Topological dynamics of automorphism groups, ultrafilter combinatorics, and the Generic Point Problem

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Abstract

For G a closed subgroup of S_∞, we provide a precise combinatorial characterization of when the universal minimal flow $M(G)$ is metrizable. In particular, each such instance fits into the framework of metrizable flows developed in [KPT] and [NVT]; as a consequence, each G with metrizable universal minimal flow has the generic point property, i.e. every minimal G-flow has a point whose orbit is comeager. This solves the Generic Point Problem raised in [AKL] for closed subgroups of S_∞.

1 Introduction

In the study of abstract topological dynamics, one is often concerned with the continuous action of a Hausdorff topological group G on a compact Hausdorff space X, often called a *G-flow.* The flow X is *minimal* if every orbit is dense and *universal* if for every G-flow Y, there is a G-map $f : X \to Y$, where a *G-map* is a continuous map which respects the G-action. It is a fact that every topological group G admits a *universal minimal flow* $M(G)$ which is unique up to G-flow isomorphism.

One common tool used to study the universal minimal flow $M(G)$ is the *greatest ambit* $(S(G), 1)$. A *G-ambit* (X, x_0) is a G-flow X with a distinguished point $x_0 \in X$ whose orbit is dense in X. The greatest ambit is then an ambit which maps onto every other G-ambit, where a map of G-ambits is a G-map which also respects the distinguished point. Since any minimal G-flow can be turned into an ambit by distinguishing any point, it follows that every minimal subflow of the greatest ambit is universal, hence isomorphic to $M(G)$.

An active field of research for the past two decades has been the attempt to classify those Polish groups G for which $M(G)$ is metrizable. The introduction of the seminal paper by Kechris, Pestov and Todorčević [KPT] contains an excellent survey of early efforts in this direction. In this paper, the authors provide a general way of constructing $M(G)$ for many closed subgroups of S_∞, the group of permutations of $\mathbb{N}$ endowed with the pointwise

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convergence topology. Interestingly, the greatest ambit is not the primary tool used to study $M(G)$ in this case.

The closed subgroups of S_∞ are exactly those Polish groups which are *non-Archimedean*, i.e. which admit a neighborhood basis at the identity consisting of open subgroups. The characterization we will find most useful is that the closed subgroups of S_∞ are exactly the automorphism groups of countably infinite, model-theoretic structures with universe $\mathbb{N}$. We can in fact narrow our scope to certain countably infinite structures known as *Fraïssé structures*. These are those countably infinite structures $\mathbf{K}$ with universe $\mathbb{N}$ which are:

- *locally finite* – there are finite substructures $\mathbf{A}_n \subseteq \mathbf{K}$ with $\mathbf{K} = \bigcup_n \mathbf{A}_n$,
- *ultrahomogeneous* – every isomorphism $f : \mathbf{A} \to \mathbf{B}$ between finite substructures of $\mathbf{K}$ extends to an automorphism of $\mathbf{K}$.

Examples of Fraïssé structures include the countably infinite set, the rational linear ordering, the random graph, and the countable atomless boolean algebra. See [KPT] for many more examples.

The most useful aspect of Fraïssé structures is that they are uniquely determined by their *age*, the class of finite structures which embed into $\mathbf{K}$. The major insight of [KPT] is that the dynamical properties of $\mathrm{Aut}(\mathbf{K})$ can be studied using the combinatorial properties of $\mathrm{Age}(\mathbf{K})$. Of particular importance is the notion of (structural) *Ramsey degree*:

- If $\mathbf{A}, \mathbf{B}$ are finite substructures, let $\binom{\mathbf{B}}{\mathbf{A}}$ denote the set of substructures of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. Let $\mathcal{K}$ be a class of finite structures, and for $n \in \mathbb{N}$, set $[n] = \{1, 2, ..., n\}$. We say that $\mathbf{A} \in \mathcal{K}$ has (structural) *Ramsey degree* $\leq k$ if for every $\mathbf{B} \in \mathcal{K}$ with $\binom{\mathbf{B}}{\mathbf{A}}$ nonempty and every $r \in \mathbb{N}$, there is $\mathbf{C}$ in $\mathcal{K}$ such that for every coloring $\gamma : \binom{\mathbf{C}}{\mathbf{A}} \to [r]$, there is $\mathbf{B}_0 \in \binom{\mathbf{C}}{\mathbf{B}}$ with $|\gamma(\binom{\mathbf{B_0}}{\mathbf{A}})| \leq k$.

If $\mathbf{A}$ has Ramsey degree 1, we say that $\mathbf{A}$ is a (structural) *Ramsey object*. We say that $\mathcal{K}$ has the (structural) *Ramsey Property* if every $\mathbf{A} \in \mathcal{K}$ is a Ramsey object. In section 4, we will introduce the (embedding) Ramsey Property, and most of this paper will use this rather than the structural version above. For now, we note the following for $\mathcal{K}$ a class of finite structures:

- $\mathcal{K}$ has the (embedding) *Ramsey Property* iff $\mathcal{K}$ has the (structural) Ramsey Property and consists of *rigid* structures, i.e. structures with no non-trivial automorphisms.
- $\mathbf{A} \in \mathcal{K}$ has finite (structural) Ramsey degree iff $\mathbf{A}$ has finite (embedding) Ramsey degree.

We can now state the first major theorem in [KPT].

Theorem 1.1. *Let $\mathbf{K}$ be a Fraïssé structure with $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$. Set $G = \mathrm{Aut}(\mathbf{K})$. Then the universal minimal flow $M(G)$ is a single point iff $\mathcal{K}$ has the embedding Ramsey Property.*

Topological groups G with $M(G)$ a single point are called *extremely amenable*. Another major theme of [KPT] is that if $\mathbf{K}$ is a Fraïssé structure with universe $\mathbb{N}$ and $G = \mathrm{Aut}(\mathbf{K})$ is not extremely amenable, we can often express $M(G)$ as a *logic action*. Let $X_{LO}^{\mathbf{K}}$ be the space of structures of the form $\langle \mathbf{K}, < \rangle$, where $<$ is a linear ordering of $\mathbb{N}$. We endow $X^{\mathbf{K}}_{LO}$ with the topology whose basic open neighborhoods are of the form $N(L) = \{\langle \mathbf{K}, < \rangle \in X^{\mathbf{K}}_{LO} : <|_{[k]} = L\}$, where L is some linear ordering of $[k] = \{1, 2, ..., k\}$. With this topology, $X^{\mathbf{K}}_{LO}$ is compact and metrizable. We let G act on $X^{\mathbf{K}}_{LO}$ via $\langle \mathbf{K}, < \rangle \cdot g = \langle \mathbf{K}, <_g \rangle$ where $(m <_g n)$ iff $(g(m) < g(n))$. This turns $X^{\mathbf{K}}_{LO}$ into a G-flow. When $\mathbf{K}$ is a Fraïssé structure, $<_0 \in X^{\mathbf{K}}_{LO}$ with $\langle \mathbf{K}, <_0 \rangle$ also a Fraïssé structure, and $G = \mathrm{Aut}(\mathbf{K})$, [KPT] provides a complete characterization of when $\overline{\langle \mathbf{K}, <_0 \rangle \cdot G} \cong M(G)$.

Nguyen Van Thé in [NVT] made the observation that by allowing more general logic actions, one can describe $M(G)$ for more groups. Let $\mathcal{S} = \{S_i\}_{i \in I}$ be a set with countably many new relation symbols of arity $n(i)$. Let $X^{\mathbf{K}}_{\mathcal{S}}$ be the space of structures of the form $\langle \mathbf{K}, \{S^{\mathbf{K}}_i : i \in I\} \rangle$. We endow $X^{\mathbf{K}}_{\mathcal{S}}$ with the topology whose basic open neighborhoods are the sets $N(\{T_i : i \in I\}) := \{\langle \mathbf{K}, \{S^{\mathbf{K}}_i : i \in I\} \rangle \in X^{\mathbf{K}}_{\mathcal{S}} : S^{\mathbf{K}}_i|_{[k]} = T_i\}$, where T_i is some interpretation of the relation symbol S_i on $[k]$. This topology is metrizable using a similar metric to the linear order case. The right logic action is also defined similarly; if $\mathbf{K}$ is a Fraïssé structure, $\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle \in X^{\mathbf{K}}_{\mathcal{S}}$ is also a Fraïssé structure, and $G = \mathrm{Aut}(\mathbf{K})$, [NVT] provides a complete characterization of when $\overline{\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle \cdot G}$ is compact and isomorphic to $M(G)$.

Given these constructions of $M(G)$, a natural question arises: suppose G is a closed subgroup of S_∞ and $M(G)$ is metrizable; then can $M(G)$ be described using the techniques of [KPT] and [NVT]? Some partial progress had been made addressing this question. If G is a topological group and $M(G)$ has a dense G_δ (i.e. generic) orbit, then we say G has the *Generic Point Property*. It can be shown (see Prop. 5.10) that if $M(G)$ has a generic orbit, then every minimal G-flow has a generic orbit. If $M(G) = \overline{\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle \cdot G}$ as above, then $M(G)$ has a dense G_δ orbit, namely $\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle \cdot G$. For Polish groups G with the generic point property and $M(G)$ metrizable, Melleray, Nguyen Van Thé, and Tsankov in [MNT] have shown that $M(G)$ must have a very particular structure (see section 9 for a brief discussion). In particular, when G is a closed subgroup of S_∞ with the generic point property and $M(G)$ metrizable, their result implies that $M(G)$ can be described using the techniques in [KPT] and [NVT]. Angel, Kechris, and Lyons in [AKL] conjectured that every Polish group G with $M(G)$ metrizable has the generic point property. This problem has come to be known as the *Generic Point Problem.*

In this paper, we show directly that if G a closed subgroup of S_∞ with $M(G)$ metrizable, then $M(G)$ can be constructed using the methods of [KPT] and [NVT]. Our main theorem (Theorem 8.14) is the following:

Theorem 1.2. *Let $\mathcal{K}$ be a Fraïssé class, $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, and $G = \mathrm{Aut}(\mathbf{K})$. Then the following are equivalent:*

1. *G has metrizable universal minimal flow,*
2. *Each $\mathbf{A} \in \mathcal{K}$ has finite Ramsey degree,*
3. *There is a countable set $\mathcal{S}$ of new relation symbols and $\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle \in X^{\mathbf{K}}_{\mathcal{S}}$ so that $\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle$ is also a Fraïssé structure and $M(G) \cong \overline{\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle \cdot G}$.*

As a consequence, this settles the Generic Point Problem for closed subgroups of S_∞ (Corollary 8.15).

Corollary 1.3. *Let G be a closed subgroup of S_∞ with metrizable universal minimal flow $M(G)$. Then G has the Generic Point Property.*

The paper is organized as follows. Sections 2 through 5 provide a review of topology, Fraïssé structures, structural Ramsey theory, and KPT correspondence, respectively. Section 6 provides a representation of the greatest ambit $(S(G), 1)$ for closed subgroups of S_∞, and section 7 gives a new proof of Theorems 1.1 and 5.1. Section 8 proves Theorem 1.2 and, for completeness, also gives a new proof of KPT-correspondence (Theorem 5.7). As a warning, sections 3,4, and 5, while mostly review, do contain some new notions. Section 3 introduces the notion of a *Fraïssé–HP* class (read "Fraïssé minus HP"), and section 5 discusses precompact expansions on Fraïssé–HP classes. Section 4 introduces the notions of (embedding) Ramsey Property/degree/object and contains some other new ideas and nonstandard vocabulary.

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2 Topological Preliminaries

In this section, we will discuss the topological tools needed going forward. We should note now that *all* topological spaces and groups are assumed to be Hausdorff unless explicitly stated otherwise; in particular, any results stated for a class of topological spaces should only be presumed to hold for those members of the class which are Hausdorff.

2.1 Topological Dynamics and Topological Semigroups

Let G be a topological group. A (right) *G-flow* is a pair (X, τ), where X is a compact space and $\tau : X \times G \to G$ is a continuous action, i.e. for every $x \in X$ and $g, h \in G$, we have $\tau(\tau(x, g), h) = \tau(x, gh)$. Typically the action τ is understood and suppressed, so we write $x \cdot g$ for $\tau(x, g)$, or simply xg when there is no confusion. Then we have the identity $x \cdot (gh) = (x \cdot g) \cdot h$. A *subflow* of X is a non-empty closed subspace $Y \subseteq X$ for which $y \cdot g \in Y$ for all $y \in Y$ and $g \in G$. As X is compact, we see that the intersection of a decreasing chain of subflows of X is itself a subflow. Applying Zorn's lemma, we see that X contains a *minimal* subflow Y, a flow containing no proper subflows. Notice that if Y is minimal and $y \in Y$, then the *orbit closure* $\overline{y \cdot G}$ is a subflow of Y, so we must have $\overline{y \cdot G} = Y$. More generally, a flow Y is minimal iff every orbit is dense.

If X and Y are G-flows, a G-map $f : X \to Y$ is a continuous map which respects the G-action, i.e. $f(x \cdot g) = f(x) \cdot g$ for each $x \in X$ and $g \in G$. Notice that the dots on the left and the right express different G-actions. An *isomorphism* of G-flows is a bijective G-map (by compactness, the inverse is continuous, hence also a G-map). A flow X is *universal* iff for each minimal flow Y, there is a G-map $f : X \to Y$. It is a fact that every topological group G admits a unique *universal minimal flow* $M(G)$ up to G-flow isomorphism. The rest of this section will be spent proving this fact. The proof we will use is to first prove the existence and uniqueness of the greatest G-ambit $S(G)$. Then any minimal subflow of $S(G)$ is universal, and we will show that any universal minimal flow is isomorphic to any minimal flow of $S(G)$.

A *G-ambit* (X, x_0) consists of a G-flow X and a distinguished point $x_0 \in X$ with dense orbit. A typical example of a G-ambit is the *orbit closure*: start with any G-flow X and any $x_0 \in X$, then $(\overline{x_0 \cdot G}, x_0)$ is a G-ambit. For ambits (X, x_0) and (Y, y_0), a map of G-ambits is a G-map $f : X \to Y$ with $f(x_0) = y_0$. Notice that if there is a map of G-ambits $f : (X, x_0) \to (Y, y_0)$, it must be unique since f is determined on the dense set $x_0 \cdot G$. The greatest ambit $(S(G), 1)$ is characterized by being universal for the class of G-ambits, i.e. for any G-ambit (X, x_0), there is a map of G-ambits $f : (S(G), 1) \to (X, x_0)$. Since maps between ambits are unique, the greatest ambit, should it exist, is unique up to a unique isomorphism of G-ambits. The following theorem is well known:

Theorem 2.1. *For any topological group G, there exists a greatest G-ambit* $(S(G), 1)$.

One of the major advantages of considering the greatest ambit is that it carries the structure of a left-topological semigroup: we say that a semigroup S is a *left-topological* semigroup if S is a compact topological space in which left multiplication is continuous, i.e. for each $s \in S$, the map $\lambda_s : S \to S$ with $\lambda_s(t) = st$ is continuous. A *right ideal* of S is a non-empty subset $I \subseteq S$ with $IS \subseteq I$. A right ideal is *minimal* if it does not properly contain any right ideals. Equivalently, I is a minimal right ideal iff $xS = I$ for every $x \in I$; in particular, since $xS = \lambda_x(S)$ and S is compact, minimal right ideals are always closed (closed always refers to the topology; we will write $I^2 \subseteq I$ when we mean closed with respect to the operation). A quick Zorn's lemma proof shows that minimal right ideals always exist in left-topological semigroups.

