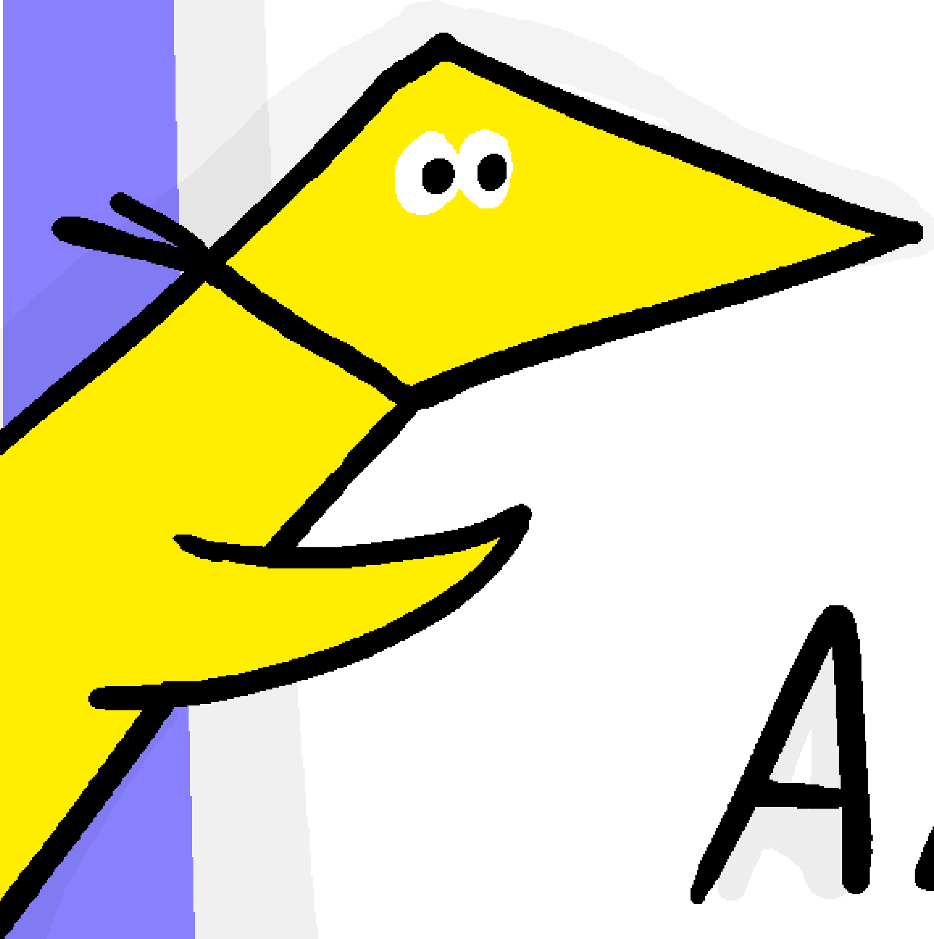
Leray numbers of Tolerance complexes



Alan Lew
Hebrew University of Jerusalem

(Joint w/ Minki Kim)

Copenhagen-Jerusalem Combinatorics Seminar

Simplicial complexes

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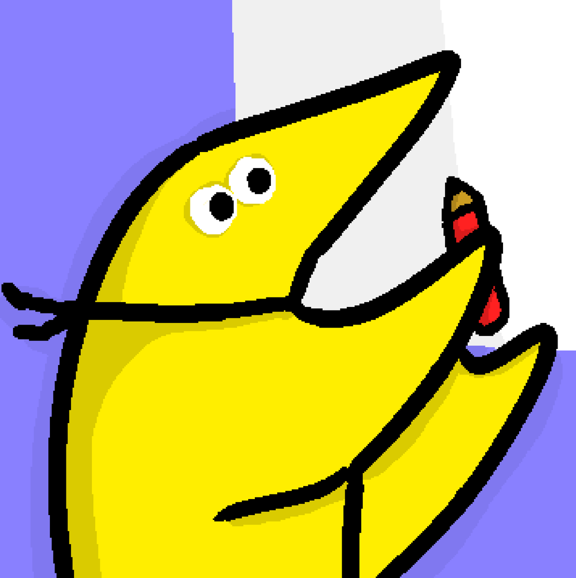

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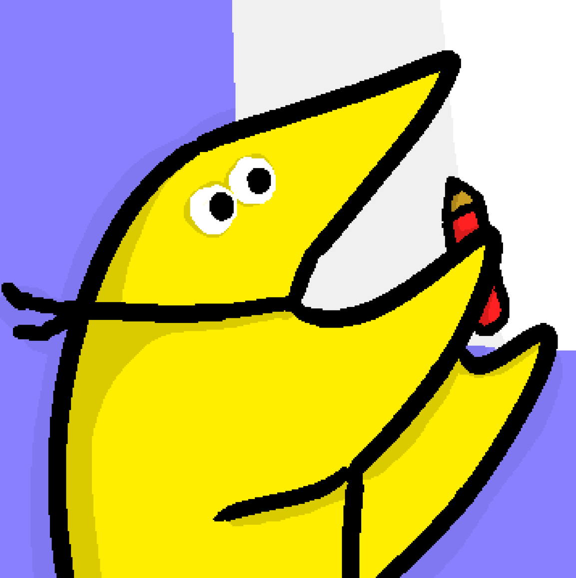

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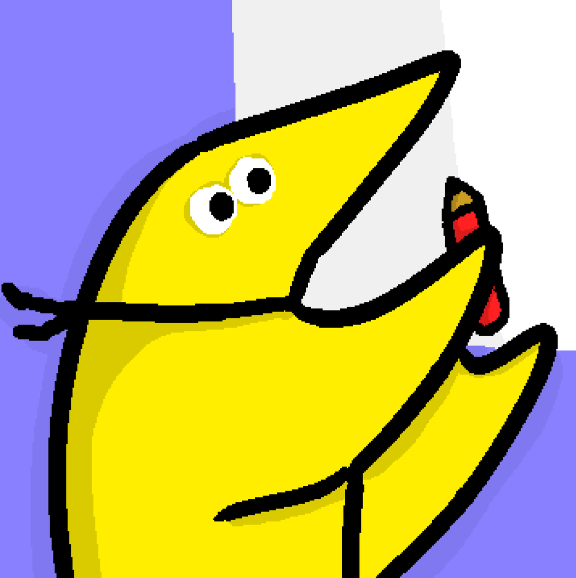

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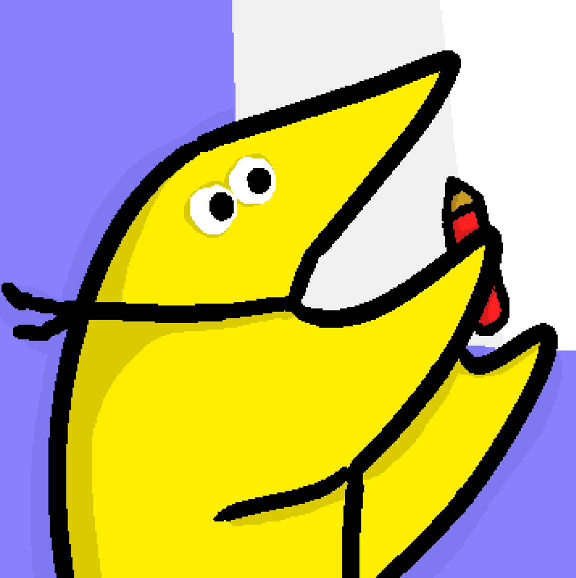

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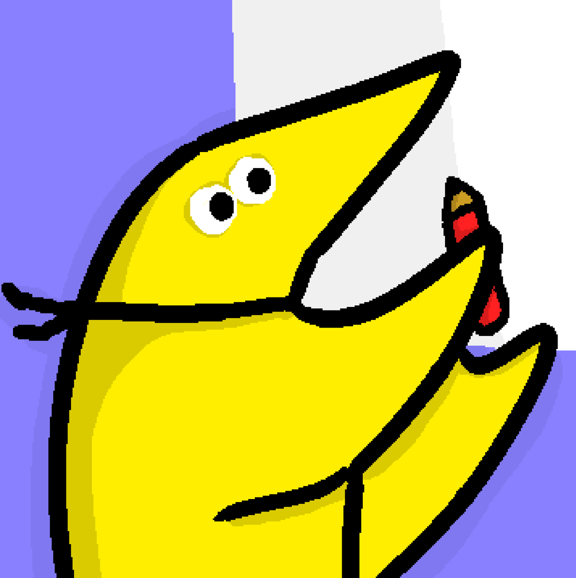

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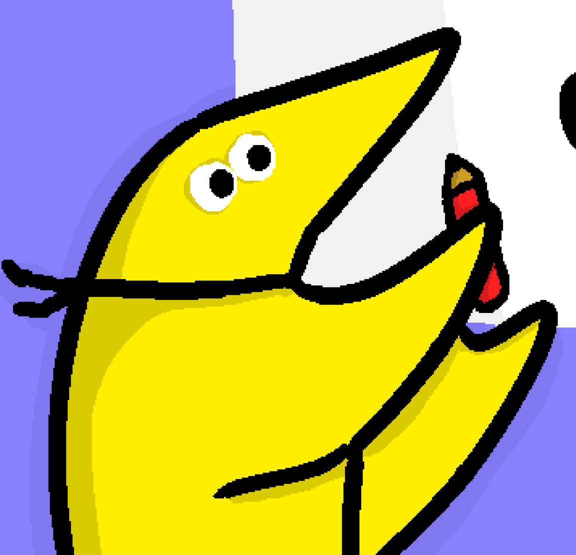

Simplicial complexes

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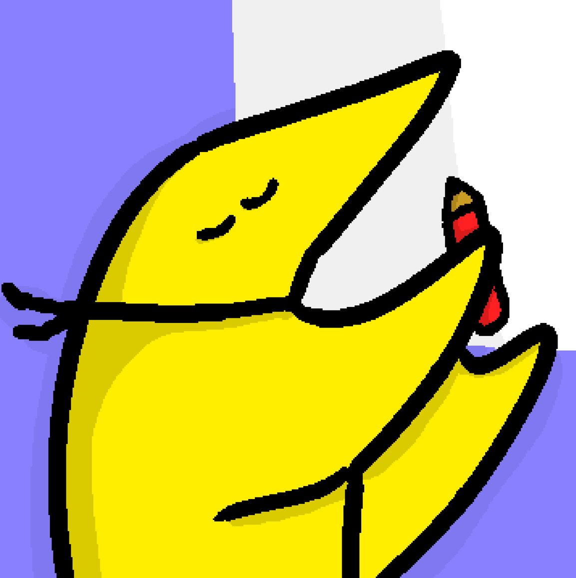

