A new proof of the 2-dimensional Halpern–Läuchli Theorem

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Abstract

We provide an ultrafilter proof of the 2-dimensional Halpern–Läuchli Theorem in the following sense. If T_0 and T_1 are trees and $T_0 \otimes T_1$ denotes their level product, we exhibit an ultrafilter $\mathcal{U} \in \beta(T_0 \otimes T_1)$ so that every $A \in \mathcal{U}$ contains a subset of the form $S_0 \otimes S_1$ for suitable strong subtrees of T_0 and T_1. We then discuss obstacles to extending our method of proof to higher dimensions.

1 Introduction

Our conventions on trees mostly follow [2]. By a *tree* $(T, \leq)$, we mean a rooted, finitely-branching tree of height ω so that each $t \in T$ has at least 2 immediate successors. If $t \in T$, we set $\text{Pred}(t, T) := \{s \in T : s \lneq t\}$. The *level* of $t \in T$, denoted $\text{Lev}(t, T)$, is the number $|\text{Pred}(t, T)|$. If $n < \omega$, we set $T(n) := \{t \in T : \text{Lev}(t, T) = n\}$. Given $s, t \in T$, we say that t is an *immediate successor* of s in T if $s \leq t$ and $\text{Lev}(t, T) = \text{Lev}(s, T) + 1$. Write $\text{IS}(s, T)$ for the immediate successors of s in T. Note that for every $s \in T$, $2 \leq |\text{IS}(s, T)| < \omega$.

A subset $S \subseteq T$ is called a *strong subtree* of T if $(S, \leq\restriction_S)$ is a tree satisfying the following two items.

1. For some increasing function $f : \omega \to \omega$, we have $S(n) \subseteq T(f(n))$.
2. For every $s \in S$ and $t \in \text{IS}(s, T)$, there is a unique $t' \in \text{IS}(s, S)$ with $t \leq t'$.

If in item (1) we have a specific $f : \omega \to \omega$ in mind, we call $S \subseteq T$ an *f-strong subtree* of T.

If $d < \omega$ and $T_0, ..., T_{d-1}$ are trees, the *level product*, denoted $T_0 \otimes \cdots \otimes T_{d-1}$, is the set $\bigcup_n T_0(n) \times \cdots \times T_{d-1}(n)$, which receives a tree structure in the obvious way.

We are now ready to state the Halpern–Läuchli Theorem [1].

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Theorem 1.1 (Halpern–Läuchli)**.** *Let $d < \omega$, and let $T_0, ..., T_{d-1}$ be trees. Let $\chi : T_0 \otimes \cdots \otimes T_{d-1} \to 2$ be a coloring. Then there are an increasing function $f : \omega \to \omega$ and f-strong subtrees $S_i \subseteq T_i$ so that $S_0 \otimes \cdots \otimes S_{d-1}$ is monochromatic for χ.*

The parameter d is referred to as the *dimension*. We will provide a new proof of the Halpern–Läuchli Theorem for $d = 2$.

2 Warmup in one dimension

As a warmup, we will first provide a new proof for $d = 1$. If T is a tree, a *branch* through T is a maximal linearly ordered subset of T. If $x \subseteq T$ is a branch, then for every $n < \omega$, there is a unique element of $T(n)$ in x, which we will denote by $x(n)$. Let $[T]$ denote the set of branches through T. We endow $[T]$ with the topology generated by the sets $\langle N_t : t \in T\rangle$, where for $t \in T$, we set $N_t := \{x \in [T] : t \in x\}$. With this topology, $[T]$ is homeomorphic to Cantor space. Of particular interest will be the ideal of nowhere dense subsets of $[T]$.