If G is a topological group, we can give the greatest ambit $(S(G), 1)$ a left-topological semigroup structure as follows. If $x, y \in S(G)$ and we want to define xy, consider the orbit closure $X := \overline{x \cdot G}$. Then (X, x) is an ambit, and there is a unique map of G-ambits $f_x : (S(G), 1) \to (X, x)$. Then we can define $xy = f_x(y)$. Associativity follows once we note that $f_x \circ f_y = f_{xy}$. Notice that f_x is continuous and for $g \in G$, we have $x \cdot g = f_x(1 \cdot g)$. Any constructive proof of the existence of the greatest ambit (see section 1 of [KPT], for instance) shows that the map $g \to 1 \cdot g$ is a homeomorphism of G onto its image, and it is common to identify G as a subspace of $S(G)$. Using this identification, we see that the closed right ideals of $S(G)$ are exactly the subflows, and the minimal right ideals are exactly the minimal subflows.

Notice that if Y is another minimal G-flow, we can turn Y into a G-ambit by distinguishing any $y \in Y$. Let $\varphi : S(G) \to Y$ be the unique map of ambits. If $I \subseteq S(G)$ is a minimal right ideal, then as Y is minimal, $\varphi|_I$ must be surjective. Hence I is a universal minimal flow. It can be shown (see, for example, Uspenskij [U] section 3) that any G-map of a minimal right ideal (i.e. minimal subflow) of $S(G)$ to itself is an isomorphism; this shows that the universal minimal flow $M(G)$ is unique up to G-flow isomorphism.

See Auslander [A] or Uspenskij [U] for a more detailed exposition of topological dynamics and the universal minimal flow. See [HS] or [EEM] for more on topological semigroups.

2.2 Filters, Ultrafilters, and the β-compactification

Let X be a set. A *filter* on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following:

- $\mathcal{F}$ is nontrivial: $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,
- $\mathcal{F}$ is upwards closed: if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$,
- $\mathcal{F}$ is closed under finite intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Notice that the union of a chain of filters is also a filter, so by Zorn's Lemma, every filter is contained in a maximal filter. These are called *ultrafilters*. Equivalently, ultrafilters are those filters which contain A or $X \setminus A$ for every $A \subseteq X$. The prototypical example of an ultrafilter is a *principal* ultrafilter, one of the form $p_x := \{A \subseteq X : x \in A\}$ for some fixed $x \in X$.

Let X and Y be sets, $f : X \to Y$ any function, $g : X \to Y$ a surjective function, $\mathcal{F}$ a filter on X, and $\mathcal{G}$ a filter on Y. Then $f(\mathcal{F})$, the *push forward* of $\mathcal{F}$, is the filter on Y with $A \in f(\mathcal{F})$ iff $f^{-1}(A) \in \mathcal{F}$. The *pre-image filter* $g^{-1}(\mathcal{G})$ is the filter on X generated by the sets $g^{-1}(B)$ for $B \in \mathcal{G}$. The push forwards of ultrafilters are ultrafilters, but pre-images of ultrafilters are typically just filters.

The dual notion to a filter is an *ideal*, a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ which is nontrivial ($\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$), downwards closed, and closed under finite unions. $\mathcal{I}$ is an ideal iff $\{A \subseteq X : X \setminus A \in \mathcal{I}\}$ is a filter; we call this the *dual filter* of $\mathcal{I}$. Every ideal is contained in a maximal ideal, and $\mathcal{I}$ is a maximal ideal iff $\{A \subset X : X \setminus A \in \mathcal{I}\}$ is an ultrafilter.

Denote the space of ultrafilters on X by βX; we endow βX with the topology whose basic open sets are of the form $\overline{A} := \{p \in \beta X : A \in p\}$ for $A \subseteq X$. Notice that each of these basic open sets is closed, $\overline{A} = cl(A)$, and $\overline{A} \sqcup \overline{X \setminus A} = \beta X$. We can identify X as a subspace of βX by identifying each $x \in X$ with the principal ultrafilter p_x. Notice that $\{p_x\} = \overline{\{x\}}$, so under the identification, X is an open, discrete subspace of βX.

The main property of the space βX is that it is the *Stone-Čech compactification* of X; this is to say that βX is a compact Hausdorff space into which X embeds densely via inclusion, and furthermore, if Y is another compact Hausdorff space and $f : X \to Y$ is any function, there is a unique continuous extension $\tilde{f} : \beta X \to Y$ making the following diagram commute:

$$\begin{tikzcd}
\beta X \arrow[dr, "\tilde{f}"] & \\
X \arrow[u, "i"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Let G be an infinite discrete group, and form βG. We can give βG a left-topological semigroup structure as follows:

- For each fixed $g \in G$, the map $h \to hg$ has a unique continuous extension to βG,
- For each fixed $p \in \beta G$, the map $h \to ph$ has a unique continuous extension to βG.

Associativity must be verified, but is straightforward. Note that it was arbitrary whether we started with left or right multiplication; however, you can only choose one of left or right multiplication to be continuous. Here, we have chosen a semigroup structure where right multiplication by elements of G is continuous and left multiplication by any $p \in \beta G$ is continuous. We can also identify elements of βG with ultrafilters on G. For $A \subseteq G$ and $p, q \in \beta G$, we have $A \in pq$ iff $\{h \in G : Ah^{-1} \in p\} \in q$. In particular, if $p \in \beta G$ and $g \in G$, we have $A \in pg$ iff $Ag^{-1} \in p$. For more on semigroup compactifications, see [HS] or [EEN].

A brief discussion of the topological properties of the space βX for X discrete is in order. For our purposes, one of the most useful facts about βX is that it is *extremely disconnected*, i.e. the closure of every open set is open. The consequence of this that we are interested in is the following: any compact, extremely disconnected space embeds no nontrivial metric spaces (see Theorem 3.40 of [HS]). A general fact from topology (see [W], p. 166) says that the continuous image of a compact metric space in a Hausdorff space is metrizable. Therefore exhibiting lots of continuous maps from a compact space of interest into βX for various X is a useful tool; we will use this in section 8.

3 Fraïssé structures

We now move towards the case we will consider, where G is a closed subgroup of S_∞. In this section, we describe a canonical way of viewing any such group. Recall that S_∞ is the group of permutations of $\mathbb{N}$ endowed with the pointwise convergence topology; a basis of open sets at the identity is given by G_n, the pointwise stabilizer of $\{1, 2, ..., n\}$. A compatible left-invariant metric is given by $d(g, h) = 1/n$ iff n is least with $g^{-1}h \notin G_n$.

A *language* $L = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$ is a collection of relation, function and constant symbols. Each relation symbol R_i has an *arity* $n_i \in \mathbb{N}$, as does each function symbol f_j. An *L-structure* $\mathbf{A} = \langle A, R_i^{\mathbf{A}}, f_j^{\mathbf{A}}, c_k^{\mathbf{A}}\rangle$ consists of a set A, relations $R_i^{\mathbf{A}} \subseteq A^{n_i}$, functions $f_j^{\mathbf{A}} : A^{n_j} \to A$, and constants $c_k^{\mathbf{A}} \in A$; we say that $\mathbf{A}$ is an L-structure on A. If $\mathbf{A}, \mathbf{B}$ are L-structures, then $g : \mathbf{A} \to \mathbf{B}$ is an *embedding* if g is a map from A to B such that $R_i^{\mathbf{A}}(x_1, ..., x_{n_i}) \Leftrightarrow R_i^{\mathbf{B}}(g(x_1), ..., g(x_{n_i}))$, $g(f_j^{\mathbf{A}}(x_1, ..., x_{n_j})) = f_j^{\mathbf{B}}(g(x_1), ..., g(x_{n_j}))$, and $g(c_k^{\mathbf{A}}) = c_k^{\mathbf{B}}$ for all relations, functions, and constants, respectively. If there is an embedding $g : \mathbf{A} \to \mathbf{B}$, we say that $\mathbf{B}$ *embeds* $\mathbf{A}$. An *isomorphism* is a bijective embedding, and an *automorphism* is an isomorphism between a structure and itself. If $A \subseteq B$, then we say that $\mathbf{A}$ is a *substructure* of $\mathbf{B}$, written $\mathbf{A} \subseteq \mathbf{B}$, if the inclusion map is an embedding. $\mathbf{A}$ is finite, countable, etc. if A is.

Let $\mathbf{K}$ be a countably infinite L-structure. We say that $\mathbf{K}$ is *locally finite* if there are finite substructures $\mathbf{A}_n \subseteq \mathbf{K}$ with $\mathbf{A}_n \subseteq \mathbf{A}_{n+1}$ and $\mathbf{K} = \bigcup_n \mathbf{A}_n$. Then $\bigcup_n \mathbf{A}_n$ is said to be an *exhaustion* of $\mathbf{K}$. We set $\mathrm{Fin}(\mathbf{K})$ to be the set of finite substructures of $\mathbf{K}$, and we set $\mathcal{K} = \mathrm{Age}(\mathbf{K})$, the *age* of $\mathbf{K}$, to be the class of finite L-structures which embed into $\mathbf{K}$, i.e. those structures isomorphic to some structure in $\mathrm{Fin}(\mathbf{K})$. It is natural to ask which classes of finite structures are the age of a countably infinite locally finite structure. If $\mathcal{K}$ is a class of finite structures, we call $\mathcal{K}$ an *age class* if $\mathcal{K}$ satisfies the following:

- $\mathcal{K}$ is closed under isomorphism, contains countably many isomorphism types, and contains structures of arbitrarily large finite cardinality

- $\mathcal{K}$ has the *Hereditary Property* (HP): if $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{A} \in \mathcal{K}$,
- $\mathcal{K}$ satisfies the *Joint Embedding Property* (JEP): if $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, then there is $\mathbf{C} \in \mathcal{K}$ which embeds both $\mathbf{A}$ and $\mathbf{B}$.

It is not hard to verify that $\mathcal{K}$ is an age class iff $\mathcal{K} = \mathrm{Age}(\mathbf{K})$ for $\mathbf{K}$ some countably infinite locally finite structure. In general, there could be many non-isomorphic $\mathbf{K}$ that work.

A countably infinite locally finite structure $\mathbf{K}$ is a *Fraïssé* structure if $\mathbf{K}$ is *ultrahomogeneous*:

- For any $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$ and any embedding $g : \mathbf{A} \to \mathbf{K}$, there is an automorphism of $\mathbf{K}$ extending g.

Another useful, equivalent definition is that $\mathbf{K}$ is a Fraïssé structure iff it is countably infinite, locally finite, and for every $\mathbf{A} \subseteq \mathbf{B} \in \mathrm{Age}(\mathbf{K})$, every embedding $g : \mathbf{A} \to \mathbf{K}$ can be extended to an embedding $h : \mathbf{B} \to \mathbf{K}$. This is often called the *extension property* for $\mathbf{K}$. The proof that this is equivalent to ultrahomogeneity uses a standard technique known as the *back and forth* method. The back and forth method is also used to show that if two Fraïssé structures have the same age, then they are isomorphic. However, not all age classes are the ages of Fraïssé structures. A class of finite structures $\mathcal{K}$ is a *Fraïssé class* if $\mathcal{K}$ is an age class which additionally satisfies the *Amalgamation Property* (AP):

- If $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ are embeddings, there is $\mathbf{D} \in \mathcal{K}$ and embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ with $r \circ f = s \circ g$.

It is actually enough in the definition of AP to take f, g, and r to be inclusion maps. The following theorem is the starting point for Fraïssé theory:

Theorem 3.1 (Fraïssé). *$\mathcal{K}$ is a Fraïssé class iff $\mathcal{K} = \mathrm{Age}(\mathbf{K})$ for some Fraïssé structure $\mathbf{K}$. Furthermore, each Fraïssé class $\mathcal{K}$ admits up to isomorphism a unique Fraïssé structure $\mathbf{K}$ with $\mathcal{K} = \mathrm{Age}(\mathbf{K})$.*

If $\mathcal{K}$ is a Fraïssé class and $\mathbf{K}$ is the unique structure guaranteed by Theorem 3.1, then we write $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, the *Fraïssé limit* of $\mathcal{K}$.

We can also define Fraïssé limits of more general classes. If $\mathbf{K}$ is a countable structure and $\mathcal{K} \subseteq \mathrm{Age}(\mathbf{K})$ is closed under isomorphism, we say that $\mathbf{K}$ is $\mathcal{K}$-homogeneous if any partial isomorphism of structures in $\mathcal{K}$ can be extended to an automorphism of $\mathbf{K}$. Most often, we will use this added generality when $\mathcal{K}$ is a *Fraïssé–HP class*; i.e. a class of finite structures which satisfies every condition of being a Fraïssé class except possibly the Hereditary Property. If $\mathcal{K}$ is a class of structures which is not necessarily hereditary, let $\mathcal{K}{\downarrow} := \{\mathbf{A} : \exists \mathbf{B} \in \mathcal{K}(\mathbf{A} \subseteq \mathbf{B})\}$. Now if $\mathcal{K}$ is a Fraïssé–HP class, a similar back and forth proof shows that up to isomorphism, there is a unique countably infinite locally finite structure with age $\mathcal{K}{\downarrow}$ which is $\mathcal{K}$-homogeneous; we will also call this the Fraïssé limit.

Our interest in Fraïssé structures stems from the following:

Theorem 3.2. *G is a closed subgroup of S_∞ iff G is the automorphism group of a relational Fraïssé structure on $\mathbb{N}$.*

Proof. If $\mathbf{K}$ is a relational Fraïssé structure and $G = \mathrm{Aut}(\mathbf{K})$, then if $g_n \in G$ and $g_n \to g$ with $g \in S_\infty$, then g must also be an automorphism of $\mathbf{K}$ and hence in G. Conversely, suppose G is a closed subgroup of S_∞. For every $\overline{a} \in {}^{<\omega}\mathbb{N}$, introduce a relational symbol $R_{\overline{a}}$ of arity $\mathrm{len}(\overline{a})$, and let $L = \{R_{\overline{a}} : \overline{a} \in {}^{<\omega}\mathbb{N}\}$. Give $\mathbb{N}$ an L-structure by declaring that $R_{\overline{a}}(b_1, ..., b_n)$ iff there is $g \in G$ with $g(a_i) = b_i$ for each $i \leq n$. Then $\mathbf{K} = \langle \mathbb{N}, \{R_{\overline{a}}^{\mathbf{K}} : \overline{a} \in {}^{<\omega}\mathbb{N}\}\rangle$ is a Fraïssé structure with $\mathrm{Aut}(\mathbf{K}) = G$. $\square$

For a more detailed exposition of Fraïssé theory, see [Ho].

4 Structural Ramsey Theory

In this section, we introduce some of the ideas underlying structural Ramsey theory. However, we begin with a discussion of Ramsey theory for embeddings, as this is what we will use in the rest of the paper. Proposition 4.4 makes the connection between the structural and embedding versions explicit.