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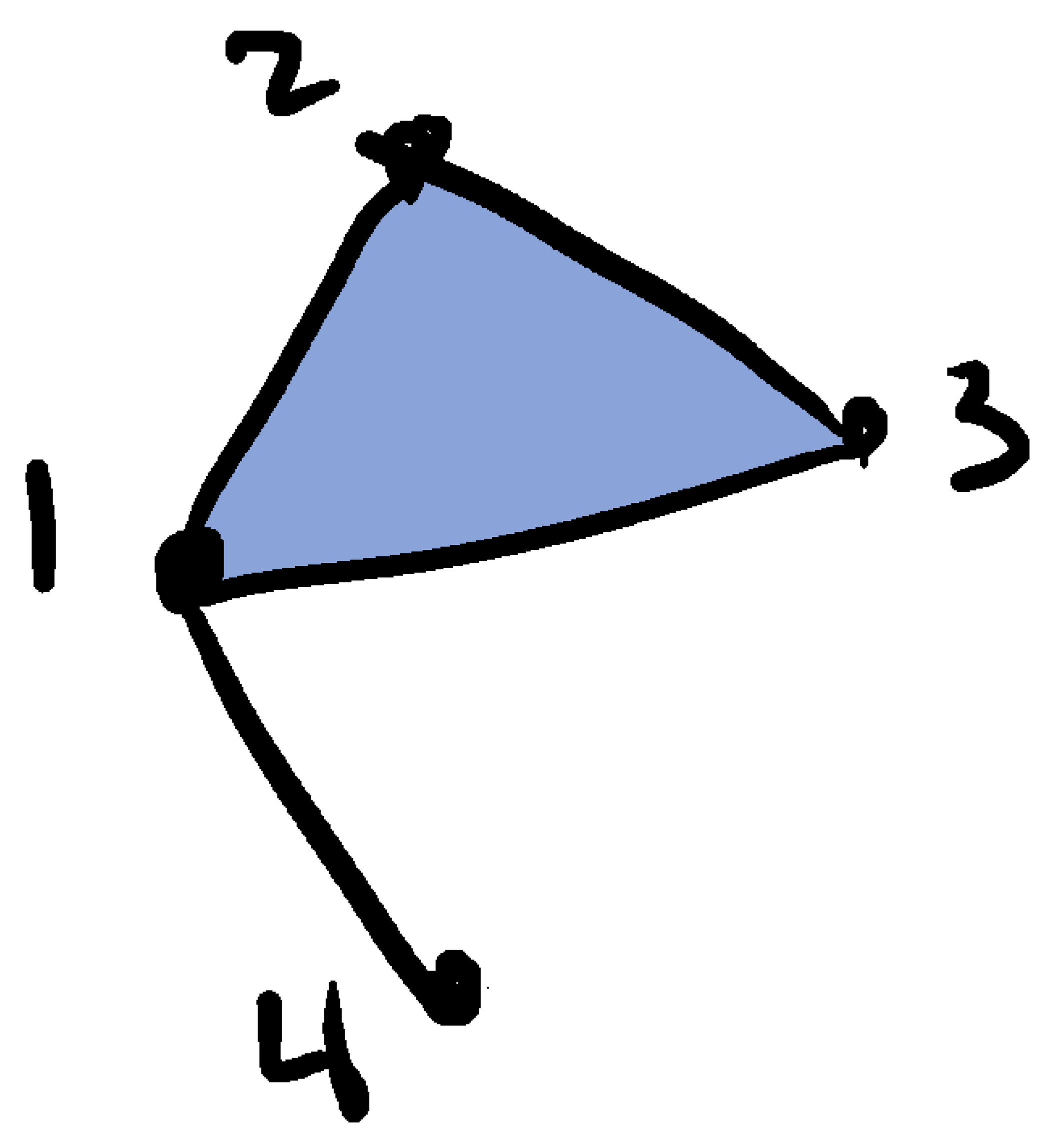

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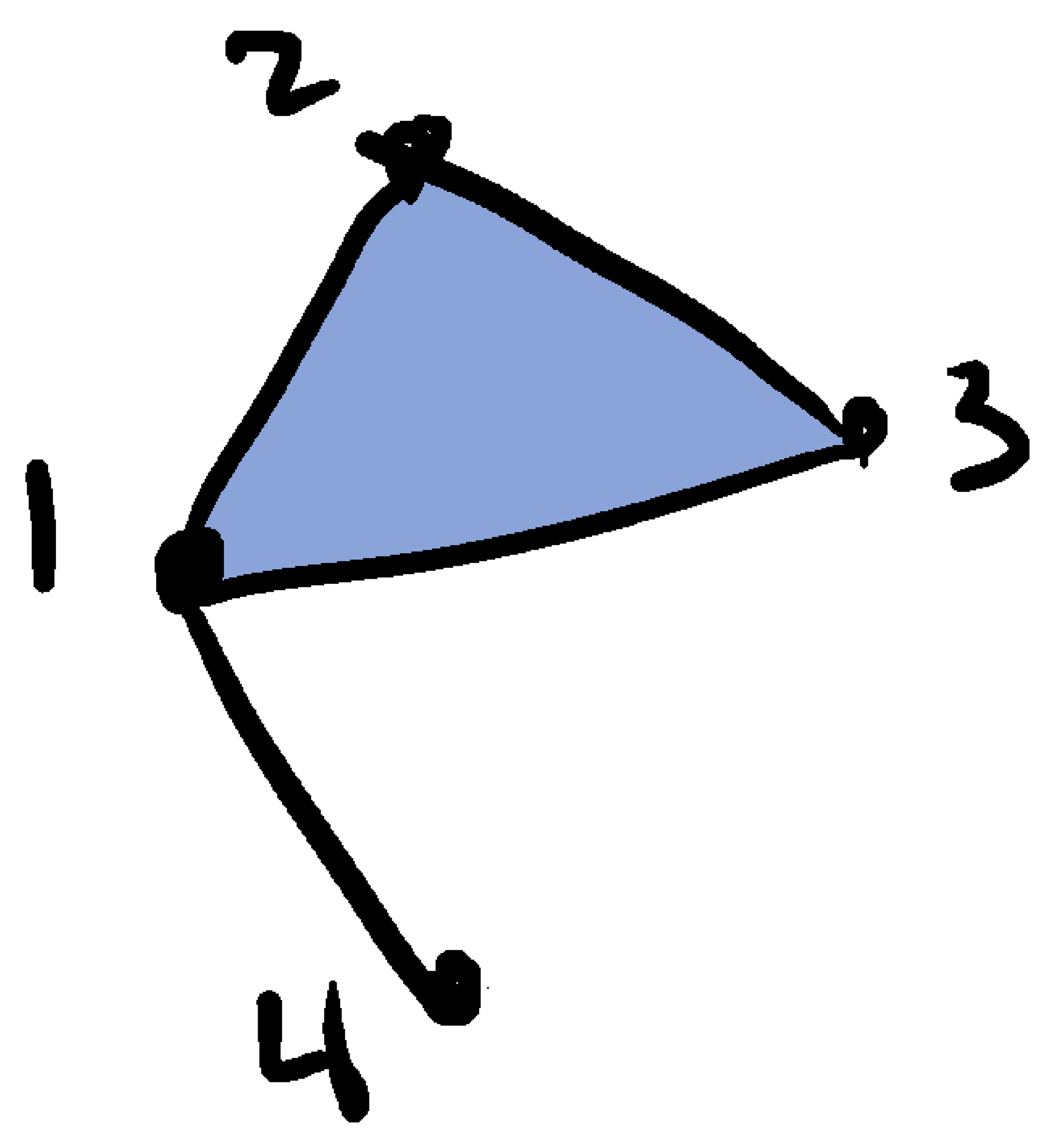

$$\iff K = \left\{ \{1,2,3\}, \{1,4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset \right\}$$

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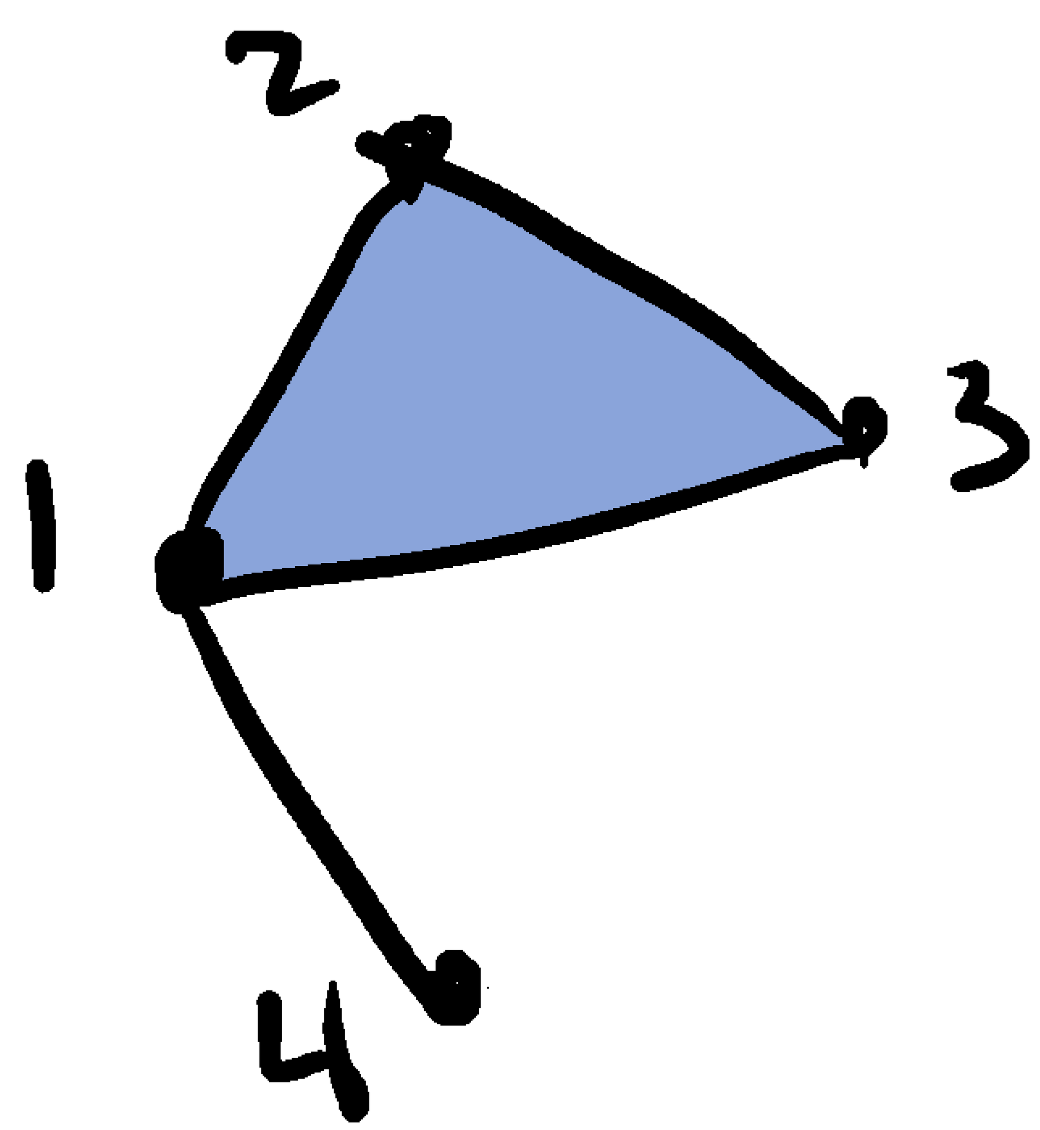
K = 

