Stochastic Volatility: Time Scales and Perturbations

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Abstract

Our goal is to address the following fundamental question in pricing and hedging derivatives. How traded call options, quoted in terms of implied volatilities, can be used to price and hedge more complicated contracts. One can approach this difficult problem in two different ways: modeling the evolution of the underlying or modeling the evolution of the implied volatility surface. In both cases one requires that the model is free of arbitrage.

Modeling the underlying usually involves the specification of a multi-factor Markovian model under the risk-neutral pricing measure. The calibration to the observed implied volatilities of the parameters of that model, including the market prices of risk, is a challenging task because of the complex relation between call option prices and model parameters (through a pricing partial differential equation for instance). A main problem with this approach is to find the "right model" which will produce a stable parameter estimation. We like to think of this problem as the "(t, T, K)" problem: for a given present time t and a fixed maturity T, it is usually easy with low dimensional models to fit the skew with respect to strikes K. Getting a good fit of the term structure of implied volatility, that is when a range of observed maturities are taken into account, is a much harder problem which can be handled with a sufficient number of parameters and eventually including jumps in the model. The main problem remains: the stability with respect to t of these calibrated parameters. However this is an highly desirable quality if one wants to use the model to

compute no-arbitrage prices of more complex path-dependent derivatives, since in this case the distribution over time of the underlying is crucial.

Modeling directly the evolution of the implied volatility surface is a promising approach but involves some complicated issues. One has to make sure that the model is free of arbitrage or, in other words, that the surface is produced by some underlying under a risk-neutral measure. This is not an obvious task, and the choice of a model and its calibration is also an important issue in this approach. But most importantly, in order to use this modeling to price other path-dependent contracts, one has to identify a corresponding underlying which typically does not lead to a low dimensional Markovian evolution.

Wouldn't it be nice to have a direct and simple connection between the observed implied volatilities and prices of more complex path-dependent contracts! Our objective is to provide such a bridge. This is done by using a combination of singular and regular perturbations techniques corresponding respectively to fast and slow time scales in volatility. We obtain a parametrization of the implied volatility surface in terms of Greeks, which involves four parameters at the first order of approximation. This procedure leads to parameters which are exactly those needed to price other contracts at this level of approximation. In our previous work presented in [1] we used only the fast volatility time scale combined with a statistical estimation of an effective constant volatility from historical data. The introduction of the slow volatility time scale enables us to capture more accurately the behavior of the term structure of implied volatility at long maturities. Moreover in the framework presented here, statistics of historical data are not needed. Thus, in summary, we directly link the implied volatilities to prices of path-dependent contracts by exploiting volatility time scales. We refer to [2] for a detailed presentation of volatility time scales in the S&P 500 index. The mathematical derivation of the singular perturbation for a call option can be found in [3], and the derivation of the combined regular and singular perturbations is in [4]. Calibration formulas are given in [5].

References

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