A *partial k-coloring* γ of a set X is a function $\gamma : Y \to [k]$, where $Y \subseteq X$ and $[k] = \{1, 2, ..., k\}$. A coloring γ of X is *full* if $\mathrm{dom}(\gamma) = X$. We will often write γ_i for $\gamma^{-1}(i)$. If $\mathrm{dom}(\gamma)$ is unspecified, then γ is presumed to be a full coloring. If γ is a coloring of X and $Y \subseteq \mathrm{dom}(\gamma)$, we say that Y is *monochromatic* if $Y \subseteq \gamma_i$ for some i.

If $\mathbf{A}, \mathbf{B}$ are L-structures, write $\mathrm{Emb}(\mathbf{A}, \mathbf{B})$ for the set of embeddings from $\mathbf{A}$ to $\mathbf{B}$, and write $\mathbf{A} \leq \mathbf{B}$ if $\mathrm{Emb}(\mathbf{A}, \mathbf{B}) \neq \emptyset$. If $\mathcal{C}$ is a class of finite L-structures, we say that $\mathbf{A} \in \mathcal{C}$ is a *Ramsey object* if for any $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$, there is $\mathbf{C} \in \mathcal{C}$, $\mathbf{A} \leq \mathbf{C}$, such that for any full 2-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{C})$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ monochromatic. We say that $\mathcal{C}$ has the *Ramsey Property* (RP) if each $\mathbf{A} \in \mathcal{C}$ is a Ramsey object. The choice of 2 colors is arbitrary; a straightforward induction on the number of colors shows that if $\mathbf{A} \in \mathcal{C}$ is a Ramsey object, then for any $k \geq 2$ and any $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$, there is a $\mathbf{C} \in \mathcal{C}$ such that for any k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{C})$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ monochromatic.

Once again, we are using an embedding version of Ramsey object/Ramsey property, as opposed to the structural version defined in the introduction. A useful translation between the two versions is as follows: suppose $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to [r]$ is a coloring which additionally has $\gamma(f) = \gamma(g)$ whenever $f = g \circ h$ for some $h \in \mathrm{Aut}(\mathbf{A})$. Let us call such a γ a *structural coloring*. Then we may define $\gamma' : \binom{\mathbf{C}}{\mathbf{A}} \to [r]$ via $\gamma'(\mathbf{A}_0) = \gamma(f)$ for any $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{C})$ with $\mathrm{Im}(f) = \mathbf{A}_0$. Conversely, suppose $\gamma : \binom{\mathbf{C}}{\mathbf{A}} \to [r]$ is a coloring. Then we can define $\gamma' : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to [r]$ via $\gamma'(f) = \gamma(\mathrm{Im}(f))$. Notice that this γ' is a structural coloring. In what follows, should "embedding" or "structural" not be specified, "Ramsey" will always refer to embedding Ramsey. We will borrow the hook-arrow notation used in [MP],

$$\mathbf{C} \hookrightarrow (\mathbf{B})_k^{\mathbf{A}}$$

to mean that for any full k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{C})$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ monochromatic. We use the standard arrow notation,

$$\mathbf{C} \to (\mathbf{B})_k^{\mathbf{A}}$$

to mean that for any full k-coloring of $\binom{\mathbf{C}}{\mathbf{A}}$, there is $\mathbf{B}_0 \in \binom{\mathbf{C}}{\mathbf{B}}$ with $\binom{\mathbf{B}_0}{\mathbf{A}}$ monochromatic.

If $\mathcal{C}$ and $\mathcal{D}$ are classes of structures, we say that $\mathcal{C}$ is *cofinal* in $\mathcal{D}$ if for any $\mathbf{A} \in \mathcal{D}$, there is $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$. Suppose that $\mathcal{D}$ is the age of a countably infinite locally finite structure $\mathbf{D}$ and that $\mathcal{C}$ is cofinal in $\mathcal{D}$. For $\mathbf{A} \in \mathcal{C}$, we say that $S \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is *thick* if for any $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$, there is an $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}) \subseteq S$. We say a partial coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is *large* if $\mathrm{dom}(\gamma)$ is thick.

Proposition 4.1. *Suppose* $\mathbf{D}$ *is a countably infinite locally finite structure,* $\mathcal{D} = \mathrm{Age}(\mathbf{D})$*, and* $\mathcal{C}$ *is cofinal in* $\mathbf{D}$. *Let* $\mathbf{A} \in \mathcal{C}$ *and fix any* $k \geq 2$. *Then the following are equivalent:*

1. $\mathbf{A}$ *is a Ramsey object in* $\mathcal{C}$*,*
2. $\mathbf{A}$ *is a Ramsey object in* $\mathcal{D}$*,*
3. *For any full* k*-coloring* γ *of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$*, there is some* γ_i *which is thick,*
4. *For any large* k*-coloring* γ *of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$*, there is some* γ_i *which is thick.*

Proof. $(1 \Leftrightarrow 2)$ and $(4 \Rightarrow 3)$ are straightforward.

For $(2 \Rightarrow 4)$, fix γ a large k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$. Say $\mathbf{A} \leq \mathbf{B} \in \mathcal{D}$, and fix $\mathbf{C} \in \mathcal{D}$ for which $\mathbf{C} \hookrightarrow (\mathbf{B})^{\mathbf{A}}_k$ holds. Since γ is large, find $f \in \mathrm{Emb}(\mathbf{C}, \mathbf{D})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \subseteq \mathrm{dom}(\gamma)$. Then find $x \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $f \circ x \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ monochromatic. For each $i \leq k$, let $\mathcal{D}_i \subseteq \mathcal{D}$, where $\mathbf{B} \in \mathcal{D}_i$ iff there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}) \subseteq \gamma_i$. We have just shown that each $\mathbf{B} \geq \mathbf{A}$ is in some $\mathcal{D}_i$.

Suppose for sake of contradiction that no $\mathcal{D}_i$ was cofinal in $\mathcal{D}$. For each $i \leq k$, pick $\mathbf{A}_i \in \mathbf{D}$, $\mathbf{A} \leq \mathbf{A}_i$, so that any $\mathbf{B}' \in \mathcal{D}$ which embeds $\mathbf{A}_i$ is not in $\mathcal{D}_i$. Now use JEP for $\mathcal{D}$ to find $\mathbf{A}'$ embedding each $\mathbf{A}_i$. But $\mathbf{A}' \in \mathcal{D}_i$ for some $i \leq k$, so this is a contradiction. Now observe that each $\mathcal{D}_i$ is hereditarily closed, so if $\mathcal{D}_i$ is cofinal, then $\mathcal{D}_i = \mathcal{D}$. This means that γ_i must be thick, so we are done.

For $(3 \Rightarrow 2)$, let $\mathbf{D} = \bigcup_n \mathbf{B}_n$ be an exhaustion with $\mathbf{A} \leq \mathbf{B}_1$. Suppose $\mathbf{B} \in \mathcal{D}$ witnesses the fact that $\mathbf{A}$ is not a Ramsey object. Call a coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_n)$ *bad* if there is no $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ monochromatic. So for each n, there is a bad k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_n)$. In particular, if γ is a bad k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_n)$ and $m \leq n$, the restriction of γ to $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_m)$ is also bad. We can now use König's lemma to find a bad full k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$. □

Often, we will use Proposition 4.1 with a Fraïssé structure $\mathbf{K}$, where we can say more.

Lemma 4.2. *Let* $\mathbf{K}$ *be a Fraïssé structure with* $\mathcal{K} = \mathrm{Age}(\mathbf{K})$. *Suppose* $\mathbf{A}, \mathbf{B} \in \mathcal{K}$*, and let* $f : \mathbf{A} \to \mathbf{B}$ *be an embedding. If* $S \subseteq \mathrm{Emb}(\mathbf{B}, \mathbf{K})$ *is thick, then* $T := \{x \circ f : x \in S\} \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ *is also thick.*

Proof. Fix $\mathbf{C} \in \mathrm{Fin}(\mathbf{K})$. By repeated use of the extension property, find $\mathbf{D} \in \mathrm{Fin}(\mathbf{K})$, $\mathbf{C} \subseteq \mathbf{D}$, such that for each $g \in \mathrm{Emb}(\mathbf{A}, \mathbf{C})$, there is $h \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$ with $g = h \circ f$ (here we view $\mathrm{Emb}(\mathbf{A}, \mathbf{C}) \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ in the natural way). Now as S is thick, find $x \in \mathrm{Emb}(\mathbf{D}, \mathbf{K})$ with $x \circ \mathrm{Emb}(\mathbf{B}, \mathbf{D}) \subseteq S$. Then $x|_{\mathbf{C}} \in \mathrm{Emb}(\mathbf{C}, \mathbf{K})$, and $x|_{\mathbf{C}} \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \subseteq T$. □

Proposition 4.3. *Let* $\mathbf{K}$ *be a Fraïssé structure with* $\mathcal{K} = \mathrm{Age}(\mathbf{K})$*, and suppose* $\mathbf{B} \in \mathcal{K}$ *is a Ramsey object. Then if* $\mathbf{A} \leq \mathbf{B}$*, then* $\mathbf{A}$ *is a Ramsey object.*

Proof. Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding, and fix γ a full 2-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$. Let δ be the full 2-coloring of $\mathrm{Emb}(\mathbf{B}, \mathbf{K})$ defined by $\delta(x) = \gamma(x \circ f)$. For some i, δ_i is thick. Then by Lemma 4.2, $\gamma_i = \{x \circ f : x \in \delta_i\}$ is thick. □

We say that $\mathbf{A} \in \mathcal{C}$ has *Ramsey degree* k if k is least such that for any $\mathbf{B}$ in $\mathcal{C}$ with $\mathbf{A} \le \mathbf{B}$ and any $r > k$, there is $\mathbf{C} \in \mathcal{C}$ such that for any r-coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{C})$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ such that $|\gamma(f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}))| \le k$. One could define the notion of an (r, k)-Ramsey object, which would be defined just as above for some particular $r > k$. However, this is unnecessary; an induction on r shows that $\mathbf{A}$ is an (r, k)-Ramsey object iff $\mathbf{A}$ is a $(k + 1, k)$-Ramsey object. Therefore the notion of Ramsey degree is sufficient. We use a similar hook-arrow notation,

$$\mathbf{C} \hookrightarrow (\mathbf{B})^{\mathbf{A}}_{r,k}$$

to mean that for every r-coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{C})$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $|\gamma(f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}))| \le k$. We use the standard arrow notation,

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{r,k}$$

to mean that for every r-coloring γ of $\binom{\mathbf{C}}{\mathbf{A}}$, there is $\mathbf{B}_0 \in \binom{\mathbf{C}}{\mathbf{B}}$ with $|\gamma(\binom{\mathbf{B}_0}{\mathbf{A}})| \le k$.

Proposition 4.4. *$\mathbf{A} \in \mathcal{C}$ has structural Ramsey degree k iff $\mathbf{A}$ has embedding Ramsey degree $k \cdot |\mathrm{Aut}(\mathbf{A})|$.*

Proof. Set $t = |\mathrm{Aut}(\mathbf{A})|$, and fix $r > kt$. Assume $\mathbf{A}$ has structural-Ramsey degree k, and let $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \le \mathbf{B}$. Find $\mathbf{C} \in \mathcal{C}$ with $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{2^r,k}$. Fix any r-coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{C})$. Fix a bijection $\varphi : \mathcal{P}([r]) \to [2^r]$, and define a (2^r)-coloring δ with $\delta(f) = \varphi(\{i \in [r] : \exists h \in \mathrm{Aut}(\mathbf{A})(f \circ h \in \gamma_i)\})$. Then δ is a structural coloring, so find $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ at most k-colored for δ. Then $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ is at most (kt)-colored for γ.

Now since $\mathbf{A}$ has structural-Ramsey degree k, find $\mathbf{D} \in \mathcal{C}$ such that for every $\mathbf{E} \in \mathcal{C}$, there is a structural r-coloring $\gamma_{\mathbf{E}}$ of $\mathrm{Emb}(\mathbf{A}, \mathbf{E})$ where for every $f \in \mathrm{Emb}(\mathbf{D}, \mathbf{E})$, the set $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is at least k-colored. Then one can use $\gamma_{\mathbf{E}}$ to find an (rt)-coloring $\delta_{\mathbf{E}}$ of $\mathrm{Emb}(\mathbf{A}, \mathbf{E})$ where each $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is at least (kt)-colored. One way to do this is as follows: first fix a bijection $\rho : \mathrm{Aut}(\mathbf{A}) \to [t]$. Form the equivalence relation $\sim$ on $\mathrm{Emb}(\mathbf{A}, \mathbf{E})$ where $f \sim g$ iff $f = g \circ h$ for some $h \in \mathrm{Aut}(\mathbf{A})$. Let $\langle f \rangle$ denote the equivalence class of f. Let $\psi : (\mathrm{Emb}(\mathbf{A}, \mathbf{E})/\sim) \to \mathrm{Emb}(\mathbf{A}, \mathbf{E})$ choose a member of each equivalence class. Now if $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{E})$ and $h \in \mathrm{Aut}(\mathbf{A})$ is the unique automorphism with $\psi(\langle f \rangle) \circ h = f$, then set $\delta_{\mathbf{E}}(f) = (\gamma_{\mathbf{E}}(f) - 1)t + \rho(h)$.

Conversely, if $\mathbf{A}$ has finite embedding-Ramsey degree, then $\mathbf{A}$ also has finite structural-Ramsey degree, completing the proof. □

Corollary 4.5. *$\mathbf{A} \in \mathcal{C}$ is an embedding Ramsey object iff $\mathbf{A}$ is a structural Ramsey object and is rigid.*

The proof of the following proposition is nearly identical to the proof of Proposition 4.1 and is therefore omitted:

Proposition 4.6. *Suppose $\mathbf{D}$ is a countably infinite locally finite structure, $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and $\mathcal{C}$ is cofinal in $\mathcal{D}$. Let $\mathbf{A} \in \mathcal{C}$, and fix $r > k$. Then the following are equivalent:*

1. *$\mathbf{A}$ has Ramsey degree $t \le k$ in $\mathcal{C}$,*

2. *$\mathbf{A}$ has Ramsey degree $t \le k$ in $\mathcal{D}$,*

3. *Any full r-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ has some subset of k or fewer colors which form a thick subset,*

4. *Any large r-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ has some subset of k or fewer colors which form a thick subset.*

There is a similar analogue to Proposition 4.3, which we also state without proof.

Proposition 4.7. *Suppose $\mathbf{K}$ is a Fraïssé structure, $\mathcal{K} = \mathrm{Age}(\mathbf{K})$, and suppose $\mathbf{B} \in \mathcal{K}$ has Ramsey degree k. Then if $\mathbf{A} \le \mathbf{B}$, then $\mathbf{A}$ has Ramsey degree $t \le k$.*

Using Proposition 4.6, we can provide another definition of Ramsey degree which will be extremely useful going forward. Let $\mathbf{D}$ be a countably infinite locally finite structure with $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and let $\mathbf{A} \in \mathcal{D}$. Call a subset $S \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ *syndetic* if $S \cap X \neq \emptyset$ for every thick $X \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$. Call a coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ a *syndetic coloring* if each γ_i is syndetic. Now we have:

Proposition 4.8. *$\mathbf{A}$ has Ramsey degree $t \ge k$ (t possibly infinite) iff there is a syndetic k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$.*

Remark. The words *thick* and *syndetic* come from topological dynamics. If G is an infinite group and $S, T \subseteq G$, we say T is *thick* if the collection $\{Tg : g \in G\}$ has the finite intersection property, and we say S is *syndetic* if $G \setminus S$ is not thick.