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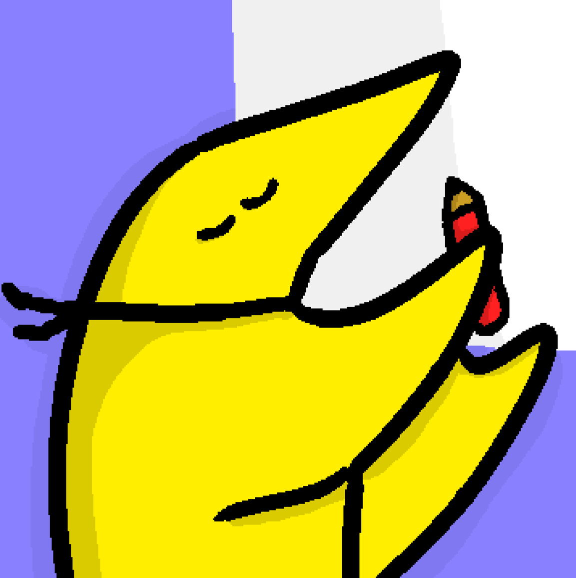
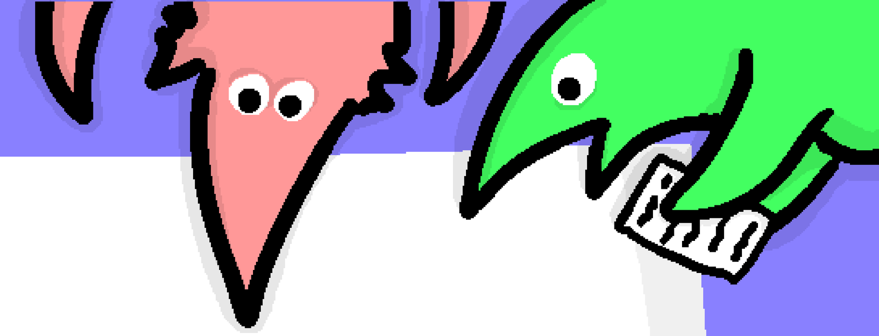
- we can study the topology of a complex.



Homology

- $\tilde{H}_i(K)$ = i-dimensional reduced homology group of K (with real coefficients)



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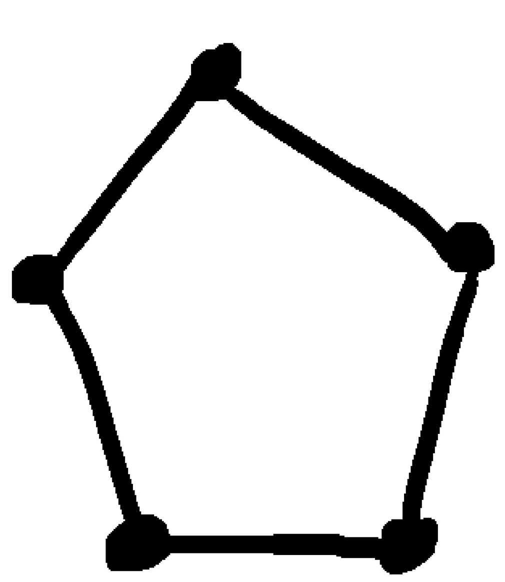
$K =$ 

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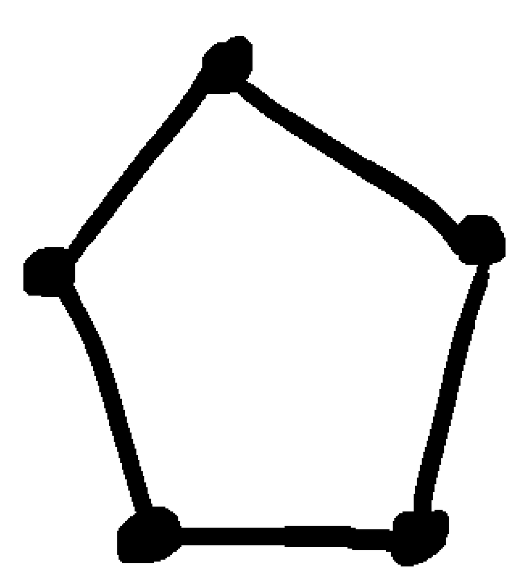
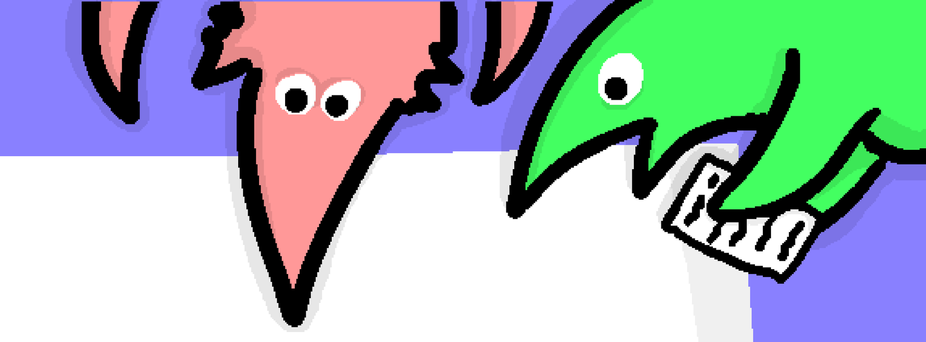
$K =$ 

$$\tilde{H}_i(K) = \begin{cases} \mathbb{R}, & i=1 \\ 0, & \text{otherwise} \end{cases}$$



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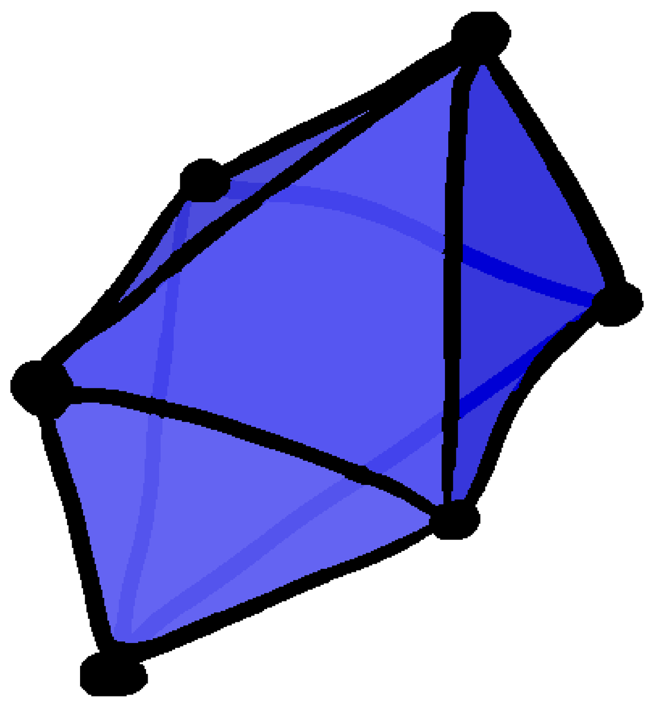
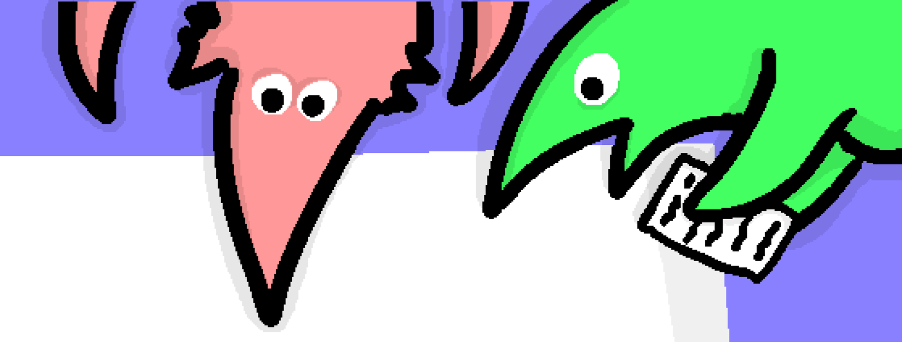
K = 



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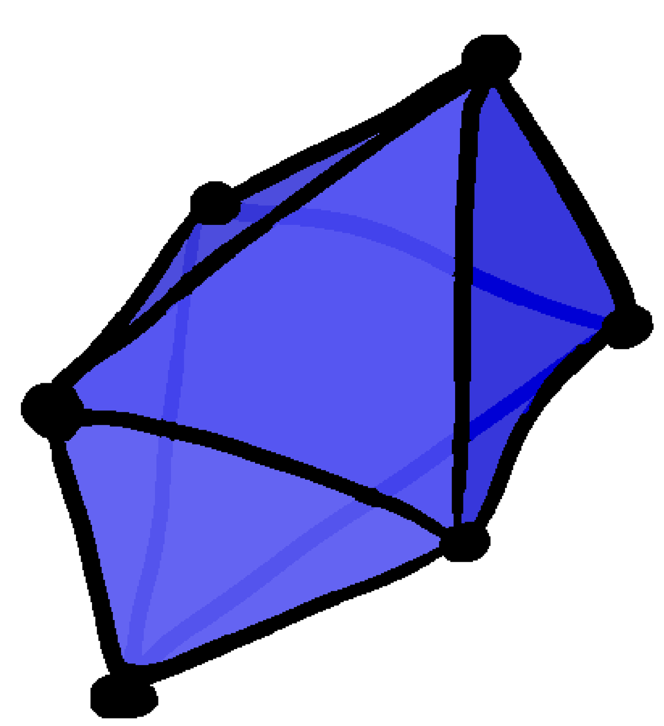
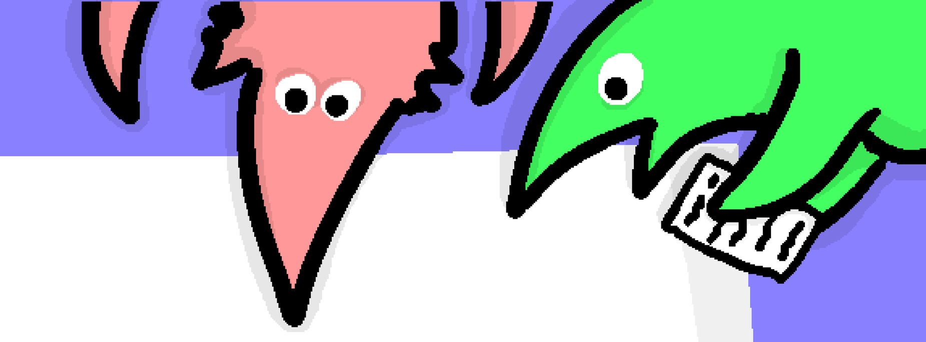
$K =$ 

$$\tilde{H}_i(K) = \begin{cases} \mathbb{R} & ; \; i=2 \\ 0 & ; \; \text{otherwise} \end{cases}$$