For X any nonempty set, let βX denote the set of ultrafilters on X. In particular, let $\beta([T])$ denote the set of ultrafilters on $[T]$, where we now view $[T]$ as just a set. Fix $\mathcal{U} \in \beta([T])$ avoiding the nowhere dense ideal. Also fix any nonprincipal ultrafilter $\mathcal{V} \in \beta\omega$. We define the ultrafilter $\mathcal{U} \otimes \mathcal{V} \in \beta T$ as follows. If $A \subseteq T$, we have

$$A \in \mathcal{U} \otimes \mathcal{V} \Leftrightarrow \forall^{\mathcal{U}} x \in [T]\ \forall^{\mathcal{V}} n < \omega\ (x(n) \in A)\,.$$

Fix $A \in \mathcal{U} \otimes \mathcal{V}$. We will show that A contains a strong subtree of T. To see this, first set

$$A_{\mathcal{V}} := \{x \in [T] : \{n < \omega : x(n) \in A\} \in \mathcal{V}\}$$

By the definition of $\mathcal{U} \otimes \mathcal{V}$, we have $A_{\mathcal{V}} \in \mathcal{U}$. As $\mathcal{U}$ avoids the nowhere dense ideal, $A_{\mathcal{V}}$ is somewhere dense. This means that for some $t \in T$, $A_{\mathcal{V}}$ is dense in N_t. Pick any $x \in A_{\mathcal{V}}$ with $t \in x$. Then $\{n < \omega : x(n) \in A\} \in \mathcal{V}$. So for some $n < \omega$, we have $t \leq x(n)$ and $x(n) \in A$. Set $S(0) = \{x(n)\}$.

Assume $S(m) = \{s_0, ..., s_{k-1}\}$ has been determined. Let $\bigcup_{i<k} \mathrm{IS}(s_i, T) = \{t_0, ..., t_{\ell-1}\}$. For each $i < \ell$, we can find $x_i \in A_{\mathcal{V}}$ with $t_i \in x_i$. Then $\bigcap_{i<\ell}\{n < \omega : x_i(n) \in A\} \in \mathcal{V}$. So for some suitably large n, set $S(m+1) = \{x_i(n) : i < \ell\}$.

3 The proof for 2 dimensions

The proof for $d = 2$ will be very similar to the proof for $d = 1$. We will choose ultrafilters $\mathcal{U} \in \beta([T_0 \otimes T_1])$ and $\mathcal{V} \in \beta\omega$ and form $\mathcal{U} \otimes \mathcal{V}$ as before, and argue that every $A \in \mathcal{U} \otimes \mathcal{V}$ contains a subset of the form $S_0 \otimes S_1$ for some f-strong subtrees S_0 and S_1. The added difficulty in dimension 2 is that we must choose $\mathcal{U}$ more carefully.

Notice first that $[T_0 \otimes T_1] \cong [T_0] \times [T_1]$. Let $\pi_i : [T_0] \times [T_1] \to [T_i]$ be the projection maps. We call $Z \subseteq [T_0] \times [T_1]$ a *dense-by-dense-filter*, or DDF for short, if

1. $\pi_0(Z) \subseteq [T_0]$ is dense.

2. Letting $(Z)_x = \{y \in [T_1] : (x, y) \in z\}$, the collection $\{(Z)_x : x \in \pi_0(Z)\}$ generates a filter of dense subsets of $[T_1]$.

If $s \in T_0$ and $t \in T_1$, we say $Z \subseteq N_s \times N_t$ is (s,t)-DDF if the relativized analogs of items (1) and (2) hold. We call $Z \subseteq [T_0] \times [T_1]$ *somewhere DDF* if Z is (s,t)-DDF for some $s \in T_0$ and $t \in T_1$.

Proposition 3.1. *The collection of somewhere DDF subsets of $[T_0] \times [T_1]$ is weakly partition regular, i.e. for any $k < \omega$ and partition $[T_0] \times [T_1] = P_0 \cup \cdots \cup P_{k-1}$, some P_k contains a somewhere DDF subset.*

Proof. We prove a "relativized" version. First suppose that $X \subseteq [T_0]$ is non-meager and $Y \subseteq [T_1]$ is somewhere dense. By zooming in to a suitable $N_s \subseteq [T_0]$ and $N_t \subseteq [T_1]$, we may assume that $X \subseteq [T_0]$ is nowhere meager and $Y \subseteq [T_1]$ is dense.