Suppose $\mathbf{K}$ is a Fraïssé structure with $G = \mathrm{Aut}(\mathbf{K})$ and $\mathbf{A} \in \mathrm{Fin}(\mathbf{K})$. If $X \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$, then X is thick (resp. syndetic) according to our definition iff $\{g \in G : g|_{\mathbf{A}} \in X\}$ is thick (resp. syndetic) in the usual sense.

5 KPT Correspondence

We have now developed enough background to state the results in [KPT]. Our discussion will have two notable differences however. First we will be using embedding Ramsey throughout. Second, we will develop the theory using Fraïssé–HP classes, as this will allow us more flexibility in section 8. Later, we will provide new proofs of Theorems 5.1 and 5.7 (see Theorem 7.3 and the discussion after Corollary 8.15).

Theorem 5.1. *If $\mathcal{K}$ is a Fraïssé–HP class, $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, and $G = \mathrm{Aut}(\mathbf{K})$, then $M(G)$ is a singleton iff $\mathcal{K}$ has the Ramsey Property.*

Let L be a language and $L^* = L \cup \mathcal{S}$, where $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ and the S_i are new relational symbols of arity $n(i)$. If $\mathbf{A}$ is an L^*-structure, write $\mathbf{A}|_L$ for the structure obtained by throwing away the interpretations of the S_i. If $\mathcal{K}^*$ is a class of L^*-structures, set $\mathcal{K}^*|_L = \{\mathbf{A}^*|_L : \mathbf{A}^* \in \mathcal{K}^*\}$. If $\mathcal{K} = \mathcal{K}^*|_L$ and $\mathcal{K}^*$ is closed under isomorphism, we say that $\mathcal{K}^*$ is an *expansion* of $\mathcal{K}$. If $\mathbf{A}^* \in \mathcal{K}^*$ and $\mathbf{A}^*|_L = \mathbf{A}$, then we say that $\mathbf{A}^*$ is an expansion of $\mathbf{A}$. If $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ and $\mathbf{B}^*$ is an expansion of $\mathbf{B}$, we let $\mathbf{A}(f, \mathbf{B}^*)$ be the unique expansion of $\mathbf{A}$

so that $f \in \mathrm{Emb}(\mathbf{A}(f, \mathbf{B}^*), \mathbf{B}^*)$. The expansion $\mathcal{K}^*$ is *precompact* if for each $\mathbf{A} \in \mathcal{K}$, the set $\{\mathbf{A}^* \in \mathcal{K}^* : \mathbf{A}^*|_L = \mathbf{A}\}$ is finite.

If $\mathcal{K}^*$ is an expansion of the class $\mathcal{K}$, it will be useful to think of the pair $(\mathcal{K}^*, \mathcal{K})$ as a category as follows. If $X \subseteq \mathcal{K}$ is a set (as opposed to a proper class), say that X is *adequate* for $\mathcal{K}$ if X contains at least one representative of each isomorphism type in $\mathcal{K}$. For X adequate, let $\mathrm{Cat}_X(\mathcal{K}^*, \mathcal{K})$ be the category C with $\mathrm{Ob}(C) = \{\mathbf{A}^* : \mathbf{A}^* \text{ is an expansion of some } \mathbf{A} \in X\}$ and with $\mathrm{Arr}(C)$ the set of embeddings between structures in $\mathrm{Ob}(C)$. If $\mathcal{K}^*$ and $\mathcal{K}^{**}$ are two expansions of the class $\mathcal{K}$ in languages L^* and L^{**}, we say that $\mathcal{K}^*$ and $\mathcal{K}^{**}$ are *isomorphic expansions* if for some adequate X, there is a fully faithful functor $\Phi_X : \mathrm{Cat}_X(\mathcal{K}^*, \mathcal{K}) \to \mathrm{Cat}_X(\mathcal{K}^{**}, \mathcal{K})$ with fully faithful inverse satisfying

- $\Phi_X(\mathbf{A}^*)|_L = \mathbf{A}^*|_L$ for any $\mathbf{A}^* \in \mathrm{Ob}(\mathrm{Cat}_X(\mathcal{K}^*, \mathcal{K}))$,
- $\Phi_X(f) = f$ for any $f \in \mathrm{Arr}(\mathrm{Cat}(\mathcal{K}^*, \mathcal{K}))$.

We will call such a Φ_X an *isomorphism of expansions*. Notice that L^* need not equal L^{**} for $\mathcal{K}^*$ and $\mathcal{K}^{**}$ to be isomorphic.

Proposition 5.2. *If $\mathcal{K}^*$ and $\mathcal{K}^{**}$ are isomorphic, then for any adequate X, there is an isomorphism of expansions $\Phi_X : \mathrm{Cat}_X(\mathcal{K}^*, \mathcal{K}) \to \mathrm{Cat}_X(\mathcal{K}^{**}, \mathcal{K})$.*

Proof. Let X and Y be adequate, and suppose that there is an isomorphism of expansions Φ_X. Since isomorphisms of expansions are trivial on embeddings, it is enough to define Φ_Y on objects. For $\mathbf{A} \in Y$, choose isomorphic $\mathbf{A}_X$ and an isomorphism $f_{\mathbf{A}} : \mathbf{A}_X \to \mathbf{A}$. Let $\mathbf{A}^*$ be a $\mathcal{K}^*$-expansion of $\mathbf{A}$. Then $\Phi_X(\mathbf{A}_X(f_{\mathbf{A}}, \mathbf{A}^*))$ is a $\mathcal{K}^{**}$-expansion of $\mathbf{A}_X$. Set $\Phi_Y(\mathbf{A}^*) = \mathbf{A}(f_{\mathbf{A}}^{-1}, \Phi_X(\mathbf{A}_X(f_{\mathbf{A}}, \mathbf{A}^*)))$. It is straightforward to check that this works. $\square$

If $\mathcal{K}^*$ is an expansion of the Fraïssé–HP class $\mathcal{K}$, we say that the pair $(\mathcal{K}^*, \mathcal{K})$ is *reasonable* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, embedding $f : \mathbf{A} \to \mathbf{B}$, and expansion $\mathbf{A}^*$ of $\mathbf{A}$, then there is an expansion $\mathbf{B}^*$ of $\mathbf{B}$ with $f : \mathbf{A}^* \to \mathbf{B}^*$ an embedding. When $\mathcal{K}^*$ is also a Fraïssé–HP class, we have the following equivalent definition.

Proposition 5.3. *Let $\mathcal{K}^*$ be a Fraïssé–HP expansion class of the Fraïssé–HP class $\mathcal{K}$ with Fraïssé limits $\mathbf{K}^*, \mathbf{K}$ respectively. Then the pair $(\mathcal{K}^*, \mathcal{K})$ is reasonable iff $\mathbf{K}^*|_L \cong \mathbf{K}$.*

Proof. Assume that $\mathbf{K}^*|_L \cong \mathbf{K}$. Fix $\mathbf{A}^* \in \mathrm{Fin}(\mathbf{K}^*) \cap \mathcal{K}^*$ and $\mathbf{B} \in \mathcal{K}$ with $\mathbf{A}^*|_L \subseteq \mathbf{B}$. Use the extension property for $\mathbf{K}$ to find an embedding $f : \mathbf{B} \to \mathbf{K}^*|_L$ extending the inclusion of $\mathbf{A}^*|_L$. Now define the expansion $\mathbf{B}^*$ of $\mathbf{B}$ by declaring that $S_i^{\mathbf{B}^*}(x_1, ..., x_{n_i})$ holds iff $S_i^{\mathbf{K}^*}(f(x_1), ..., f(x_{n_i}))$ holds. This is enough to show that $(\mathcal{K}^*, \mathcal{K})$ is reasonable.

Conversely, suppose $(\mathcal{K}^*, \mathcal{K})$ is reasonable. Let $\mathbf{K}^* = \bigcup_n \mathbf{A}_n^*$ be a $\mathcal{K}^*$-exhaustion, and let $\mathbf{K} = \bigcup_n \mathbf{B}_n$ be a $\mathcal{K}$-exhaustion. Set $\mathbf{A}_n = \mathbf{A}_n^*|_L$; let $f_1 : \mathbf{A}_1 \to \mathbf{B}_{n_1}$ for some large enough n_1. Using the reasonable property, choose an expansion $\mathbf{B}_{n_1}^*$ of $\mathbf{B}_{n_1}$ with $f_1 : \mathbf{A}_1^* \to \mathbf{B}_{n_1}^*$ an embedding. Then use the extension property for $\mathbf{K}^*$ to find an embedding $f_2 : \mathbf{B}_{n_1} \to \mathbf{A}_{n_2}$ extending f_1^{-1} for some large enough n_2. If f_k is defined and k is even, use the extension property for $\mathbf{K}$ to find $f_{k+1} : \mathbf{A}_{n_k} \to \mathbf{B}_{n_{k+1}}$ extending f_k^{-1}. If k is odd, use the reasonable property and the extension property for $\mathbf{K}^*$ to define f_{k+1} extending f_k^{-1}. We proceed in this manner, building an isomorphism $\bigcup_n f_{2n} : \mathbf{K} \to \mathbf{K}^*|_L$. $\square$

Now suppose $(\mathcal{K}^*, \mathcal{K})$ is reasonable and precompact. Set

$$X_{\mathcal{K}^*} := \{\langle \mathbf{K}, \vec{S}\rangle : \langle \mathbf{A}, \vec{S}|_A\rangle \in \mathcal{K}^* \text{ whenever } \mathbf{A} \in \mathrm{Fin}(\mathbf{K}) \cap \mathcal{K}\}$$

We topologize this space by declaring the basic open neighborhoods to be of the form $N(\mathbf{A}^*) := \{\mathbf{K}' \in X^*_{\mathcal{K}} : \mathbf{A}^* \subseteq \mathbf{K}'\}$, where $\mathbf{A}^*$ is an expansion of some $\mathbf{A} \in \mathrm{Fin}(\mathbf{K}) \cap \mathcal{K}$. We can view $X_{\mathcal{K}^*}$ as a closed subspace of

$$\prod_{\mathbf{A} \in \mathrm{Fin}(\mathbf{K}) \cap \mathcal{K}} \{\mathbf{A}^* : \mathbf{A}^* \text{ is an expansion of } \mathbf{A}\}.$$

Notice that since $(\mathcal{K}^*, \mathcal{K})$ is precompact, $X_{\mathcal{K}^*}$ is compact. If $\bigcup_n \mathbf{A}_n = \mathbf{K}$ is an exhaustion, a compatible metric is given by

$$d(\langle \mathbf{K}, \vec{S}\rangle, \langle \mathbf{K}, \vec{T}\rangle) = 1/k(\vec{S}, \vec{T})$$

where $k(\vec{S}, \vec{T})$ is the largest k for which $\langle \mathbf{A}_k, \vec{S}|_{A_k}\rangle \cong \langle \mathbf{A}_k, \vec{T}|_{A_k}\rangle$.

We can now form the (right) logic action of $G = \mathrm{Aut}(\mathbf{K})$ on $X_{\mathcal{K}^*}$ by setting $\mathbf{K}' \cdot g$ to be the structure where $S_i^{(\mathbf{K}'g)}(x_1, ..., x_{n_i})$ holds iff $S_i^{\mathbf{K}'}(g(x_1), ..., g(x_{n_i}))$ holds. It is easy to check that this action is jointly continuous, turning $X_{\mathcal{K}^*}$ into a G-flow. For readers used to left logic actions, acting on the right by g is the same as acting on the left by g^{-1}.

Proposition 5.4. *If $\mathcal{K}$ is a Fraïssé–HP class and $\mathcal{K}^*$ and $\mathcal{K}^{**}$ are isomorphic expansions of $\mathcal{K}$ with each reasonable and precompact, then $X_{\mathcal{K}^*} \cong X_{\mathcal{K}^{**}}$.*

Proof. Let $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, and fix a $\mathcal{K}$-exhaustion $\bigcup_n \mathbf{A}_n$. Set $X = \mathrm{Fin}(\mathbf{K}) \cap \mathcal{K}$, and let $\Phi_X : \mathrm{Cat}_X(\mathcal{K}^*, \mathcal{K}) \to \mathrm{Cat}_X(\mathcal{K}^{**}, \mathcal{K})$ be an isomorphism of expansions. Define a map $\varphi : X_{\mathcal{K}^*} \to X_{\mathcal{K}^{**}}$ via $\varphi(\langle \mathbf{K}, \vec{S}\rangle) = \bigcup_n \Phi_X(\langle \mathbf{A}_n, \vec{S}|_{A_n}\rangle)$. Notice that since Φ_X respects embeddings, the right hand side is a member of $X_{\mathcal{K}^{**}}$. It is straightforward to check that this is a continuous bijection which respects G-action. ☐

First let us consider when $X_{\mathcal{K}^*}$ is a minimal G-flow. We say that the pair $(\mathcal{K}^*, \mathcal{K})$ has the *Expansion Property* (ExpP) when for any $\mathbf{A}^* \in \mathcal{K}^*$, there is $\mathbf{B} \in \mathcal{K}$ such that for any expansion $\mathbf{B}^*$ of $\mathbf{B}$, there is an embedding $f : \mathbf{A}^* \to \mathbf{B}^*$.

Proposition 5.5. *Let $\mathcal{K}^*$ be a reasonable, precompact Fraïssé–HP expansion class of the Fraïssé–HP class $\mathcal{K}$ with Fraïssé limits $\mathbf{K}^*, \mathbf{K}$ respectively. Let $G = \mathrm{Aut}(\mathbf{K})$. Then the G-flow $X_{\mathcal{K}^*}$ is minimal iff the pair $(\mathcal{K}^*, \mathcal{K})$ has the ExpP.*

Proof. Suppose the pair has ExpP. Let $\mathbf{A}^* \in \mathcal{K}^*$, and find $\mathbf{B} \in \mathcal{K}$ witnessing the ExpP for $\mathbf{A}^*$. Pick any $\mathbf{K}' \in X_{\mathcal{K}^*}$, and find $\mathbf{B}' \subseteq \mathbf{K}'$ with $\mathbf{B}'|_L = \mathbf{B}$. Then there is an embedding $f : \mathbf{A}^* \to \mathbf{B}' \subseteq \mathbf{K}'$, hence $\mathcal{K}^* \subseteq \mathrm{Age}(\mathbf{K}')$. Now fix a basic open neighborhood $N(\mathbf{C}^*)$ of $X_{\mathcal{K}^*}$, where $\mathbf{C}^*$ expands $\mathbf{C} \in \mathrm{Fin}(\mathbf{K}) \cap \mathcal{K}$. Let $f : \mathbf{C}^* \to \mathbf{D}'$ be an embedding. Use ultrahomogeneity in $\mathbf{K}$ to find $g \in G$ extending f. Then $\mathbf{K}' \cdot g \in N(\mathbf{C}^*)$; hence $X_{\mathcal{K}^*}$ is minimal.