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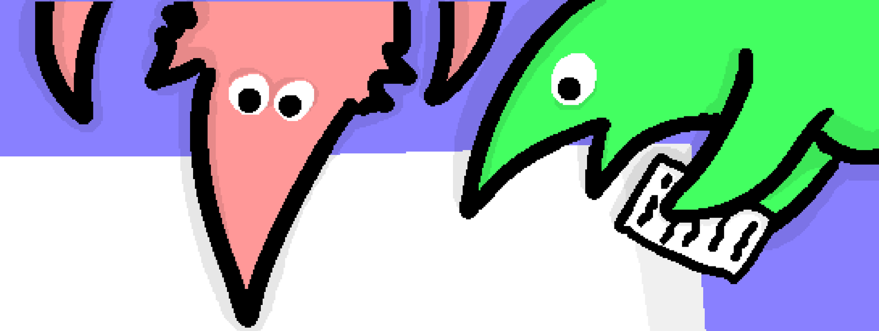
K = d-dimensional sphere



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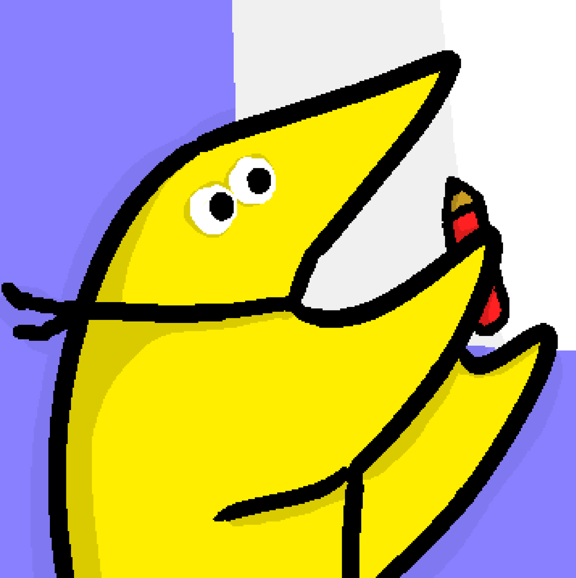

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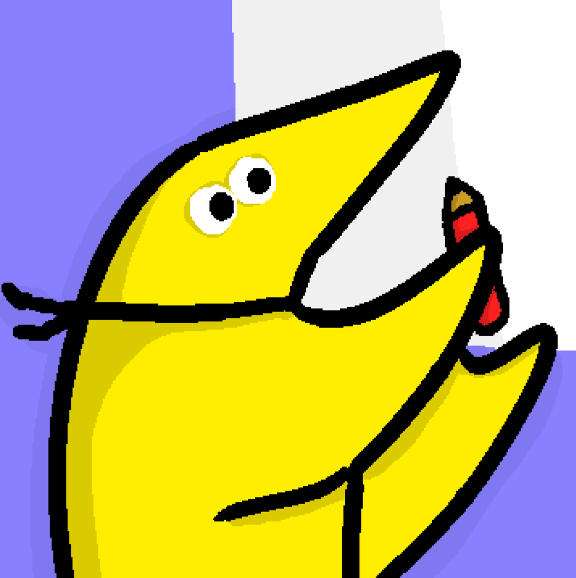

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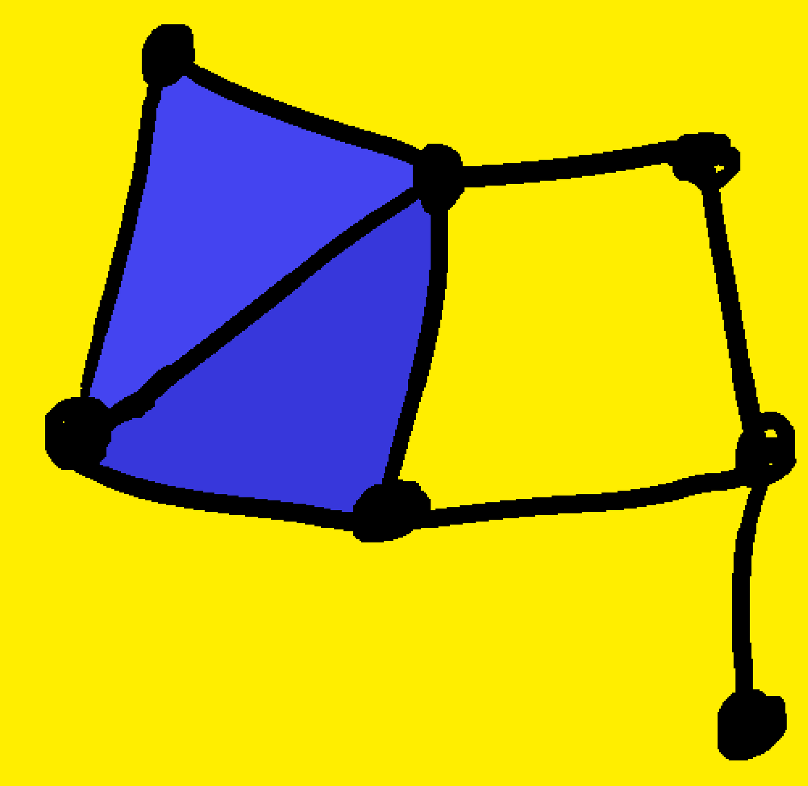
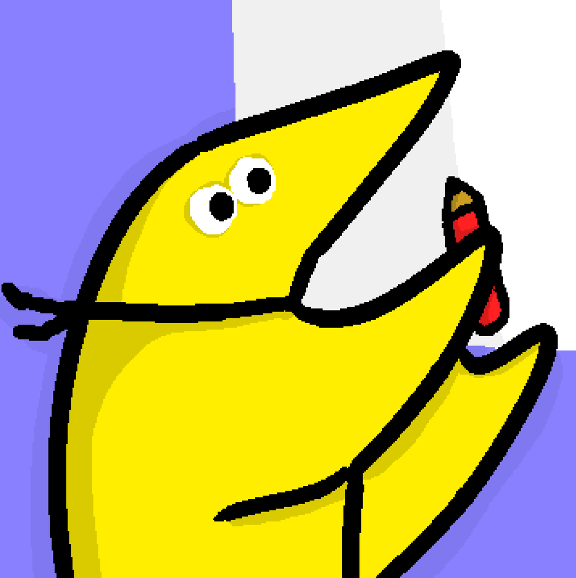
$K =$ 



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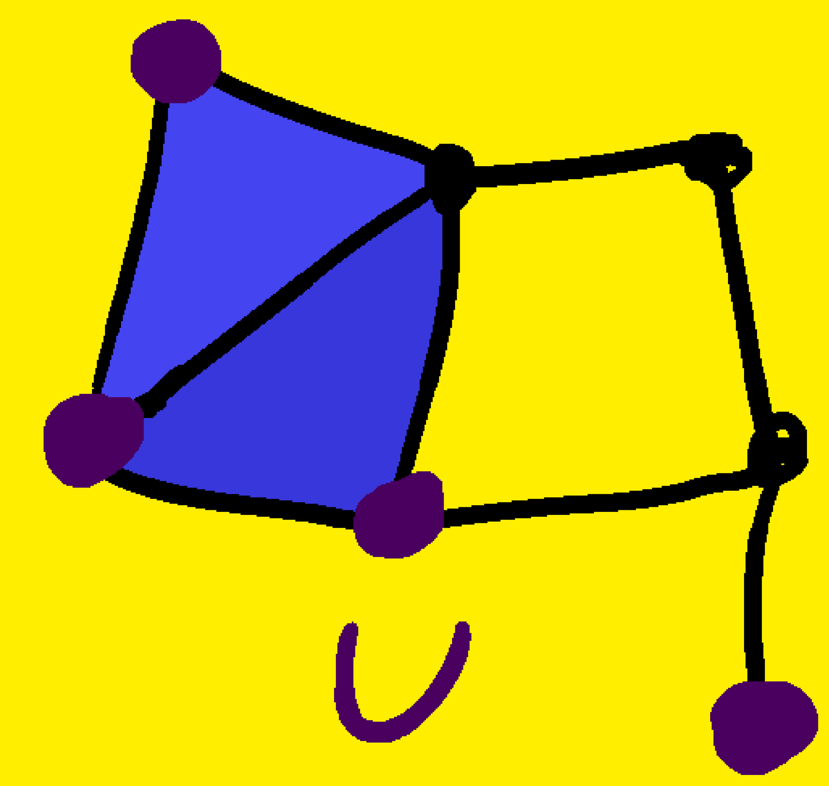
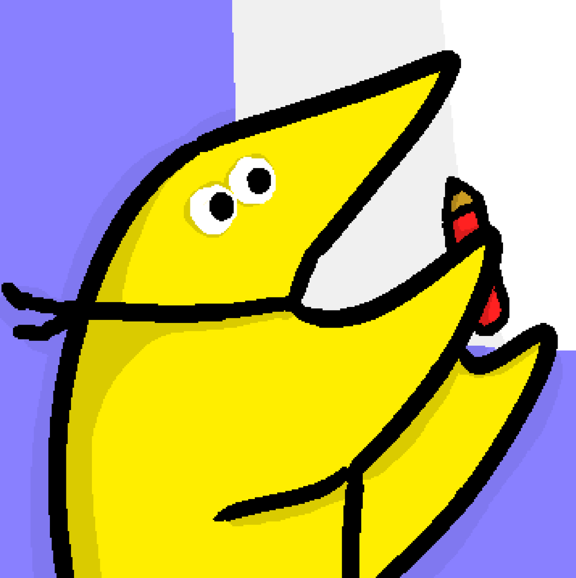
$K =$ 



