}$$

where the infimum is taken over all probability measures Q on $(\Omega, \mathcal{F}_T)$ and $H(Q \mid P)$ is the relative entropy of Q with respect to P. In fact, it can easily be checked that the penalty function in the representation (5.28) is the conditional relative entropy of f_Z with respect to f. For more details and relations to pricing in incomplete markets we refer to Frittelli (2000), Rouge et al. (2000) and Delbaen et al. (2002). More on the entropic risk measure $-C$ can be found in Föllmer and Schied (2002a) and Weber (2003). A conditional version of the entropic risk measure is studied in Detlefsen (2003). In Frittelli and Rosazza Gianin (2004) it is shown that the dynamic entropic risk measure is time-consistent in continuous time.

5.6 Time-consistent monetary utility functional processes and worst stopping

For finite time horizon the coherent utility functional processes of Subsection 5.2 can be generalized as follows:

Let $T \in \mathbb{N}$ and $(\hat{\phi}_{t,T})_{t=0}^T$ a time-consistent monetary utility functional process on $R^\infty_{0,T}$ such that for all $t = 0, \ldots, T$ and $X \in \mathcal{R}^\infty_{t,T}$, $\hat{\phi}_{t,T}(X)$ depends only on the final value X_T of X, that is, for all $t = 0, \ldots, T$,

$$\hat{\phi}_{t,T}(X) = \psi_t(X_T)\,,$$

where ψ_t is a mapping from $L^\infty(\mathcal{F}_T)$ to $L^\infty(\mathcal{F}_t)$ that satisfies the following conditions:

(0) $\psi_t(1_A Y) = 1_A \psi(Y)$ for all $Y \in L^\infty(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$

(1) $\psi_t(Y) \leq \psi_t(Z)$ for all $Y, Z \in L^\infty(\mathcal{F}_T)$ such that $Y \leq Z$

(2) $\psi_t(Y + m) = \psi_t(Y) + m$ for all $Y \in L^\infty(\mathcal{F}_T)$ and $m \in L^\infty(\mathcal{F}_t)$

(tc) $\psi_t(\psi_{t+1}(Y)) = \psi_t(Y)$, for all $t = 0, \dots, T-1$ and $Y \in L^\infty(\mathcal{F}_T)$.

Denote by $\Theta_{t,T}$ the set of all $(\mathcal{F}_t)$-stopping times ξ such that $t \leq \xi \leq T$, and define a new monetary utility functional process by

$$\phi_{t,T}(X) := \operatorname{ess\,inf}_{\xi \in \Theta_{t,T}} \psi_t(X_\xi)\,.$$

For a given $X \in \mathcal{R}^\infty_{0,T}$, define the process $(S_t(X))_{t=0}^T$ recursively by

$$S_T(X) := X_T$$

and

$$S_t(X) := X_t \wedge \psi_t(S_{t+1}(X))\,, \quad \text{for } t \leq T-1\,.$$

For all $t = 0, \dots, T$, denote by ξ^t the stopping time given by

$$\xi^t := \inf\left\{j = t, \dots, T \mid S_j(X) = X_j\right\}\,.$$

It can easily be checked that

$$\phi_{t,T}(X) = \psi_t(X_{\xi^t}) = S_t(X)\,.$$

For a $t \in [0, T-1] \cap \mathbb{N}$, set $Y := X 1_{\{t\}} + \phi_{t,T}(X) 1_{[t+1,\infty)}$. It is easy to see that $S_{t+1}(Y) = S_{t+1}(X)$. Therefore,

$$\phi_{t,T}(Y) = S_t(Y) = Y_t \wedge \psi_t(S_{t+1}(Y)) = X_t \wedge \psi_t(S_{t+1}(X)) = S_t(X) = \phi_{t,T}(X)\,,$$

which shows that $(\phi_{t,T})_{t=0}^T$ is time-consistent.

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