Conversely, suppose the pair does not have ExpP. Find $\mathbf{A}^* \in \mathrm{Fin}(\mathbf{K}^*) \cap \mathcal{K}^*$ for which there is no $\mathbf{B} \in \mathcal{K}$ witnessing ExpP. Now use König's Lemma to find a $\mathbf{K}' \in X_{\mathcal{K}^*}$ with $\mathbf{A}^* \notin \mathrm{Age}(\mathbf{K}')$. It follows that $\mathbf{K}' \cdot G \cap N(\mathbf{A}^*) = \emptyset$, so $X_{\mathcal{K}^*}$ is not minimal. ☐

Corollary 5.6. *With the assumptions of Proposition 5.5, suppose* $(\mathcal{K}^*, \mathcal{K})$ *has ExpP. Let* $\mathbf{A} \in \mathcal{K}$*, and let* $\mathbf{A}^*$ *be an expansion of* $\mathbf{A}$*. Then* $\mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$ *is a syndetic subset of* $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$*.*

Proof. Let $\mathbf{B} \in \mathcal{K}$ witness the ExpP for $\mathbf{A}^*$. If $X \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ is thick, find $g \in \mathrm{Emb}(\mathbf{B}, \mathbf{K})$ with $g \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}) \subseteq X$. Let $\mathbf{B}^* = \mathbf{B}(g, \mathbf{K}^*)$. Since $\mathbf{B}$ witnesses ExpP for $\mathbf{A}^*$, find $f \in \mathrm{Emb}(\mathbf{A}^*, \mathbf{B}^*)$. Now $g \circ f \in \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*) \cap X$. $\square$

The following extends Theorem 5.1 and is one of the major theorems in [KPT]. This theorem in its full generality is proven in [NVT]:

Theorem 5.7. *Let* $\mathcal{K}^*$ *be a reasonable, precompact Fraïssé–HP expansion class of the Fraïssé–HP class* $\mathcal{K}$ *with Fraïssé limits* $\mathbf{K}^*, \mathbf{K}$*, respectively. Let* $G = \mathrm{Aut}(\mathbf{K})$*. Then* $X_{\mathcal{K}^*} \cong M(G)$ *iff the pair* $(\mathcal{K}^*, \mathcal{K})$ *has the ExpP and* $\mathcal{K}^*$ *has the RP.*

Remark. Pairs $(\mathcal{K}^*, \mathcal{K})$ of Fraïssé–HP classes which are reasonable, precompact, and satisfy the ExpP and RP are called *excellent*.

When $(\mathcal{K}^*, \mathcal{K})$ is an excellent pair, the Ramsey Property for $\mathcal{K}^*$ tells us something about the combinatorics of $\mathcal{K}$

Proposition 5.8. *Let* $(\mathcal{K}^*, \mathcal{K})$ *be an excellent pair. Then every* $\mathbf{A} \in \mathcal{K}$ *has finite Ramsey degree. In particular, the Ramsey degree of* $\mathbf{A}$ *is equal to the number of expansions of* $\mathbf{A} \in \mathcal{K}^*$*.*

We will prove this by using some of the ideas developed in section 4.

Lemma 5.9. *With* $(\mathcal{K}^*, \mathcal{K})$ *as in Proposition 5.8, let* $\mathbf{A} \in \mathcal{K}$ *and* $\mathbf{A}^*$ *be an expansion of* $\mathbf{A}$*. Then* $X \subseteq \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$ *is thick iff* $X = T \cap \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$ *with* $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ *thick.*

Remark. "Thick" above is referring to two different notions of thickness. In general, when we say $X \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is thick/syndetic, this means with respect to the class $\mathcal{D} = \mathrm{Age}(\mathbf{D})$.

Proof. $(\Rightarrow)$ If $X \subseteq \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$ is thick, fix $\mathbf{B} \in \mathcal{K}$ with $\mathbf{A} \leq \mathbf{B}$. Pick any expansion $\mathbf{B}^*$ of $\mathbf{B}$, then as $X \subseteq \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$ is thick, find $g_{\mathbf{B}} \in \mathrm{Emb}(\mathbf{B}^*, \mathbf{K}^*)$ with $g_{\mathbf{B}} \circ \mathrm{Emb}(\mathbf{A}^*, \mathbf{B}^*) \subseteq X$. Now set $T' = \bigcup_{\mathbf{A} \leq \mathbf{B}} g_{\mathbf{B}} \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$. Then $T' \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ is thick, as is $T = T' \cup X$. Then $X = T \cap \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$ as desired.

$(\Leftarrow)$ If $X = T \cap \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$ with $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ thick, then fix $\mathbf{B}^* \in \mathcal{K}^*$ with $\mathbf{A}^* \leq \mathbf{B}^*$. Then find $\mathbf{C} \in \mathcal{K}$ witnessing the ExpP for $\mathbf{B}^*$. As $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ is thick, find $g \in \mathrm{Emb}(\mathbf{C}, \mathbf{K})$ with $g \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \subseteq T$. Let $\mathbf{C}^* = \mathbf{C}(g, \mathbf{K}^*)$ be the unique expansion of $\mathbf{C}$ so that $g \in \mathrm{Emb}(\mathbf{C}^*, \mathbf{K}^*)$. As $\mathbf{C}$ witnesses ExpP for $\mathbf{B}^*$, pick $f \in \mathrm{Emb}(\mathbf{B}^*, \mathbf{C}^*)$. Now we have $g \circ f \in \mathrm{Emb}(\mathbf{B}^*, \mathbf{K}^*)$ and $g \circ f \circ \mathrm{Emb}(\mathbf{A}^*, \mathbf{B}^*) \subseteq T \cap \mathrm{Emb}(\mathbf{A}^*, \mathbf{K}^*)$. $\square$

Proof of Proposition 5.8. Fix $\mathbf{A} \in \mathcal{K}$, and let $\mathbf{A}_1, ..., \mathbf{A}_k$ list the expansions of $\mathbf{A}$. We can now write

$$\mathrm{Emb}(\mathbf{A}, \mathbf{K}) = \bigsqcup_{i \leq k} \mathrm{Emb}(\mathbf{A}_i, \mathbf{K}^*).$$

Fix a $k+1$-coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$. Find a thick $T_1 \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ so that $T_1 \cap \mathrm{Emb}(\mathbf{A}_1, \mathbf{K}^*)$ is monochromatic. If thick $T_i \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ has been determined, find thick $T_{i+1} \subseteq$

$\mathrm{Emb}(\mathbf{A}, \mathbf{K})$, $T_{i+1} \subseteq T_i$, so that $T_{i+1} \cap \mathrm{Emb}(\mathbf{A}_{i+1}, \mathbf{K}^*)$ is monochromatic. Then $T_k \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ is thick and $|\gamma(T_k)| \leq k$. This shows that $\mathbf{A}$ has Ramsey degree $\leq k$.

For the other bound, note that by Corollary 5.6, $\mathrm{Emb}(\mathbf{A}_i, \mathbf{K}^*)$ is syndetic. Let γ be the coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$ with $\gamma(f) = i$ iff $f \in \mathrm{Emb}(\mathbf{A}_i, \mathbf{K}^*)$. Then γ is a syndetic k-coloring, so by Proposition 4.8, $\mathbf{A}$ has Ramsey degree $\geq k$. □

In the setting of Theorem 5.7, we can consider the orbit of $\mathbf{K}^*$ in $X_{\mathcal{K}^*} = M(G)$. Notice that $\mathbf{K}' = \mathbf{K}^* \cdot g$ iff $\mathbf{K}' \cong \mathbf{K}^*$ iff $\mathbf{K}'$ has age $\mathcal{K}^*$ and satisfies the extension property. We can write out which $\mathbf{K}'$ satisfy these assumptions as a countable intersection of open conditions in $X_{\mathcal{K}^*}$. Since $X_{\mathcal{K}^*}$ is minimal, each $\mathbf{K}'$ has age $\mathcal{K}^*$.

Now fix $\mathbf{A} \in \mathrm{Fin}(\mathbf{K}) \cap \mathcal{K}$, and let $\mathbf{A}^*$ be an expansion. If $\mathbf{B}* \in \mathcal{K}^*$ and $\mathbf{A}^* \subseteq \mathbf{B}^*$, let $N(\mathbf{A}^* \subseteq \mathbf{B}^*) \subseteq X_{\mathcal{K}^*}$ be defined as follows: $\mathbf{K}' \in N(\mathbf{A}^* \subseteq \mathbf{B}^*)$ iff either $\mathbf{A}(i_{\mathbf{A}}, \mathbf{K}') \neq \mathbf{A}^*$ or if there is an embedding $f : \mathbf{B}^* \to \mathbf{K}'$ with $f|_{\mathbf{A}^*} = i_{\mathbf{A}}$. Then $N(\mathbf{A}^* \subseteq \mathbf{B}^*)$ is open, and $\mathbf{K}' \in X_{\mathcal{K}^*}$ satisfies the extension property iff it is in each $N(\mathbf{A}^* \subseteq \mathbf{B}^*)$. The orbit $\mathbf{K}^* \cdot G$ is also dense since $X_{\mathcal{K}^*}$ is minimal; hence $\mathbf{K}^* \cdot G$ is a generic orbit in $X_{\mathcal{K}^*}$. Note that any G-flow can have at most one generic orbit as the intersection of two generic subsets of any Baire space is nonempty. The following proposition is proved in [AKL] (Prop. 14.1).

Proposition 5.10. *Let G be a Polish topological group and suppose $M(G)$ has a generic orbit. Then if Y is a minimal G-flow, then Y has a generic orbit.*

If G is Polish and $M(G)$ has a generic orbit, G is said to have the *generic point property.* Angel, Kechris, and Lyons posed the following question ([AKL] Question 15.2):

Question 5.11 (Generic Pont Problem)**.** *Let G be a Polish group with metrizable universal minimal flow. Does G have the generic point property?*

Our new proof of Theorems 5.1 and 5.7 will have the added benefit of solving the Generic Point Problem for G a closed subgroup of S_∞.

6 The Greatest Ambit

In the remaining sections, we fix once and for all a relational Fraïssé class $\mathcal{K}$ with Fraïssé limit $\mathbf{K}$, which we suppose has universe $\mathbb{N}$. Set $G = \mathrm{Aut}(\mathbf{K})$. For each $n \in \mathbb{N}$ we also let $\mathbf{A}_n \in \mathrm{Fin}(\mathbf{K})$ with $A_n = \{1, 2, ..., n\}$. As a shorthand, write H_n for $\mathrm{Emb}(\mathbf{A}_n, \mathbf{K})$; let i_m^n denote the inclusion $\mathbf{A}_m \hookrightarrow \mathbf{A}_n$ for $m \leq n$ and i_n denote the inclusion embedding $\mathbf{A}_n \subseteq \mathbf{K}$.

Suppose $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$. As $\mathbf{K}$ is a Fraïssé structure, the map $\hat{f} : H_n \to H_m$ given by $\hat{f}(g) = g \circ f$ is surjective. Let βH_n denote the β-compactification of the discrete space H_n. Then $\hat{f}$ has a unique continuous extension $\tilde{f} : \beta H_n \to \beta H_m$ which is also surjective. If $q \in \beta H_n$ and $S \subseteq H_m$, then $S \in \tilde{f}(q)$ iff $\hat{f}^{-1}(S) \in q$, i.e. $\tilde{f}(q)$ is just the pushforward of q by $\hat{f}$. We will primarily be interested in the case when $f = i_m^n$.

Form the space $\varprojlim \beta H_n := \{\alpha \in \prod_n \beta H_n : \tilde{i}_m^n(\alpha(n)) = \alpha(m)\}$. Topologically, we view $\varprojlim \beta H_n$ as a subspace of $\prod_n \beta H_n$. Let $1 \in \varprojlim \beta H_n$ denote the element where on each level n, the ultrafilter is principal on the embedding i_n. Our goal is to give $\varprojlim \beta H_n$ a G-flow structure; then $(\varprojlim \beta H_n, 1)$ will be the greatest G-ambit. To do this, we first take the peculiar step of stripping away the pointwise convergence topology on G, replacing it with the discrete topology. Form βG, which here will always refer to the compactification of G as a discrete space. Endow βG with the left-topological semigroup structure extending G. For the most part, we will only need the right G-action that arises from this structure; if $p \in \beta G$, $g \in G$, and $S \subseteq G$, then $S \in pg$ iff $Sg^{-1} \in p$

Let $\tilde{\pi}_n : \beta G \to \beta H_n$ be the unique continuous extension of the map $\pi_n(g) = g|_{\mathbf{A}_n}$, and let $\tilde{\pi} : \beta G \to \varprojlim \beta H_n$ be given by $(\tilde{\pi}(p))(n) = \tilde{\pi}_n(p)$. Implicit in this definition is that $\tilde{\pi}_m = \tilde{\imath}^n_m \circ \tilde{\pi}_n$, which follows since $\pi_m = \hat{\imath}^n_m \circ \pi_n$. Notice that $\tilde{\pi}$ is continuous, $1 = \tilde{\pi}(1_G)$, and $\{\tilde{\pi}(g) : g \in G\}$ is dense, hence $\tilde{\pi}$ is surjective. We can use the semigroup structure on βG to give $\varprojlim \beta H_n$ a G-action.