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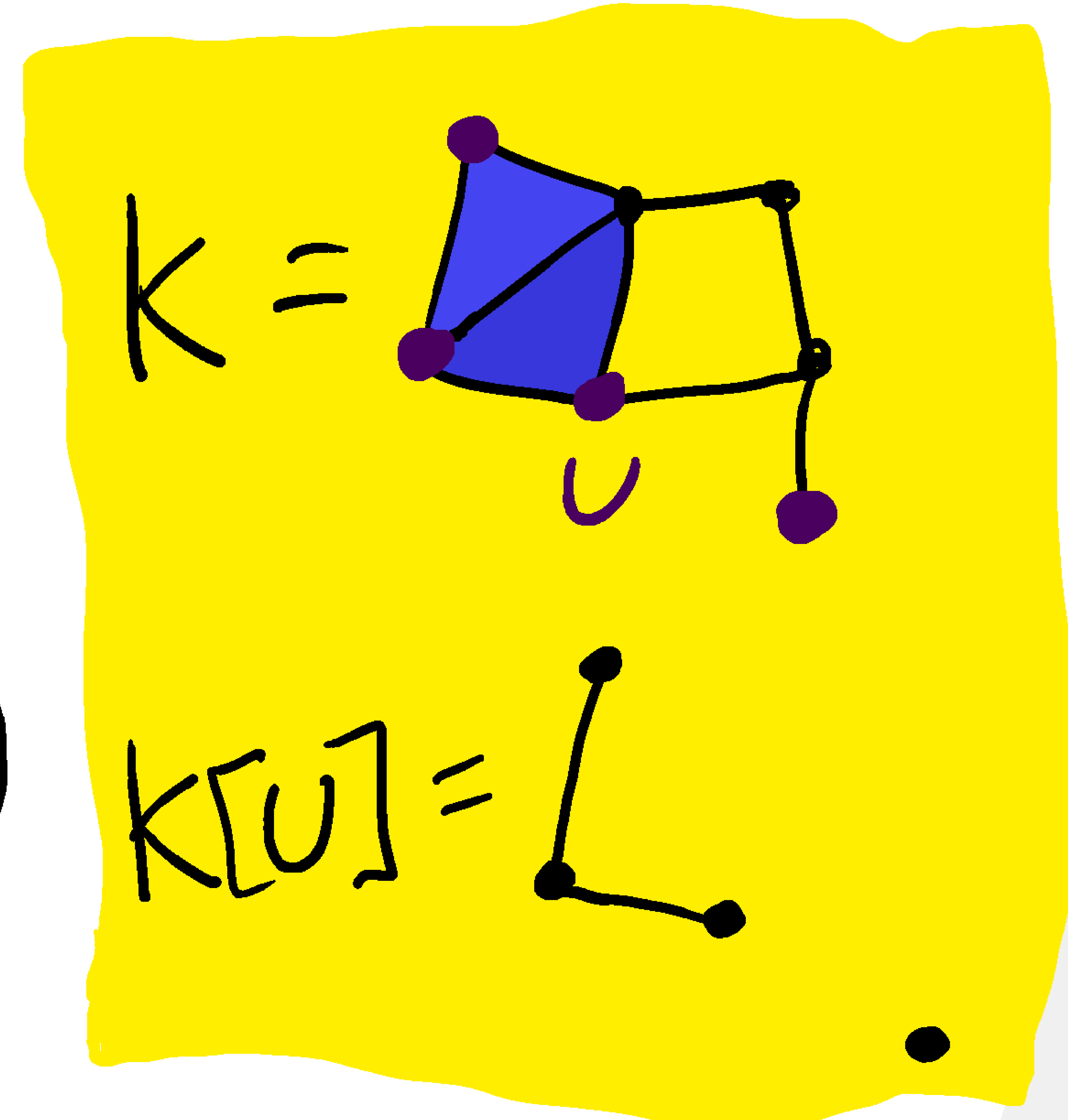

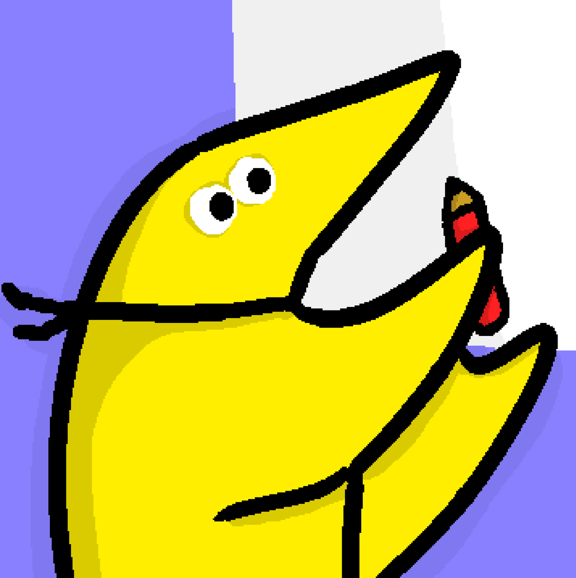

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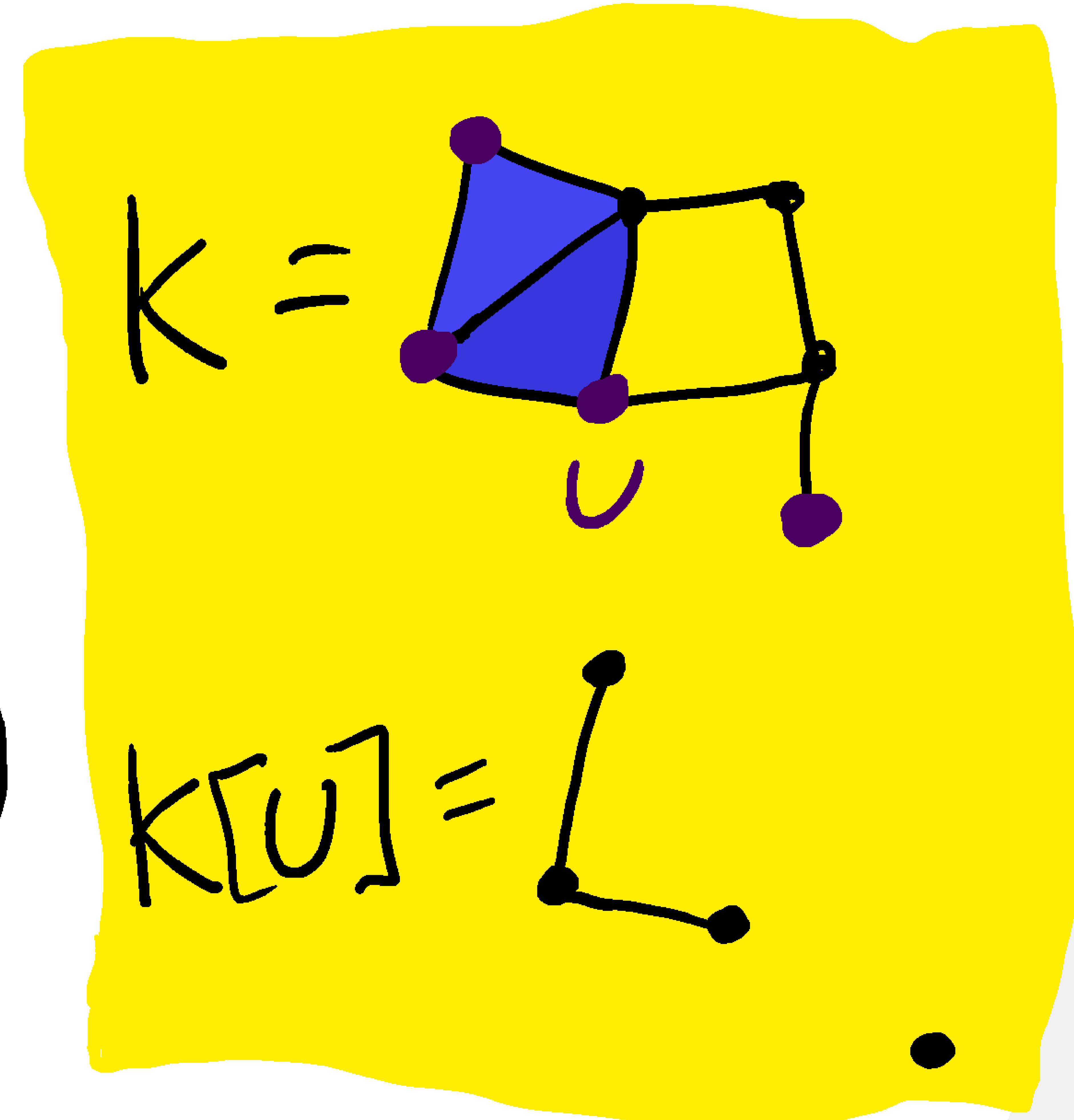

- K is d-Leray if

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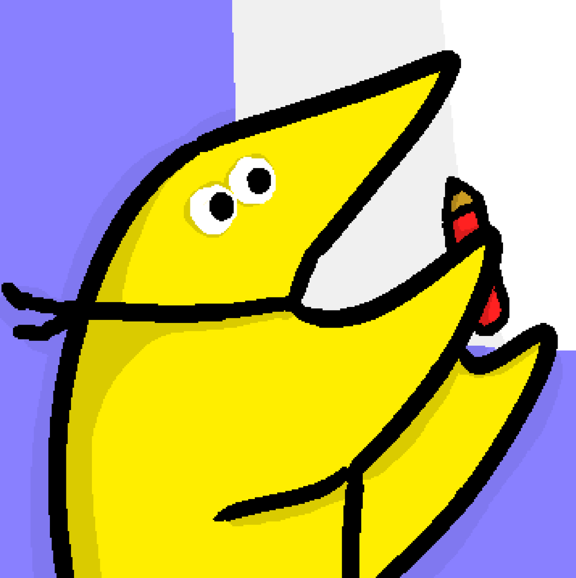

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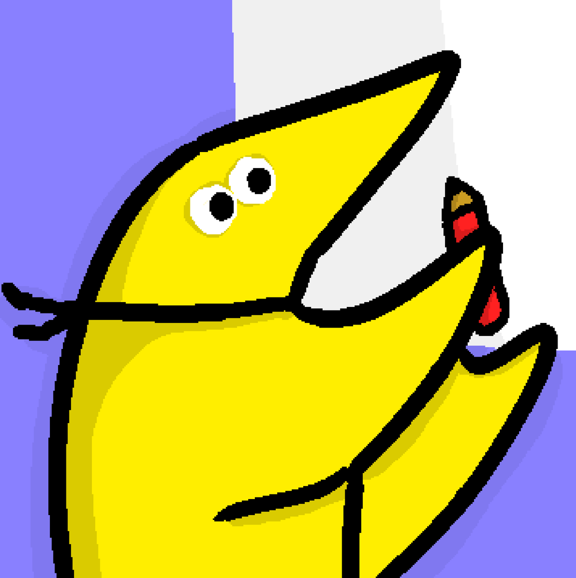