Fix a partition $X \times Y = P_0 \cup \cdots \cup P_{\ell-1}$. We will attempt to find $D \subseteq P_0$ which is DDF. Enumerate $T_0 := \{s_n : n < \omega\}$. First set $Y = Y_0$. At stage k, starting with $k = 0$, we find if possible some $x_k \in X \cap N_{s_k}$ so that $Y_{k+1} := (P_0)_{x_k} \cap Y_k$ is dense. If we can do this for every $k < \omega$, then P_0 contains a DDF subset as desired.

Suppose we fail at stage k. This means that for every $x \in X \cap N_{s_k}$, there is some $t_x \in T_1$ so that $(P_0)_x \cap Y_k \cap N_{t_x} = \emptyset$. Since X is nowhere meager, there is some $t \in T_1$ so that $X' := \{x \in X \cap N_{s_k} : t_x = t\}$ is non-meager. Setting $Y' = Y_k \cap N_t$, we have X' non-meager, Y' somewhere dense, and the partition relative to $X' \times Y'$ has one fewer piece. □

We can now complete the proof of Halpern–Läuchli for $d = 2$. Let $\mathcal{U} \in \beta([T_0] \times [T_1])$ be an ultrafilter chosen so that every large set contains a somewhere DDF subset. Let $\mathcal{V} \in \beta\omega$ be any non-principal ultrafilter, and define $\mathcal{U} \otimes \mathcal{V}$ exactly as before.

Fix $A \in \mathcal{U} \otimes \mathcal{V}$. We will show that A contains a subset of the form $S_0 \otimes S_1$ for suitable strong subtrees $S_0 \subseteq T_0$ and $S_1 \subseteq T_1$. First set

$$A_{\mathcal{V}} := \{(x, y) \in [T_0] \times [T_1] : \{n < \omega : (x(n), y(n)) \in A\} \in \mathcal{V}\}.$$

By definition of $\mathcal{U} \otimes \mathcal{V}$, we have $A_{\mathcal{V}} \in \mathcal{U}$. Let $D \subseteq A_{\mathcal{V}}$ be an (s,t)-DDF subset for some $s \in T_0$ and $t \in T_1$. Pick some $(x, y) \in D$; then $\{n < \omega : (x(n), y(n)) \in A\} \in \mathcal{V}$. Pick $n > \max(\mathrm{Lev}(s, T_0), \mathrm{Lev}(t, T_1))$ with $(x(n), y(n)) \in A$, and set $S_0(0) = \{x(n)\}$, $S_1(0) = \{y(n)\}$.

Assume $S_0(m) = \{s_0, ..., s_{k_0-1}\}$ and $S_1(m) = \{t_0, ..., t_{k_1-1}\}$ have been determined. Let $\bigcup_{i<k_0} \mathrm{IS}(s_i, T_0) = \{s'_0, ..., s'_{\ell_0-1}\}$. For each $i < \ell_0$, we can find $x_i \in \pi_0(D)$ with $s'_i \in x_i$. Since D is (s,t)-DDF, the set $\bigcap_{i<\ell_0} (D)_{x_i}$ is dense in N_t. Let $\bigcup_{i<k_1} \mathrm{IS}(t_i, T_1) = \{t'_0, ..., t'_{\ell_1-1}\}$. For each $j < \ell_1$, we can find $y_j \in [T_1]$ so that $(x_i, y_j) \in D$ for each $i < \ell_0$. Now observe that $\{n < \omega : \forall i < \ell_0 \forall j < \ell_1 (x_i(n), y_j(n)) \in A\} \in \mathcal{V}$. For a suitably large n, set $S_0(m+1) = \{x_i(n) : i < \ell_0\}$ and $S_1(m+1) = \{y_j(n) : j < \ell_1\}$.