Proposition 6.1. *If $p, q \in \beta G$ are such that $\tilde{\pi}(p) = \tilde{\pi}(q)$, then $\tilde{\pi}(pg) = \tilde{\pi}(qg)$ for any $g \in G$.*

Proof. Fix $S \subseteq H_m$ and $g \in G$. Choose n large enough so that $\mathbf{A}_m \cup g(\mathbf{A}_m) \subseteq \mathbf{A}_n$. Set $T = \{f \in H_n : f \circ g|_{\mathbf{A}_m} \in S\}$. Then we have:

$$\begin{aligned}
\pi_m^{-1}(S) \in pg &\Leftrightarrow \{h \in G : hg \in \pi_m^{-1}(S)\} \in p \\
&\Leftrightarrow \{h \in G : hg|_{\mathbf{A}_m} \in S\} \in p \\
&\Leftrightarrow \{h \in G : h|_{\mathbf{A}_n} \in T\} \in p \\
&\Leftrightarrow \{h \in G : h|_{\mathbf{A}_n} \in T\} \in q \\
&\Leftrightarrow \pi_m^{-1}(S) \in qg
\end{aligned}$$
□

We now can define $\tilde{\pi}(p) \cdot g := \tilde{\pi}(pg)$. That this is an action follows from associativity of βG. More explicitly, if $\alpha \in \varprojlim \beta H_n$, $g \in G$, and $S \subseteq H_m$, we have for large n that

$$S \in \alpha g(m) \Leftrightarrow \{f \in H_n : f \circ g|_{\mathbf{A}_m} \in S\} \in \alpha(n).$$

Our use of right actions instead of left actions is justified by the following:

Proposition 6.2. *The right action of G on $\varprojlim \beta H_n$ is jointly continuous when G is given the pointwise convergence topology.*

Proof. First note that a basis for the topology on $\varprojlim \beta H_n$ is given by sets of the form $\tilde{S} := \{\alpha : S \in \alpha(m)\}$, where $S \subseteq H_m$ and $m \in \mathbb{N}$. So suppose $S \subseteq H_m$ and $\alpha g \in \tilde{S}$. Fix n large enough so that $g(\mathbf{A}_m) \subseteq \mathbf{A}_n$, and let $T = \{f \in H_n : f \circ g|_{\mathbf{A}_m} \in S\}$. Letting $U_m = \{h \in G : h|_{\mathbf{A}_m} = g|_{\mathbf{A}_m}\}$, then $\tilde{T} \cdot U_m \subseteq \tilde{S}$. □

Theorem 6.3. $(\varprojlim \beta H_n, 1)$ *is the greatest G-ambit when G is given the pointwise convergence topology.*

Proof. Let (X, x_0) be a G-ambit, and let $\rho : \beta G \to X$ be the continuous extension of the map $g \to x_0 \cdot g$ (remember that βG is the compactification constructed from the discrete topology on G). Write $\rho(p) = x_0 \cdot p$; notice that if $U \ni x_0 \cdot p$ is an open neighborhood, then $\{g \in G : x_0 \cdot g \in U\} \in p$. We will show that if $\tilde{\pi}(p) = \tilde{\pi}(q)$, then $x_0 \cdot p = x_0 \cdot q$. So suppose $x_0 \cdot p \neq x_0 \cdot q$. As compact Hausdorff spaces are normal, pick U_p, V_p, U_q, V_q open neighborhoods of $x_0 \cdot p, x_0 \cdot q$ with $\overline{U_p} \subseteq V_p$, $\overline{U_q} \subseteq V_q$, and $V_p \cap V_q = \emptyset$. For each $y \in X \setminus V_p$, let W_y^p be a neighborhood avoiding U_p. For each n, let $G_n \subseteq G$ denote the pointwise stabilizer of $\mathbf{A}_n$; use joint continuity to find Y_y^p a neighborhood of y and $n_y^p \in \mathbb{N}$ with $Y_y^p \cdot G_{n_y^p} \subseteq W_y^p$. As the Y_y^p cover $X \setminus V_p$, find a finite subcover $\mathcal{C}_p$. Repeat these steps for q, and let N be the largest of any n_y^p, n_y^q mentioned in $\mathcal{C}_p, \mathcal{C}_q$.

Now if $x_0 \cdot g \in U_p$, we must have $x_0 \cdot (gG_N) \subseteq V_p$; if this were not the case, then for $h \in gG_N$ with $x_0 \cdot h \in X \setminus V_p$, we have $x_0 \cdot h \in Y_y^p$ for some $Y_y^p \in \mathcal{C}_p$. Hence $x_0 \cdot (gG_N) = (x_0 h)G_N \subseteq W_y^p$, a contradiction since $W_y^p \cap U_p = \emptyset$. Similarly for q. Let $S_p := \{f \in H_N : \exists g \in G(g|_{\mathbf{A}_N} = f \text{ and } x_0 \cdot g \in U_p)\}$. Do likewise for q. Then $\{g \in G : x_0 \cdot g \in U_p\} \subseteq \pi_N^{-1}(S_p) \in p$, $\{g \in G : x_0 \cdot g \in U_q\} \subseteq \pi_N^{-1}(S_q) \in q$, and $\pi_N^{-1}(S_p) \cap \pi_N^{-1}(S_q) = \emptyset$. Hence $\tilde{\pi}(p) \neq \tilde{\pi}(q)$.

Thus there is a well-defined map $\varphi : \varprojlim \beta H_n \to X$ with $\rho = \varphi \circ \tilde{\pi}$. To show that φ is a G-map, we need to show that φ is continuous and respects G-action. Since $\tilde{\pi}$ is a continuous and closed map, we see that φ is continuous. For fixed $g \in G$, observe that $p \to x_0 \cdot (pg)$ and $p \to (x_0 \cdot p) \cdot g$ are two continuous extensions of $h \to x_0 \cdot hg$, so are equal. Hence $\rho(pg) = \rho(p) \cdot g$ for any $p \in \beta G$, $g \in G$. Now let $\alpha \in \varprojlim \beta H_n$, $g \in G$, and pick $p \in \beta G$ with $\tilde{\pi}(p) = \alpha$. Then $\varphi(\alpha \cdot g) = \varphi(\tilde{\pi}(p) \cdot g) = \varphi(\tilde{\pi}(pg)) = \rho(pg) = \rho(p) \cdot g = \varphi(\alpha) \cdot g$. Hence φ is a G-map. □

As a first application, we easily obtain the following corollary, originally due to Pestov [P].

Corollary 6.4. *If G is a closed subgroup of S_∞ with infinite metrizable universal minimal flow $M(G)$, then as a topological space, $M(G) \cong 2^{\mathbb{N}}$.*

Proof. Let $Y \subseteq \varprojlim \beta H_n$ be infinite metrizable. Notice that if Y is a subflow, then Y has no isolated points. Recall that βH_n embeds no infinite compact metric space. Consider the projection of $\varprojlim \beta H_n$ onto the n-th coordinate; the image of Y is metrizable, hence finite. Hence for each n, there is a finite $Y_n \subseteq \beta H_n$ with $\alpha(n) \in Y_n$ for any $\alpha \in Y$. It follows that $Y = \varprojlim Y_n$. □

Remark. Lionel Nguyen Van Thé has pointed out to me that the construction in this section is essentially a more explicit version of the original construction of the greatest ambit $S(G)$ (of any topological group G) given by Pierre Samuel [Sa]. His construction proceeds as follows: let $\mathcal{V}$ be a basis of open neighborhoods of the identity. For $p \in \beta G$ (as a discrete group), let p^* be the filter generated by sets of the form $\{SV : S \in p, V \in \mathcal{V}\}$. Now set $p \sim q$ iff $p^* = q^*$; we then obtain $S(G) \cong \beta G/\sim$. Dana Bartošová uses this approach to extend some of the results from [KPT] to uncountable structures (see [B]). The representation of the greatest ambit presented here was also discovered by Pestov (Corollary 3.3 in [P]).

7 Extreme Amenability

We now turn to the proof of Theorem 5.1; though logically the material in section 8 does not depend on this section, this will provide an introduction to many of the ideas used there. Since $M(G)$ is isomorphic to any minimal subflow of $\varprojlim \beta H_n$, it is enough to characterize when $\varprojlim \beta H_n$ has a fixed point.

We say that an ultrafilter $p \in \beta H_n$ is *thick* if each $S \in p$ is thick. Denote the set of thick ultrafilters on H_n by R_n.

Proposition 7.1. *$R_n \neq \emptyset$ iff $\mathbf{A}_n$ is a Ramsey object in $\mathcal{K}$.*

Proof. To see this, we need to show that the non-thick subsets of H_n form an ideal iff $\mathbf{A}_n$ is a Ramsey object. If $\mathbf{A}_n$ is a Ramsey object, suppose $S \subseteq H_n$ is thick, and suppose $S = T_1 \sqcup T_2$. By the equivalence of (1) and (4) in Proposition 4.1, we see that one of T_1 or T_2 is thick.

Conversely, if $\mathbf{A}_n$ is not a Ramsey object, then use the equivalence of (1) and (3) in Proposition 4.1 to find disjoint $S, T \subseteq H_n$ with $S \sqcup T = H_n$ and neither S nor T thick. $\square$

Proposition 7.2. *Suppose $m \leq n$, $\mathbf{A}_n$ is a Ramsey object, and $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$. Then if $p \in R_m$, there is $q \in R_n$ with $\tilde{f}(q) = p$.*

Proof. Form the preimage filter $\hat{f}^{-1}(p)$. If $T \in \hat{f}^{-1}(p)$, then $T \supseteq \hat{f}^{-1}(S)$ for some $S \in p$. Suppose T is not thick; find a large enough N so that for each $g \in H_N$, we have $g \circ \mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N) \not\subseteq T$. It follows that for any $g \in H_N$, $g \circ \mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N) \circ f \not\subseteq S$. As $\mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N) \circ f \subseteq \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_N)$, we see that S is not thick, a contradiction. Now extend $\hat{f}^{-1}(p)$ to any $q \in \beta H_n$ avoiding the non-thick ideal. $\square$

It follows from Propositions 7.1 and 7.2 that $\varprojlim R_n \neq \emptyset$ iff $\mathcal{K}$ has the Ramsey Property. It is natural to ask whether this is a subflow of $\varprojlim \beta H_n$; in fact, we can do even better. The following theorem implies Theorem 5.1.

Theorem 7.3. *$\alpha \in \varprojlim \beta H_n$ is a fixed point iff $\alpha \in \varprojlim R_n$.*

Proof. Suppose $\alpha \in \varprojlim R_n$, and let $S \in \alpha(m)$. Fix $g \in G$; we want to show that $S \in \alpha g(m)$. Let $n \geq m$ be large enough so that $\mathbf{A}_m \cup g(\mathbf{A}_m) \subseteq \mathbf{A}_n$, and set $T_1 = \{f \in H_n : f|_{\mathbf{A}_m} \in S\}$, $T_2 = \{f \in H_n : f \circ g|_{\mathbf{A}_m} \notin S\}$. If $S \notin \alpha g(m)$, then we have $T_1 \cap T_2 \in \alpha(n)$. Now pick N large enough so that $\mathbf{A}_n \cup g(\mathbf{A}_n) \subseteq \mathbf{A}_N$. As $\alpha(n)$ is thick, fix $h \in H_N$ with $h \circ \mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N) \subseteq T_1 \cap T_2$. But now set $x = h \circ g|_{\mathbf{A}_n} \circ i_m^n = h \circ i_n^N \circ g|_{\mathbf{A}_m}$. Since $g|_{\mathbf{A}_n} \in \mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N)$, we have $h \circ g|_{\mathbf{A}_n} \in T_1 \cap T_2$, hence $x \in S$. Similarly $h \circ i_n^N \in T_1 \cap T_2$, implying $x \notin S$. This is a contradiction.

Conversely, if $\alpha(m)$ is not thick, suppose $S \in \alpha(m)$ is not thick, and find $n \geq m$ such that $f \circ \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n) \not\subseteq S$ for each $f \in H_n$. Then we have

$$\bigcap_{r \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)} \{f \in H_n : f \circ r \in S\} = \emptyset.$$

Hence for some $g \in G$, we must have $S \notin \alpha g(m)$, and α cannot be a fixed point. $\square$

Remark. Müller and Pongrácz in [MP] use different methods to show the following: let $\mathbf{K}$ be a Fraïssé structure, $\mathcal{K} = \mathrm{Age}(\mathbf{K})$, and $G = \mathrm{Aut}(\mathbf{K})$. Suppose each $\mathbf{A} \in \mathcal{K}$ has Ramsey degree $\leq d$ for some fixed $d \in \mathbb{N}$. Then $|M(G)| \leq d$.

8 Metrizability of $M(G)$

We now consider the case where $M(G)$ is metrizable. Corollary 6.4 tells us that if $M(G)$ is metrizable, then $M(G) = \varprojlim Y_n$, where Y_n is a finite subset of βH_n. To characterize the ultrafilters that can appear in such a Y_n, we need to introduce some new terminology.

If $F_1, ..., F_k$ are filters on H_n, we say that $\{F_1, ..., F_k\}$ is *thick* if every $S \in (F_1 \cap \cdots \cap F_k)$ is thick. Note that $S \in (F_1 \cap \cdots \cap F_k)$ iff $S = S_1 \cup \cdots \cup S_k$ with $S_i \in F_i$. It will often be the case that each F_i is a filter on some $X_i \subseteq H_n$; when there is no confusion, we will identify F_i with its pushforward to a filter on H_n. Note that if $\{F_1, ..., F_k\}$ is thick and F' is another filter on H_n, then $\{F_1, ..., F_k, F'\}$ is also thick. We will frequently consider the following thick set of filters:

Proposition 8.1. *Let $\mathbf{A}_n$ have Ramsey degree k, and let γ be a full syndetic k-coloring of H_n. Let $F_i \subseteq \mathcal{P}(\gamma_i)$ consist of those $X \subseteq \gamma_i$ which are syndetic. Then $\{F_1, ..., F_k\}$ is a thick set of filters.*

Proof. First we show each F_i is a filter; we prove this for F_1. Certainly F_1 is upward closed. Suppose $S, T \subseteq \gamma_1$ are syndetic. Form the $(k+3)$-coloring δ by letting $\delta_1 = (S \cap T)$, $\delta_{k+1} = S \setminus T$, $\delta_{k+2} = T \setminus S$, $\delta_{k+3} = \gamma_1 \setminus (S \cup T)$, and $\delta_i = \gamma_i$ for $2 \leq i \leq k$. If $(S \cap T)$ is not syndetic, then $H_n \setminus (S \cap T) = \delta_2 \sqcup \cdots \sqcup \delta_{k+3}$ is thick. So some subset of k colors among $\delta_2, ..., \delta_{k+3}$ must form a thick subset. Since each δ_j for $2 \leq j \leq k$ is syndetic, for one of $X = (S \setminus T), (T \setminus S), (\gamma_i \setminus (S \cup T))$ we have $\delta_2 \sqcup \cdots \sqcup \delta_k \sqcup X$ thick. But this contradicts the fact that S and T are syndetic. Hence $S \cap T$ is syndetic, and F_1 is a filter. To see that $\{F_1, ..., F_k\}$ is thick, pick $S_i \in F_i$ for $1 \leq i \leq k$. Then consider a full $2k$-coloring of H_n with colors $S_i, (\gamma_i \setminus S_i)$ for $i \leq k$. Some k equivalence classes form a thick subset; as each S_i is syndetic, S_i must be one of the equivalence classes. □

Remark. We will call $\{F_1, ..., F_k\}$ as in Proposition 8.1 the *syndetic filters for γ*.

If $X \subseteq H_n$ is thick, we say that $S \subseteq H_n$ is *syndetic relative to* X if $X \setminus S$ is not thick. Notice that if $Y \subseteq H_n$ is thick and S is syndetic relative to X for some $X \supseteq Y$, then S is also syndetic relative to Y.

Proposition 8.2. *The following are equivalent:*

1. *$\mathbf{A}_n$ has Ramsey degree $t \leq k$,*
2. *There is a thick set $\{\alpha_n^1, ..., \alpha_n^k\}$ of k ultrafilters on H_n.*

Proof. $(2 \Rightarrow 1)$ Fix a $(k+1)$-coloring γ of H_n. For each $1 \leq i \leq k$, there is some $\gamma_{j_i} \in \alpha_n^i$. Then $\bigsqcup_{1 \leq i \leq k} \gamma_{j_i}$ must be thick.

$(1 \Rightarrow 2)$ Fix γ a syndetic t-coloring of H_n. Let $\{F_1, ..., F_t\}$ be the syndetic filters for γ. We will be done once we prove the following lemma; we distinguish this lemma because it is somewhat stronger than what we need and we will use it later.