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"homological dimension of K"

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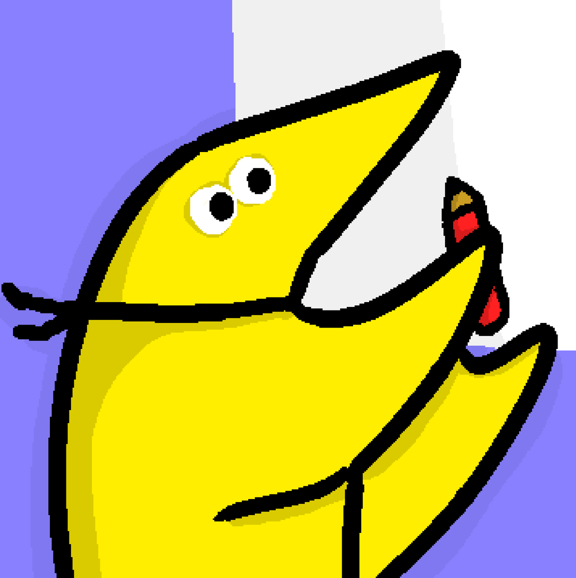

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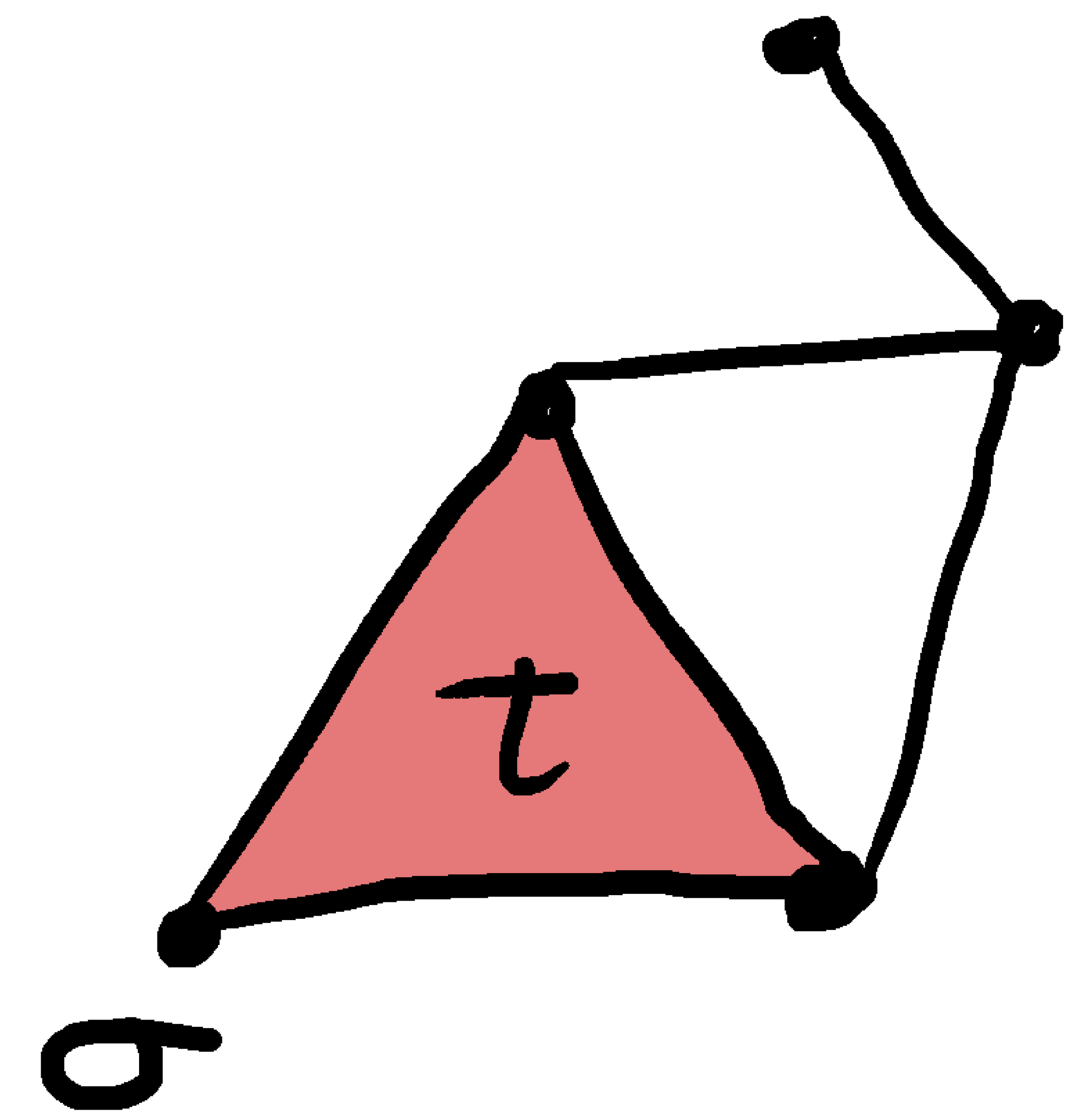

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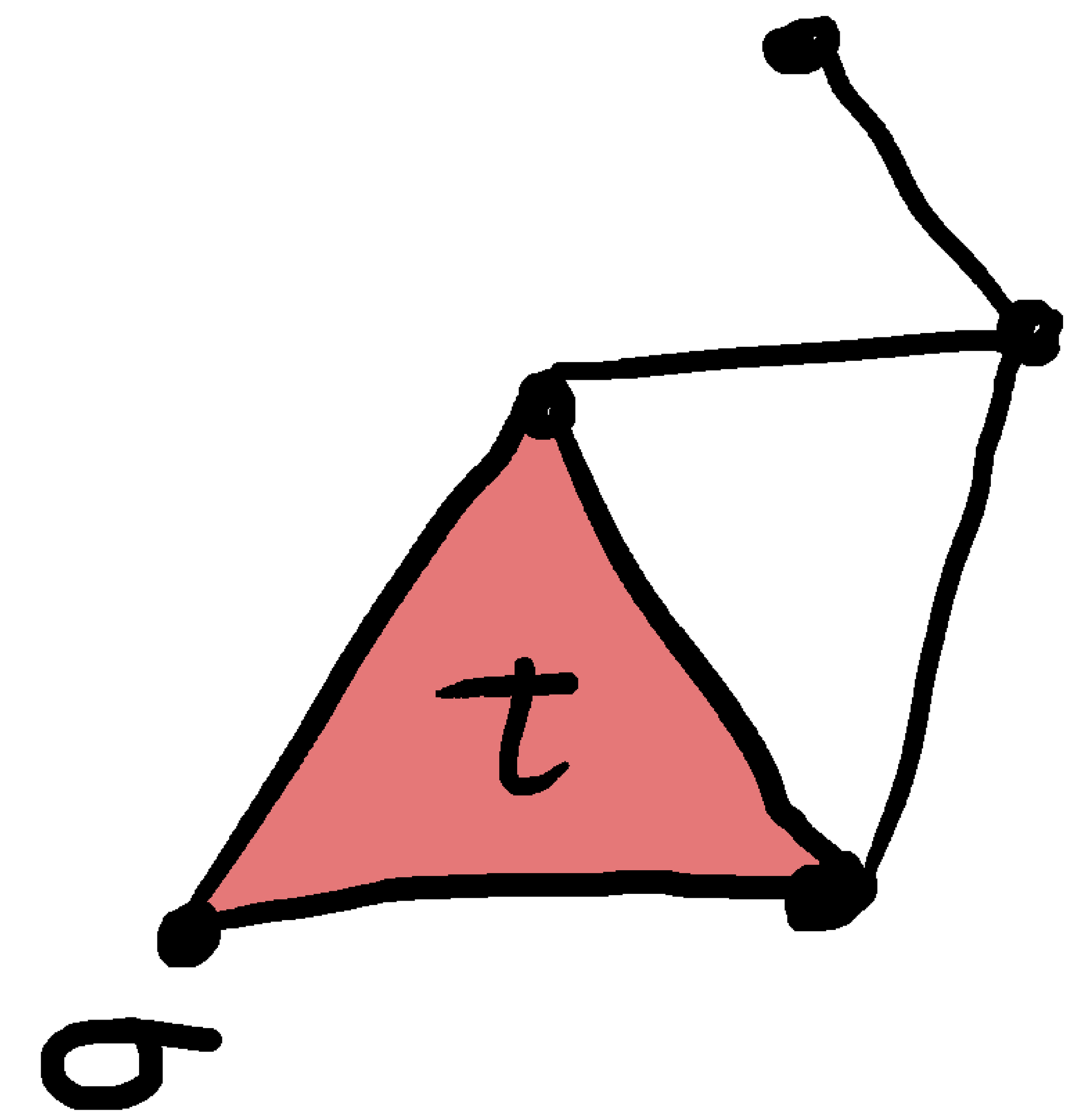

Elementary d-collapse:

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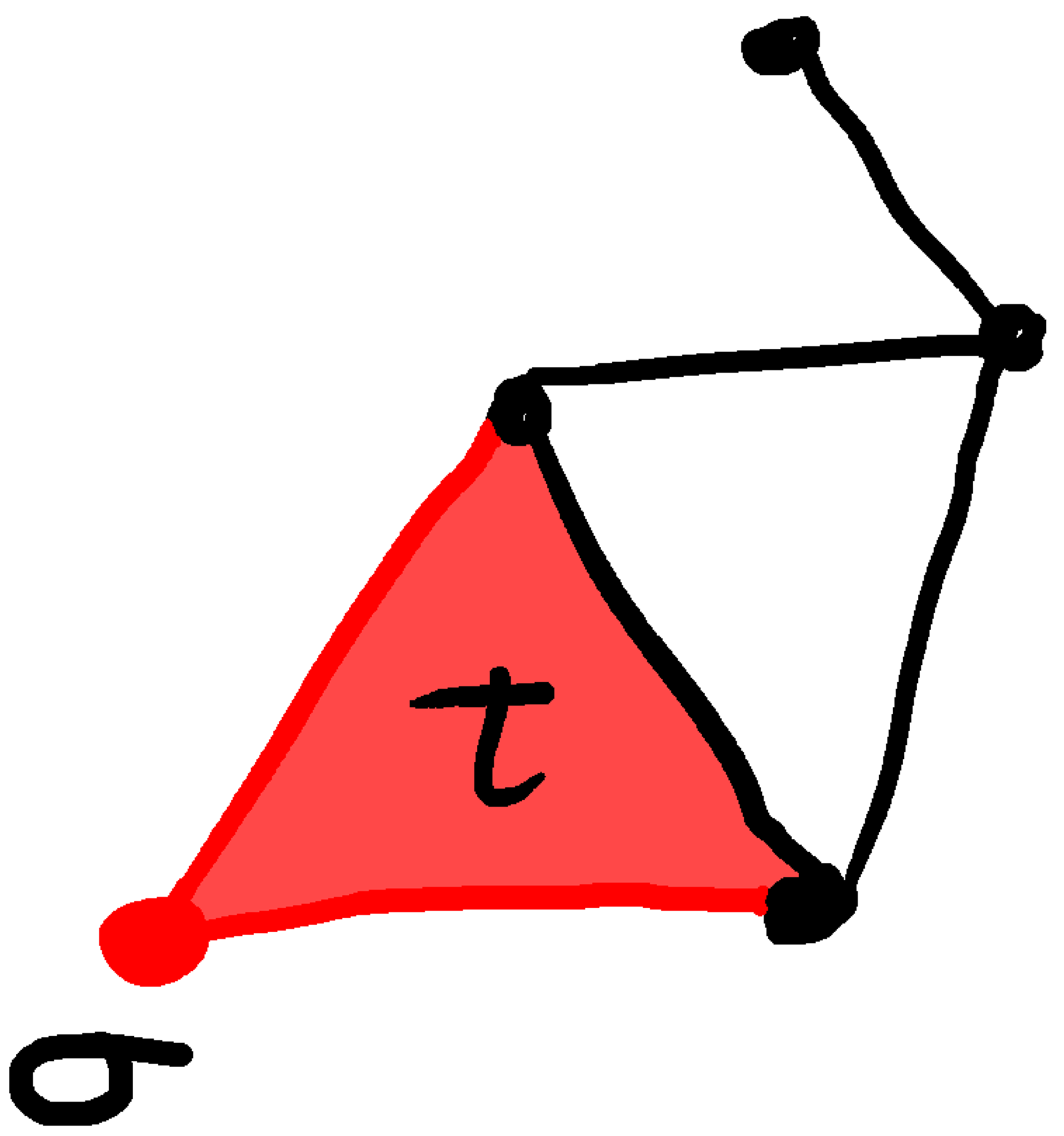

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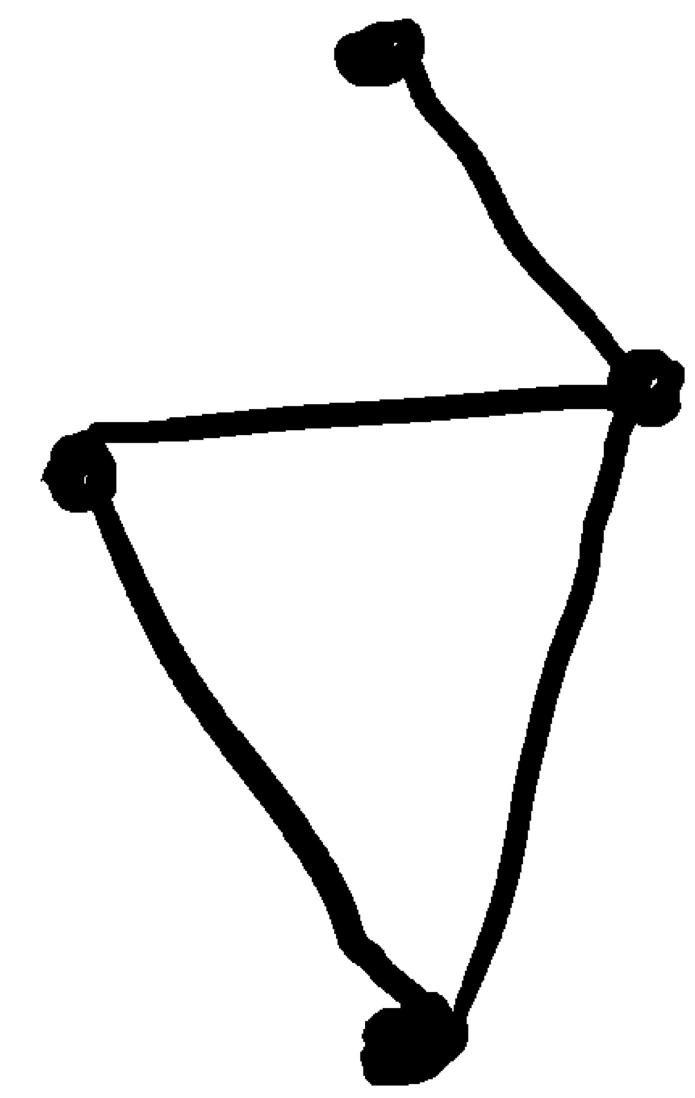

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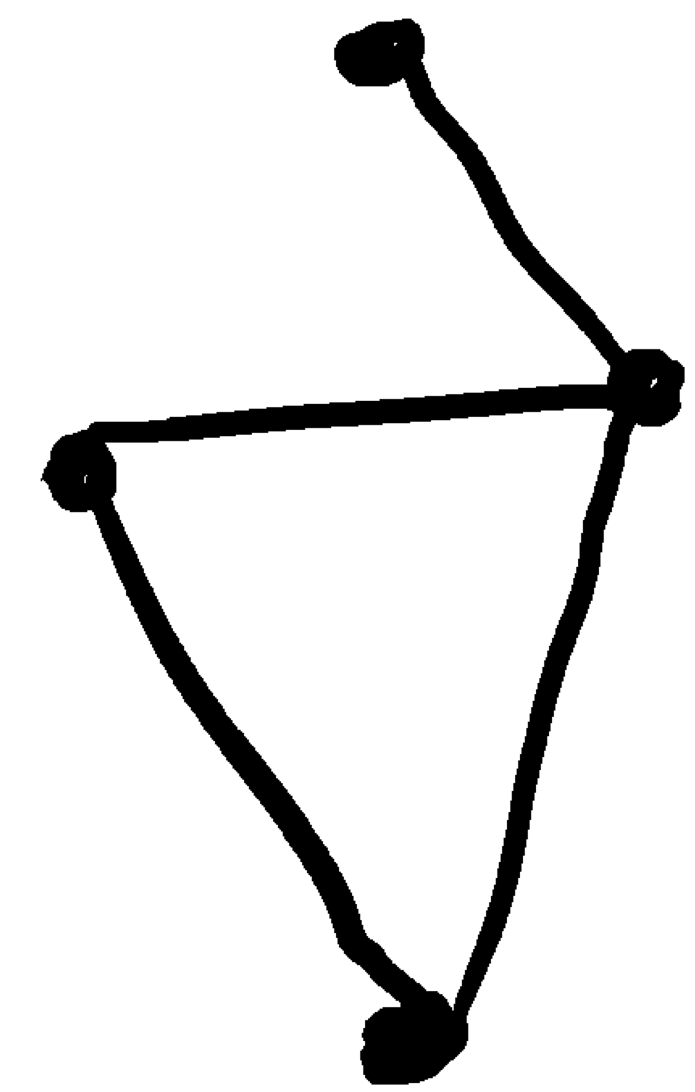
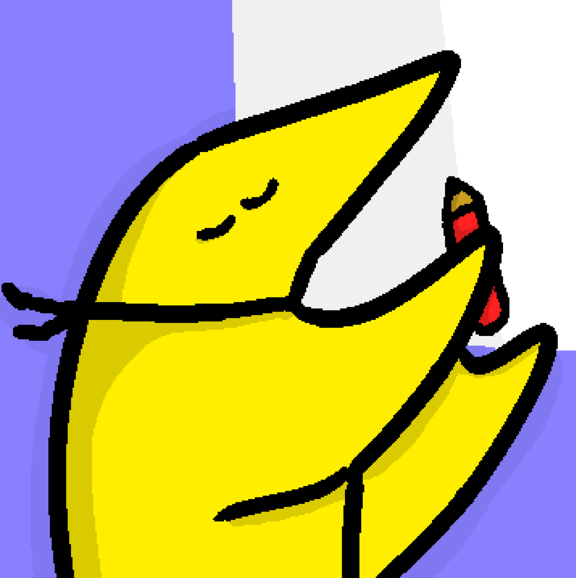


Elementary d-collapse:

- If $\exists$ sequence of elem. d-coll. from K to $\emptyset$: K is **d-collapsible**



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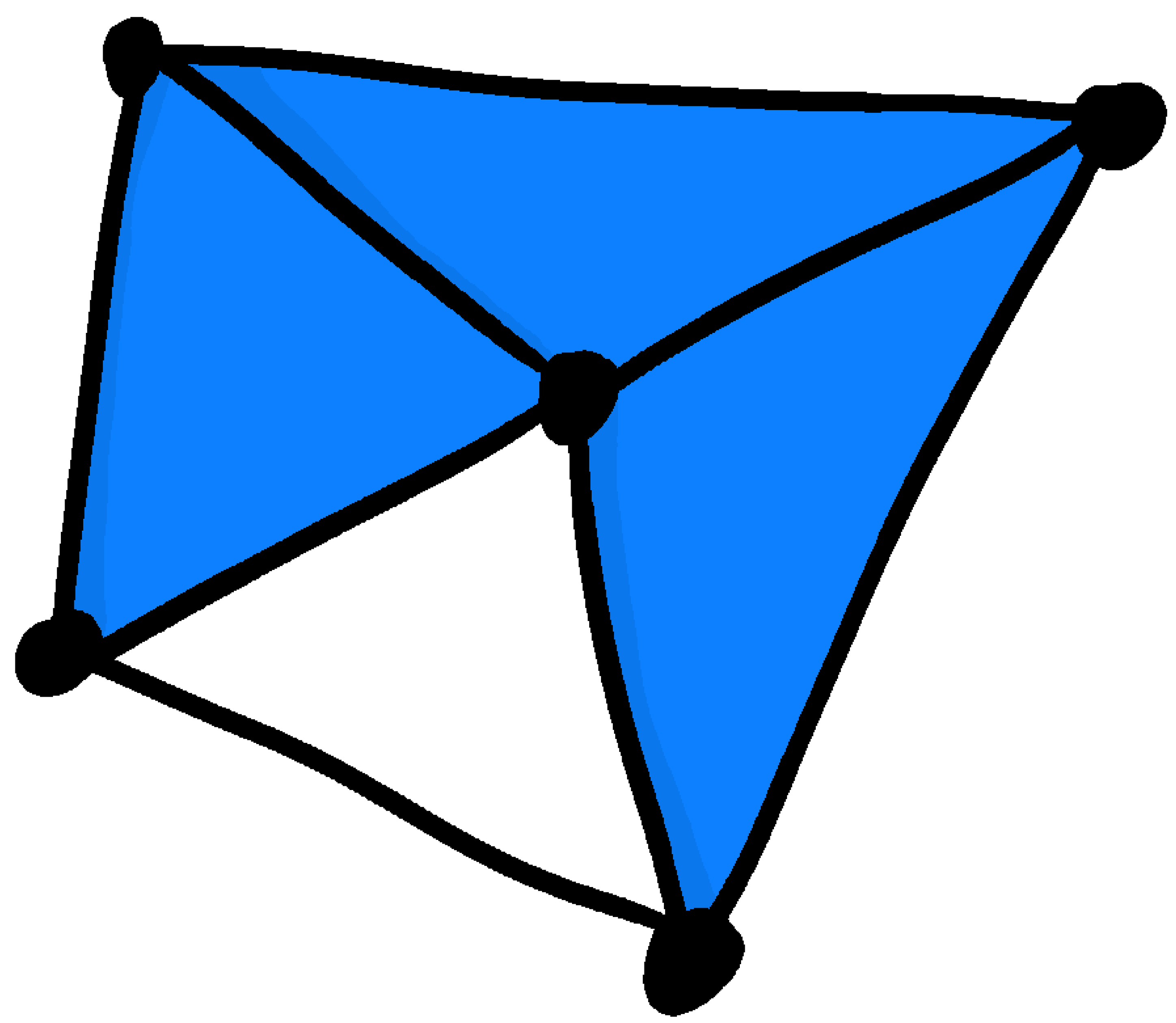
$K =$ 