4 Obstacles to higher dimensions

In this last section, we show that the appropriate notion of "somewhere DDF" subset of $2^\omega \times 2^\omega \times 2^\omega$ is consistently not weakly partition regular, which prevents the proof for $d = 2$ from being generalized. To be precise, let us call the notion of DDF from the last section DDF(2) to emphasize the dimension.

For $i < j < 3$, let $\pi_{i,j} : (2^\omega)^3 \to 2^\omega \times 2^\omega$ be the corresponding projection. Let us call $Z \subseteq (2^\omega)^3$ *DDF*(3) if the following conditions are met.

1. $\pi_{0,1}(Z) \subseteq 2^\omega \times 2^\omega$ is DDF(2).
2. Letting $(Z)_{(x,y)} = \{z \in 2^\omega : (x, y, z) \in Z\}$, the collection $\{(Z)_{(x,y)} : (x, y) \in \pi_{0,1}(Z)\}$ generates a filter of dense subsets of 2^ω.

The notion of *somewhere DDF*(3) is defined similarly to the last section.

Proposition 4.1. *ZFC does not prove that the collection of somewhere DDF*(3) *subsets of* $(2^\omega)^3$ *is weakly partition regular. In particular, under CH there is a 2-coloring of* $(2^\omega)^3$ *so that neither color class contains a somewhere DDF*(3) *subset.*

Proof. For each $n \in \omega$, let $B_n = \{z \in 2^\omega : z(n) = 0\}$. Our coloring $(2^\omega)^3 = P_0 \cup P_1$ will be such that for every $x, y \in 2^\omega$, we have $(P_0)_{(x,y)} = B_n$ for some n. So our construction is just to describe the map $\varphi : 2^\omega \times 2^\omega \to \omega$ so that $(P_0)_{(x,y)} = B_{\varphi(x,y)}$. We will more-or-less use an Ulam matrix to describe φ.[4]

Identify 2^ω with $\omega_1 \setminus \omega$. For each infinite ordinal $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega \setminus \{0\}$ be a bijection. We then set

$$\varphi(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ f_\alpha(\beta) & \text{if } \beta < \alpha, \\ f_\beta(\alpha) & \text{if } \alpha < \beta \end{cases}$$

For any distinct infinite $\alpha_0, \alpha_1 < \omega_1$ and $n < \omega$, there is at most 1 ordinal β with $\varphi(\alpha_0, \beta) = \varphi(\alpha_1, \beta) = n$.

Now suppose $D \subseteq P_0$ is (s, t, u)-DDF(3) for some $s, t, u \in 2^{<\omega}$. This implies that for some n and every $(x, y) \in \pi_{0,1}(D)$, we have $\varphi(x, y) < n$. For any $N < \omega$, we can find $\{x_i : i < N\}$ and $\{y_i : i < N\}$ so that $(x_i, y_j) \in \pi_{0,1}(D)$ for each $i, j < N$. By making N large enough, we can find x'_0, x'_1 and y'_0, y'_1 so that for every $i, j < 2$, we have $\varphi(x'_i, y'_j) = k$ for some fixed k. This is a contradiction. □

References

[1] J. D. Halpern and H. Läuchli, A partition theorem, *Trans. Amer. Math. Soc.*, **124** (1966), 360–367.

[4] I thank MathOverflow user Ashutosh [3] for suggesting the use of an Ulam matrix. Ashutosh also provides evidence which suggests that working with MA + $\mathfrak{c} = \omega_2$ might provide a different outcome.

[2] K. R. Milliken, A Ramsey Theorem for Trees, *Journal of Combinatorial Theory*, Series A **26** (1979), 215–237.

[3] A. Zucker, "Countable partitions of Cantor space mod meager," MathOverflow question (2017) http://mathoverflow.net/q/258804.