Lemma 8.3. *Suppose $\mathbf{A}_n$ has Ramsey degree k, γ is a syndetic k-coloring, and $\{F_1, ..., F_k\}$ are the syndetic filters for γ. Let G_i be a filter on γ_i extending F_i such that $\{G_1, ..., G_k\}$ is thick. Then each G_i can be extended to an ultrafilter U_i on γ_i such that $\{U_1, ..., U_k\}$ is thick.*

Proof. We will show that G_1 can be extended to an ultrafilter U_1 such that $\{U_1, G_2, ..., G_k\}$ is thick; by relabeling and repeating, this is enough. Let P_1 consist of those subsets $T \subseteq \gamma_1$ for which there are $S_i \in G_i$, $2 \leq i \leq k$, such that T is syndetic relative to $\gamma_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$. I claim P_1 is a filter. Certainly P_1 is upward closed, so suppose $T_1, T_2 \in P$. By taking intersections, we may suppose that there are $S_i \in \gamma_i$, $i \geq 2$, such that both T_1 and T_2 are syndetic relative to $\gamma_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$. Now the proof that $T_1 \cap T_2$ is syndetic relative to $\gamma_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$ mimics the proof of Proposition 8.1.

Now let $S \in G_1$, and suppose $T \in P_1$ as witnessed by $S_i \in G_i$, $2 \leq i \leq k$. Then I claim $(S \cap T) \cup S_2 \cup \cdots \cup S_k$ is thick. Consider the $(k+1)$-coloring δ with $\mathrm{dom}(\delta) = S \sqcup S_2 \sqcup \cdots \sqcup S_k$ and with $\delta_1 = (S \cap T)$, $\delta_{k+1} = (S \setminus T)$, and $\delta_i = S_i$ for $2 \leq i \leq k$. δ is large, and we cannot have $(S \setminus T) \cup S_1 \cup \cdots \cup S_k$ thick since T is syndetic relative to $\gamma_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$. So as each γ_i is syndetic, we must have $(S \cap T) \cup S_2 \cup \cdots \cup S_k$ thick. In particular, since γ_1 is syndetic, $S \cap T$ is non-empty.

We can now extend the filter generated by P_1 and G_1 to an ultrafilter U_1. Since this ultrafilter extends P_1, $\{U_1, G_2, ..., G_k\}$ is thick. □

We need to develop a few ideas related to colorings before proceeding. If γ is a k-coloring and δ is an ℓ-coloring both with domain X, the *product coloring* $\gamma * \delta$ is the $k\ell$-coloring with domain X with $\gamma * \delta(x) = \gamma(x)(\ell - 1) + \delta(x)$. We say that δ *refines* γ if $\delta(x) = \delta(y)$ implies $\gamma(x) = \gamma(y)$. If γ is a coloring of H_m and $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$, then $f(\gamma)$ is the coloring of H_n with $\mathrm{dom}(f(\gamma)) = \hat{f}^{-1}(\mathrm{dom}(\gamma))$ and $f(\gamma)(x) = \gamma(x \circ f)$.

G acts on H_n via $g \cdot x = g \circ x$. This induces a continuous (left) logic action on the compact, metrizable space of partial k-colorings with at most k colors. Explicitly, $g\gamma(g \cdot x)$ is defined iff $\gamma(x)$ is, and $g\gamma(g \cdot x) = \gamma(x)$. Below we collect some simple facts about colorings.

1. If γ is a syndetic coloring of H_m and $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$, then $f(\gamma)$ is a syndetic coloring of H_n.

2. If $\mathbf{A}_n$ has Ramsey degree k, then for every large ℓ-coloring γ of H_n with $k \leq \ell$, there is (up to relabeling colors) a full k-coloring $\gamma' \in \overline{G \cdot \gamma}$.

3. If γ is a full syndetic k-coloring of H_n, then every $\gamma' \in \overline{G \cdot \gamma}$ is a full syndetic k-coloring.

4. Let γ, δ be full colorings of H_n such that δ refines γ. If $g_N \cdot \delta \to \delta'$, then $g_N \cdot \gamma$ also converges to some γ', and δ' refines γ'.

Lemma 8.4. *Suppose $m \leq n$, and $\mathbf{A}_m$ and $\mathbf{A}_n$ have Ramsey degrees k and ℓ, respectively, with $k \leq \ell$. Then there are syndetic colorings γ, δ of H_m, H_n in k, ℓ colors, respectively, such that δ refines $i_m^n(\gamma)$.*

Proof. Choose any full syndetic k, ℓ colorings γ', δ' of H_m, H_n, respectively. Form the product coloring $P := i_m^n(\gamma') * \delta'$ on H_n. $\overline{G \cdot P}$ must contain a full ℓ-coloring δ (up to relabeling colors) which must also be syndetic. If $g_N \cdot P \to \delta$, then $g_N \cdot \gamma'$ converges to some coloring γ. γ and δ are as desired. □

If F is a filter on H_m and $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$, introduce the shorthand notation $f(F)$ for $\hat{f}^{-1}(F)$. Notice that in the proof of Proposition 7.2, we showed that if $X \subseteq H_m$ is thick, then $\hat{f}^{-1}(X) \subseteq H_n$ is also thick; it follows that if $F_1, ..., F_k$ are filters on H_m and $\{F_1, ..., F_k\}$ is thick, then $\{f(F_1), ..., f(F_k)\}$ is also thick. The following proposition is similar in spirit to Proposition 7.2.

Proposition 8.5. *Suppose $m \leq n$ and $\mathbf{A}_m$, $\mathbf{A}_n$ have Ramsey degrees $k \leq \ell$, respectively. Then if $\{\alpha_m^i : 1 \leq i \leq k\} \subseteq \beta H_m$ is thick, there is a thick set $\{\alpha_n^j : 1 \leq j \leq \ell\} \subseteq H_n$ such that $\{\tilde{\imath}_m^n(\alpha_n^j) : 1 \leq j \leq \ell\} = \{\alpha_m^i : 1 \leq i \leq k\}$.*

Proof. Fix full syndetic k, ℓ colorings γ, δ of H_m, H_n, respectively, with δ refining $i_m^n(\gamma)$ as guaranteed by Lemma 8.4. For $1 \leq j \leq \ell$, let a_j be the unique number $1 \leq a_j \leq k$ with $\delta_j \subseteq i_m^n(\gamma)_{a_j}$. Notice that since $\{\alpha_m^1, ..., \alpha_m^k\}$ is thick and since each γ_i is syndetic, without loss of generality we may suppose $\gamma_i \in \alpha_m^i$. We can then assume that α_m^i is an ultrafilter on γ_i.

Let P_j be the filter on δ_j with $T \in P_j$ iff $T \supseteq \delta_j \cap S$ for some $S \in i_m^n(\alpha_m^{a_j})$; as δ_j is syndetic and $\{i_m^n(\alpha_m^1), ..., i_m^n(\alpha_m^k)\}$ is thick, $\delta_j \cap S \neq \emptyset$ for any $S \in i_m^n(\alpha_m^{a_j})$, so P_j is indeed a filter. Let us show that $\{P_1, ..., P_\ell\}$ is thick. Pick $T_j \in P_j$ as witnessed by $S_j \in i_m^n(\alpha_m^{a_j})$ for each $1 \leq j \leq \ell$. By taking intersections, we may assume S_j depends only on a_j. It follows that $S_j \subseteq T_1 \cup \cdots \cup T_\ell$. As $j \to a_j$ is onto and $\{i_m^n(\alpha_m^1), ..., i_m^n(\alpha_m^k)\}$ is thick, we are done.

Let $\{F_1, ..., F_\ell\}$ be the syndetic filters for δ. If $T_i' \in F_i$ and $T_i \in P_i$ for each $1 \leq i \leq \ell$, then I claim that $(T_1' \cap T_1) \cup \cdots \cup (T_\ell' \cap T_\ell)$ is thick. Consider the 2ℓ-coloring of $T_1 \cup \cdots \cup T_\ell$ with colors $(T_j' \cap T_j)$ and $(T_j \setminus T_j')$ for $1 \leq j \leq \ell$. Some ℓ colors must form a thick subset, and each T_j' is syndetic.

Therefore let G_j be the filter generated by $F_j \cup P_j$; use Lemma 8.3 to obtain a thick set of ultrafilters $\{\alpha_n^j : 1 \leq j \leq \ell\}$. Since α_n^j extends P_j, we have that $\tilde{\imath}_m^n(\alpha_n^j) = \alpha_m^{a_j}$, completing the proof. ☐

Corollary 8.6. *If each $\mathbf{A} \in \mathcal{K}$ has finite Ramsey degree, there is $\varprojlim Y_n \subseteq \varprojlim \beta H_n$ with $Y_n \subseteq \beta H_n$ a finite thick set.*

Theorem 8.7. *Let $\mathbf{K}$ be a Fraïssé structure, with $\mathcal{K} = \mathrm{Age}(\mathbf{K})$ and $G = \mathrm{Aut}(\mathbf{K})$. Then the following are equivalent:*

1. *$M(G)$ is metrizable,*
2. *Each $\mathbf{A} \in \mathcal{K}$ has finite Ramsey degree.*

Proof. Suppose $Y = \varprojlim Y_n \subseteq \varprojlim \beta H_n$ with $Y_n \subseteq H_n$ finite. We will show that there is $Z \subseteq Y$ with Z a subflow of $\varprojlim \beta H_n$ iff for each n, Y_n is thick. Set $Y_n = \{\alpha_n^i : 1 \leq i \leq k_n\}$. Let F_n be the filter $(\alpha_n^1 \cap \cdots \cap \alpha_n^{k_n})$.

Suppose $S \in F_m$ is not thick. Pick $\alpha \in Y$. The proof that there is $g \in G$ with $S \notin \alpha g(m)$ now proceeds exactly the same as the second paragraph of the proof of Theorem 7.3.

Now suppose for each n that Y_n is thick. For $W \subseteq G$ finite, $m \in \mathbb{N}$, and $S \in F_m$, let $Y_{W,S}$ consist of those $\alpha \in Y$ such that $S \in \alpha g(m)$ for each $g \in W$. Notice that $Y_{W,S} \subseteq Y$ is closed, hence compact.

Claim. First, let us show that $Y_{W,S}$ is nonempty. Fix n large enough so that $g(\mathbf{A}_m) \subseteq \mathbf{A}_n$ for each $g \in W \cup \{1_G\}$. For $g \in G$, set $T_g = \{f \in H_n : f \circ g|_{\mathbf{A}_m} \in S\}$. We will show that the set $X := T_{1_G} \setminus \left(\bigcap_{g\in W} T_g\right)$ is not thick by mimicking the first paragraph of the proof of Theorem 7.3. If it were, pick N large enough so that $g(\mathbf{A}_n) \subseteq \mathbf{A}_N$ for each $g \in W \cup \{1_G\}$ and find $h \in H_N$ so that $h \circ \mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N) \subseteq X$. But now for each $g \in W$, set $x_g = h \circ g|_{\mathbf{A}_n} \circ i^n_m = h \circ i^N_n \circ g|_{\mathbf{A}_m}$. Since $g|_{\mathbf{A}_n}$ and i^N_n are both in $\mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N)$, we have $h \circ g|_{\mathbf{A}_n}$ and $h \circ i^N_n$ in X. Since $h \circ g|_{\mathbf{A}_n} \in X \subseteq T_{1_G}$, we have $x_g \in S$. But this implies that $h \circ i^N_n \in \bigcap_{g\in W} T_g$, a contradiction.

Since Y_n is thick and since $T_{1_G} = (\hat{i}^n_m)^{-1}(S) \in Y_n$, this means that $(\bigcap_{g\in W\cup\{1_G\}} T_g) \in \alpha^j_n$ for some $\alpha^j_n \in Y_n$. Now any $\alpha \in Y$ with $\alpha(n) = \alpha^j_n$ is a member of $Y_{W,S}$. This proves the claim.

Now observe that if W_1, W_2 are finite subsets of G, $S_1 \in F_m$, and $S_2 \in F_n$ $(m \leq n)$, then letting $S_3 = (\hat{i}^n_m)^{-1}(S_1) \cap S_2 \in F_n$, we have $Y_{W_1,S_1} \cap Y_{W_2,S_2} \supseteq Y_{W_1\cup W_2,S_3}$. In particular, since each $Y_{W,S}$ is compact, there is $\alpha \in Y$ a member of all of them. Hence $\overline{\alpha \cdot G} \subseteq Y$ is a metrizable subflow of $\varprojlim \beta H_n$. □

Remark. Alekos Kechris has recently pointed out to me the following application of Theorem 8.7: In [KPT], it is proved in Appendix 2 that no non-compact, locally compact group G has metrizable universal minimal flow. Now if $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ such that $G = \mathrm{Aut}(\mathbf{K})$ is non-compact and locally compact, then it follows that some object $\mathbf{A} \in \mathcal{K}$ must have infinite Ramsey degree. In particular, this answers the question raised on page 174 line 10 of [KPT], showing the existence of many Fraïssé classes containing objects of infinite Ramsey degree.

Corollary 8.8. *If $Y = \varprojlim Y_n \subseteq \varprojlim \beta H_n$, each Y_n is thick, and $\mathbf{A}_n$ has Ramsey degree $|Y_n| < \infty$, then $Y \cong M(G)$.*

Suppose $Y = \varprojlim Y_n \subseteq \varprojlim \beta H_n$ is the universal minimal flow and is metrizable. By Corollary 8.8, we may assume that $|Y_n| := k_n$ is the Ramsey degree of $\mathbf{A}_n$. It will be useful now to abuse notation and think of $\mathcal{K}$ as being the Fraïssé–HP class $\{\mathbf{B} : \mathbf{B} \cong \mathbf{A}_n \text{ for some } n\}$. Our goal is to interpret $\bigcup_{n\in\mathbb{N}} Y_n$ as $\mathrm{Cat}_X(\mathcal{K}(Y), \mathcal{K})$ for some adequate X and expansion class $\mathcal{K}(Y)$ so that $(\mathcal{K}(Y), \mathcal{K})$ is excellent.

If $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$, $\alpha^i_m \in Y_m$, $\alpha^j_n \in Y_n$, then we say that $f \in \mathrm{Emb}(\alpha^i_m, \alpha^j_n)$ if $\tilde{f}(\alpha^j_n) = \alpha^i_m$. A few things to notice – first, composition is respected, i.e. if $f_1 \in \mathrm{Emb}(\alpha^{i_1}_m, \alpha^{i_2}_n)$ and $f_2 \in \mathrm{Emb}(\alpha^{i_2}_n, \alpha^{i_3}_N)$, then $f_2 \circ f_1 \in \mathrm{Emb}(\alpha^{i_1}_m, \alpha^{i_3}_N)$. Second, if $\alpha \in \varprojlim Y_n$, then $i^n_m \in \mathrm{Emb}(\alpha(m), \alpha(n))$.

If $f \in H_m$, $\alpha^i_m \in Y_m$, and $\alpha \in Y$, we say that $f \in \mathrm{Emb}(\alpha^i_m, \alpha)$ if $f \in \mathrm{Emb}(\alpha^i_m, \alpha(n))$ for some n. Note that if $f \in \mathrm{Emb}(\alpha^i_m, \alpha(n))$, then $f \in \mathrm{Emb}(\alpha^i_m, \alpha(N))$ for all $N \geq n$.

Proposition 8.9. *Suppose we are given $g \in G$ with $g|_{\mathbf{A}_m} = f$ and some $\alpha \in Y$ with $\alpha(n) = \alpha^j_n$. If $\alpha g(m) = \alpha^i_m$, then $f \in \mathrm{Emb}(\alpha^i_m, \alpha^j_n)$. In particular, for any $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$, we have $\tilde{f}(Y_n) \subseteq Y_m$.*

Conversely, if g, f, and α are as above and $f \in \mathrm{Emb}(\alpha^i_m, \alpha^j_n)$, then $\alpha g(m) = \alpha^i_m$.

Remark. Soon we will show that $\tilde{f}(Y_n) = Y_m$.

Proof. Suppose g, f, andl α are as above; fix $S \subseteq H_m$. Let $T = \{x \in H_n : x \circ g|_{\mathbf{A}_m} \in S\}$. Then $S \in \alpha g(m)$ iff $T \in \alpha(n)$. But since $g|_{\mathbf{A}_m} = f$, we have $S \in \tilde{f}(\alpha(n))$ iff $T \in \alpha(n)$. Hence $\tilde{f}(\alpha_n^j) = \tilde{f}(\alpha(n)) = \alpha g(m) = \alpha_m^i$. $\square$

Define C to be the category with object set $\bigcup_{n\in\mathbb{N}} Y_n$ and arrows defined as above. To realize C as $\mathrm{Cat}_{\{\mathbf{A}_n:n\in\mathbb{N}\}}(\mathcal{K}(Y),\mathcal{K})$ for some expansion $\mathcal{K}(Y)$, fix an enumeration $a_n^1, ..., a_n^{N_n}$ of the elements of each $\mathbf{A}_n$. For each $\alpha_m^i \in Y_m$, introduce a new N_m-ary relation symbol R_m^i. Then the object α_n^j can now be realized as the structure $\langle \mathbf{A}_n, (R_m^i)_{m\in\mathbb{N}, 1\leq i\leq k_m}\rangle$, where $R_m^i(b_1, ..., b_{N_m})$ holds whenever the map $a_m^i \to b_i$ is in $\mathrm{Emb}(\alpha_m^i, \alpha_n^j)$. If $\alpha \in Y$, we can interpret α as the countably infinite locally finite structure $\bigcup_n \alpha(n)$. Notice that $Y \cong X_{\mathcal{K}(Y)}$.

Proposition 8.10. *$\mathcal{K}(Y)$ is a reasonable precompact expansion of $\mathcal{K}$ which has the JEP and the ExpP.*

Proof. If $\alpha_m^i, \alpha_n^j \in \mathcal{K}(Y)$, pick $\alpha \in Y$ with $\alpha(m) = \alpha_m^i$. As Y is minimal, find $g \in G$ with $\alpha g(n) = \alpha_n^j$. Let N be large enough so that $g(\mathbf{A}_n) \subseteq \mathbf{A}_N$. Then $\alpha(N)$ witnesses JEP for α_m^i, α_n^j as witnessed by the maps i_m^N and $g|_{\mathbf{A}_n}$.

Now let $f \in \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$ and $\alpha_m^i \in Y_m$. Pick $g \in G$ with $g|_{\mathbf{A}_m} = f$, and pick $\alpha \in Y$ with $\alpha(m) = \alpha_m^i$. Then $\alpha g^{-1}(n)$ is an expansion of $\mathbf{A}_n$ with $f \in \mathrm{Emb}(\alpha_m^i, \alpha_n^j)$, showing that $\mathcal{K}(Y)$ is a reasonable expansion of $\mathcal{K}$.

Suppose $\mathcal{K}(Y)$ did not have the ExpP as witnessed by α_m^i. For each $N \in \mathbb{N}$ pick $\alpha_N \in Y$ with $\mathrm{Emb}(\alpha_m^i, \alpha_N(N)) = \emptyset$. By passing to a convergent subsequence, suppose $\alpha_N \to \alpha$. Then $\overline{\alpha \cdot G} \cap \{\zeta \in Y : \zeta(m) = \alpha_m^i\} = \emptyset$, a contradiction to minimality of Y. $\square$

Notice that the ExpP implies the following useful corollary:

Corollary 8.11. *For any $\alpha \in Y$ and any $\alpha_m^i \in \mathcal{K}(Y)$, $\mathrm{Emb}(\alpha_m^i, \alpha)$ is a syndetic subset of H_m.*

Proposition 8.12. *$\mathcal{K}(Y)$ has the Ramsey Property.*

Proof. Pick any $\alpha \in Y$, and fix $\alpha_n^1 \in \mathcal{K}(Y)$. By Proposition 4.1, it is enough to show that for any full 2-coloring γ of the set $\mathrm{Emb}(\alpha_n^1, \alpha)$, there is a color γ_j which is a thick subset of $\mathrm{Emb}(\alpha_n^1, \alpha)$. This is equivalent to showing that $\gamma_j \cup \left(\bigcup_{2\leq i\leq k_n} \mathrm{Emb}(\alpha_n^i, \alpha)\right)$ is a thick subset of H_n. But consider the $(k_n + 1)$-coloring δ of H_n given by letting $\delta_1 = \gamma_1$, $\delta_{k_n+1} = \gamma_2$, and $\delta_i = \mathrm{Emb}(\alpha_i^n, \alpha)$ for $2 \leq i \leq k_n$; since $\mathbf{A}_n$ has Ramsey degree k_n and by Corollary 8.11, we are done. $\square$

To show that $\mathcal{K}(Y)$ has the AP, we note the following theorem of Nešetřil and Rödl (Lemma 1 of [NR]).

Proposition 8.13. *Let $\mathcal{C}$ be a class of finite structures with the JEP and the RP. Then $\mathcal{C}$ has the AP.*

We now have the following:

Theorem 8.14. *Let $\mathbf{K}$ be a Fraïssé structure with $\mathcal{K} = \mathrm{Age}(\mathbf{K})$ and $G = \mathrm{Aut}(\mathbf{K})$. Then the following are equivalent:*

1. *G has metrizable universal minimal flow,*

2. *Each $\mathbf{A} \in \mathcal{K}$ has finite Ramsey degree,*

3. *There is a Fraïssé precompact expansion class $\mathcal{K}^*$ with $(\mathcal{K}^*, \mathcal{K})$ excellent, i.e. $\mathcal{K}^*$ is reasonable and has both the ExpP and the RP.*

Proof. ($1 \Leftrightarrow 2$) was Theorem 8.7, and we have just shown ($2 \Rightarrow 3$). ($3 \Rightarrow 2$) is Proposition 5.8. □

In light of the discussion before Proposition 5.10, we obtain:

Corollary 8.15. *If G is a closed subgroup of S_∞ with metrizable universal minimal flow, then G has the generic point property.*

We conclude with the proof of Theorem 5.7. Suppose $\mathcal{K}^*$ is a Fraïssé precompact expansion class as in Theorem 8.14 (3). We will show that $X_{\mathcal{K}^*} \cong M(G)$. Let $\mathbf{K}^* \in X_{\mathcal{K}^*}$ be the Fraïssé limit. For each n, let $\mathbf{A}_n^1, ..., \mathbf{A}_n^{k_n}$ be the expansions of $\mathbf{A}_n$ in $\mathcal{K}^*$, with $\mathbf{K}^*|_{\mathbf{A}_n} = \mathbf{A}_n^1$. Write H_n^* for $\mathrm{Emb}(\mathbf{A}_n^1, \mathbf{K}^*)$. Notice that $\varprojlim \beta H_n^* \subseteq \varprojlim \beta H_n$ is nonempty. Pick $\alpha \in \varprojlim \beta H_n^*$ so that each $\alpha(n)$ is thick for H_n^* (we can do this as $\mathcal{K}^*$ has RP).

If $f, x \in H_m$, let us say that f and x have the *same expansion* if for large enough n, there is $r \in H_n^*$ with $r \circ f = x$. It is straightforward to check that this is an equivalence relation.

Claim. For $g, h \in G$, $g|_{\mathbf{A}_m}$ and $h|_{\mathbf{A}_m}$ have the same expansion iff $\alpha g(m) = \alpha h(m)$; we once again mimic the proof of Theorem 7.3. Suppose $g, h \in G$ are such that $g|_{\mathbf{A}_m}$ and $h|_{\mathbf{A}_m}$ have the same expansion. Pick n large enough so that $g(\mathbf{A}_m) \cup h(\mathbf{A}_m) \subseteq \mathbf{A}_n$. Let $S \in \alpha g(m)$; set $T_g = \{f \in H_n : f \circ g|_{\mathbf{A}_m} \in S\}$ and $T_h = \{f \in H_n : f \circ h|_{\mathbf{A}_m} \in S\}$. For sake of contradiction, suppose $S \notin \alpha h(m)$, so that $T_g \setminus T_h \in \alpha(n)$. Since $g|_{\mathbf{A}_m}$ and $h|_{\mathbf{A}_m}$ have the same expansion, find $r \in H_n^*$ with $r \circ g|_{\mathbf{A}_m} = h|_{\mathbf{A}_m}$. Pick N large enough so that $\mathbf{A}_n \cup r(\mathbf{A}_n) \subseteq \mathbf{A}_N$. Since $\alpha(n)$ is thick for H_n^*, find $s \in H_N^*$ so that $s \circ \mathrm{Emb}(\mathbf{A}_n^1, \mathbf{A}_N^1) \subseteq T_g \setminus T_h$. But now set $x = s \circ r \circ g|_{\mathbf{A}_m} = s \circ i_n^N \circ h|_{\mathbf{A}_m}$. This tells us that $x \in S$ and $x \notin S$, a contradiction.

For the converse, suppose $\alpha g(m) = \alpha h(m)$. Pick any $S \in \alpha g(m)$ with $S \subseteq H_m^*$. Then with T_g, T_h as above, we have $T_g \cap T_h \cap H_n^* \in \alpha(n)$. Pick $f \in T_g \cap T_h \cap H_n^*$. Then $f \circ g|_{\mathbf{A}_m}$ and $f \circ h|_{\mathbf{A}_m}$ are in H_m^*; since $\mathbf{K}^*$ is ultrahomogeneous, $f \circ g|_{\mathbf{A}_m}$ and $f \circ h|_{\mathbf{A}_m}$ have the same expansion. It follows that $g|_{\mathbf{A}_m}$ and $h|_{\mathbf{A}_m}$ also have the same expansion. This proves the claim.

It now follows that $\overline{\alpha \cdot G} \subseteq \varprojlim Y_n := Y$ with $|Y_n| \leq k_n$. By the proof of Theorem 8.7, we see that each Y_n is thick. Since the Ramsey degree of $\mathbf{A}_n$ is exactly k_n by Proposition 5.8, we must have $|Y_n| = k_n$; hence by Corollary 8.8, we have $\overline{\alpha \cdot G} = Y \cong M(G)$. Form the expansion class $\mathcal{K}(Y)$. Note that each structure in $\mathcal{K}(Y)$ is isomorphic to $\alpha g(m)$ for some $g \in G$ and $m \in \mathbb{N}$. Pick $f \in H_m$ and $g \in G$ with $g|_{\mathbf{A}_m} = f$. Then for $h_1, h_2 \in G$, we have

$$\begin{aligned} f \in \mathrm{Emb}(\alpha h_1(m), \alpha h_2(n)) &\Leftrightarrow \alpha h_2 g(m) = \alpha h_1(m) \\ &\Leftrightarrow h_2 \circ f \text{ and } h_1|_{\mathbf{A}_m} \text{ have the same expansion} \\ &\Leftrightarrow f \in \mathrm{Emb}(\mathbf{A}_m(h_1|_{\mathbf{A}_m}, \mathbf{K}^*), \mathbf{A}_n(h_2|_{\mathbf{A}_n}, \mathbf{K}^*)) \end{aligned}$$

Letting $X = \{\mathbf{A}_n : n \in \mathbb{N}\}$, this shows that $\Phi_X : \mathrm{Cat}_X(\mathcal{K}(Y), \mathcal{K}) \to \mathrm{Cat}_X(\mathcal{K}^*, \mathcal{K})$ given by $\Phi_X(\alpha g(m)) = \mathbf{A}_m(g|_{\mathbf{A}_m}, \mathbf{K}^*)$ is an isomorphism of expansions. Hence $X_{\mathcal{K}^*} \cong M(G)$.

9 Conclusion

While the new proof of KPT correspondence given here solves the Generic Point Problem for closed subgroups of S_∞, it was originally stated for any Polish group G. We briefly discuss one possible generalization of the methods presented here.

A (relational) *metric structure* is of the form $\langle X, d, \{R_i : i \in I\}\rangle$, where X is a Polish metric space, d is the metric (we assume that X has diameter less than 1), and the "relations" $R_i : X^{n_i} \to \mathbb{R}$ are n_i-ary functions which are k-Lipschitz for some k. An automorphism of the structure is then an isometry of (X, d) which in addition preserves all of the relations R_i. The *quantifier-free type* of a finite tuple $(x_1, ..., x_k)$ is just the (labelled) induced substructure on $\{x_1, ..., x_k\}$. In particular, $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$ have the same quantifier-free type iff $x_i \to y_i$ is an isomorphism of the induced substructures.

A metric structure $\mathbf{X}$ is said to be *near-ultrahomogeneous* if for any $(x_1, ..., x_k), (y_1, ..., y_k)$ with the same quantifier-free type and any $\epsilon > 0$, there is an automorphism π of $\mathbf{X}$ with $\max_i(d(\pi(x_i), y_i)) < \epsilon$. Near-ultrahomogeneous metric structures are called *metric Fraïssé structures*. One of the main theorems of metric Fraïssé theory is that for any Polish group G, there is a metric Fraïssé structure $\mathbf{X}$ with $\mathrm{Aut}(\mathbf{X}) \cong G$; here $\mathrm{Aut}(\mathbf{X})$ is given the pointwise convergence topology. One can also consider the *metric Fraïssé class* $\mathcal{X}$ of finite structures which embed into $\mathbf{X}$. By no means is this intended to be a complete introduction to the theory; the interested reader should see [MT] and [Sch].

There is evidence that metric Fraïssé theory can be used to investigate the dynamical properties of Polish groups. Melleray and Tsankov in [MT] have shown that $\mathrm{Aut}(\mathbf{X})$ is extremely amenable iff the class $\mathcal{X}$ satisfies an appropriate analogue of the Ramsey Property. Perhaps it is possible to use methods similar to those in section 6 to provide a "workable" characterization of the greatest ambit.

Problem 9.1. *Let G be a Polish group. Use metric Fraïssé theory and methods similar to those in section 6 to provide a useful characterization of the greates G-ambit.*

Some of the results in this paper are known to generalize to general Polish groups. Melleray, Nguyen Van Thé, and Tsankov [MNT] have shown that if G is a Polish group, $M(G)$ metrizable, and G has the generic point property, then $M(G)$ is of the form $\widehat{G/G_0}$, where G_0 is extremely amenable and coprecompact in G, and the completion is taken with respect to the left uniformity on G/G_0 (using the left uniformity yields a right G-action). In particular, for G a closed subgroup of S_∞, saying that $M(G)$ is of the form $\widehat{G/G_0}$ for G_0 extremely amenable and coprecompact is exactly to say that 8.14 (3) holds.

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