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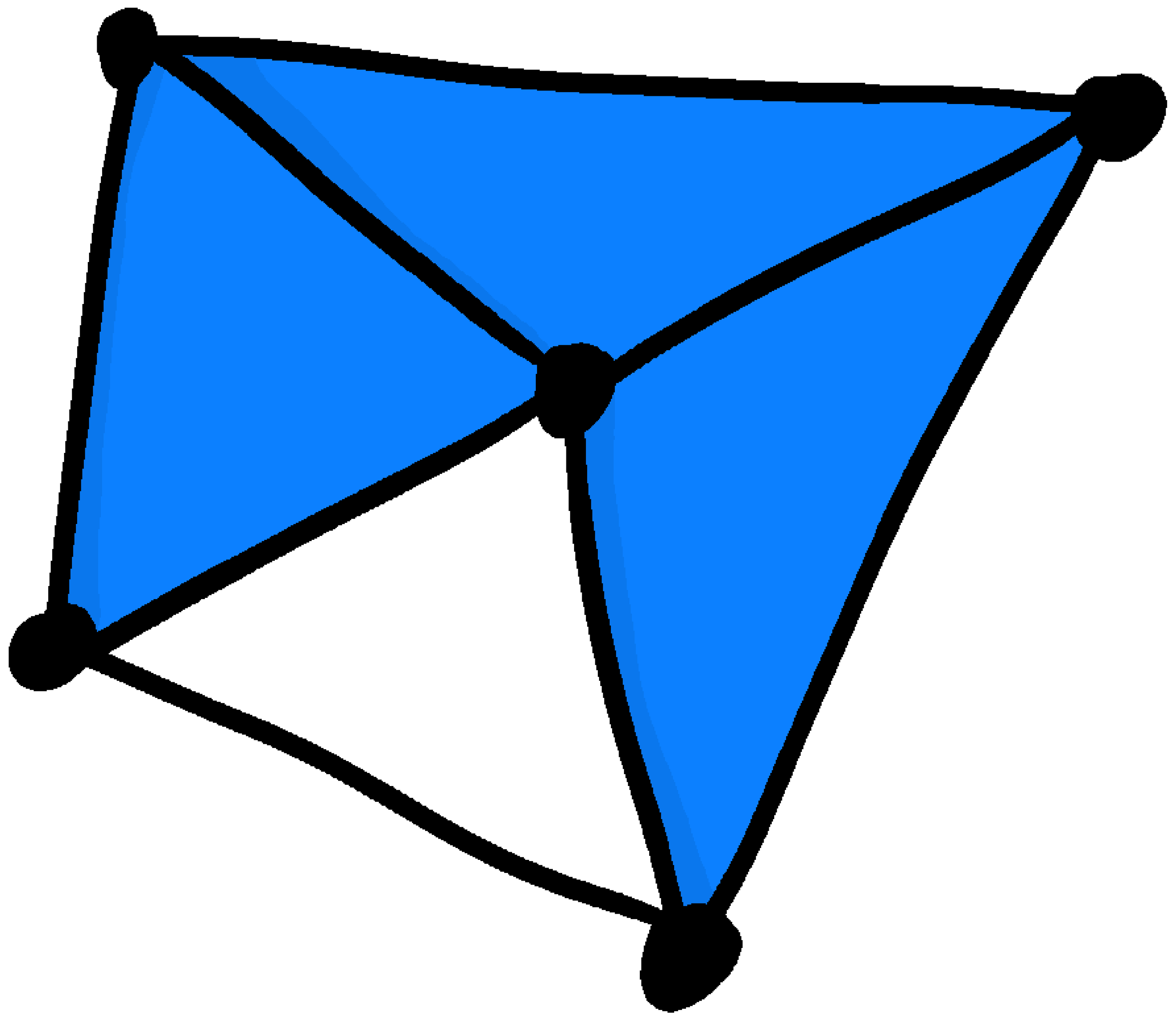
$K =$ 

K is not 1-collapsible

Collapsibility

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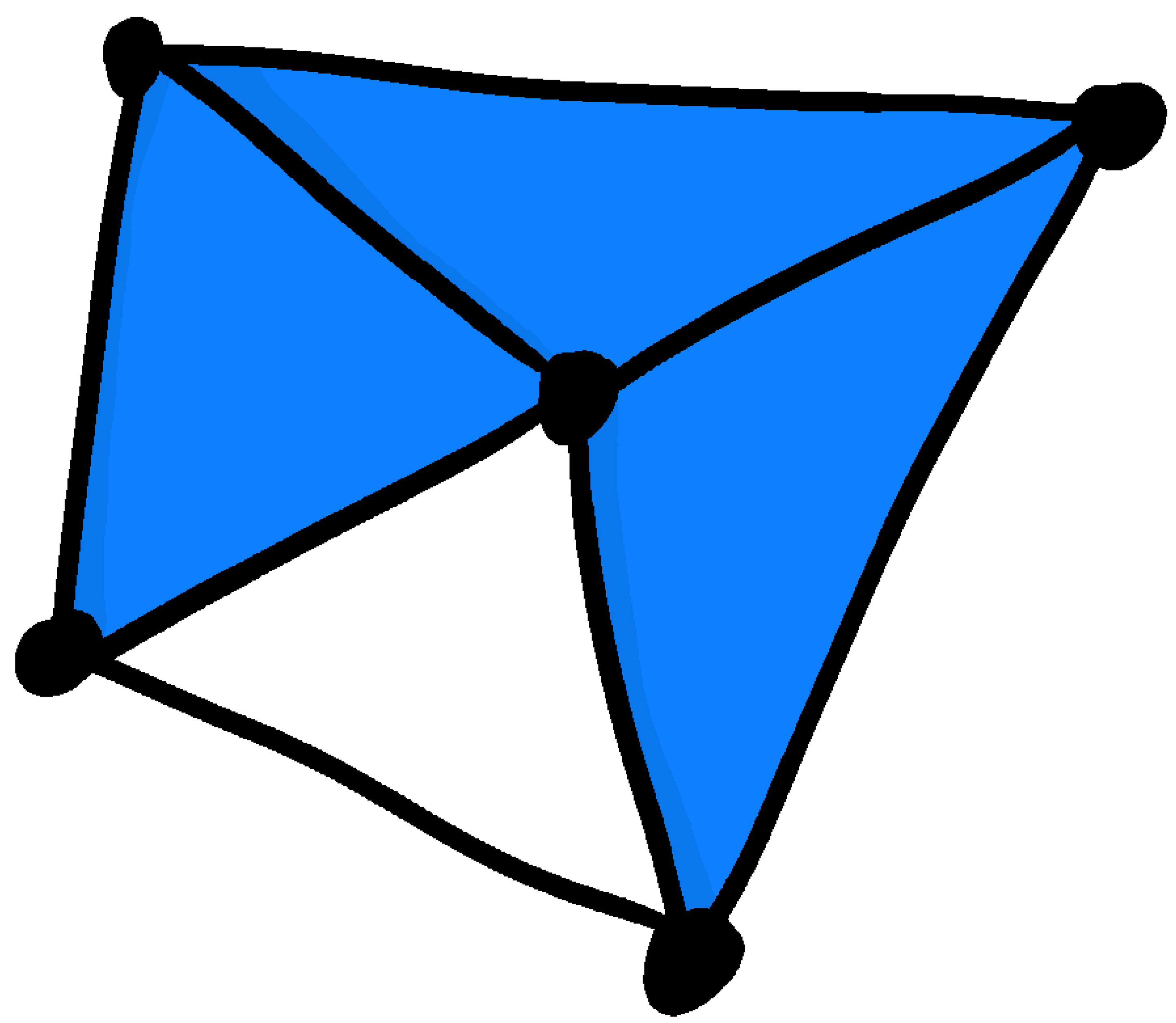
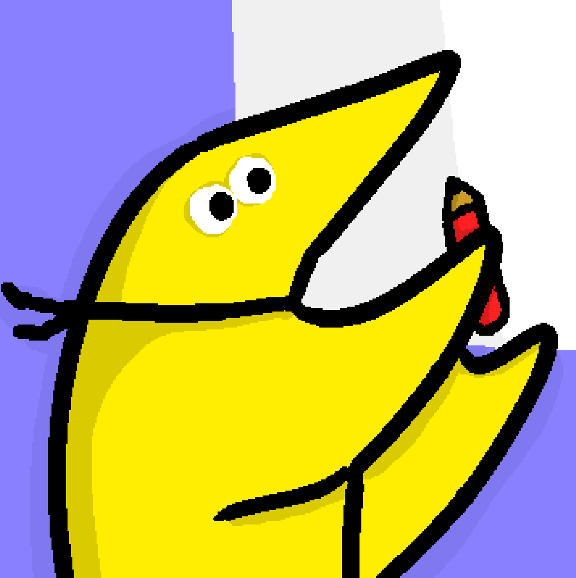
$K =$ 

We will show that K is 2-collapsible.



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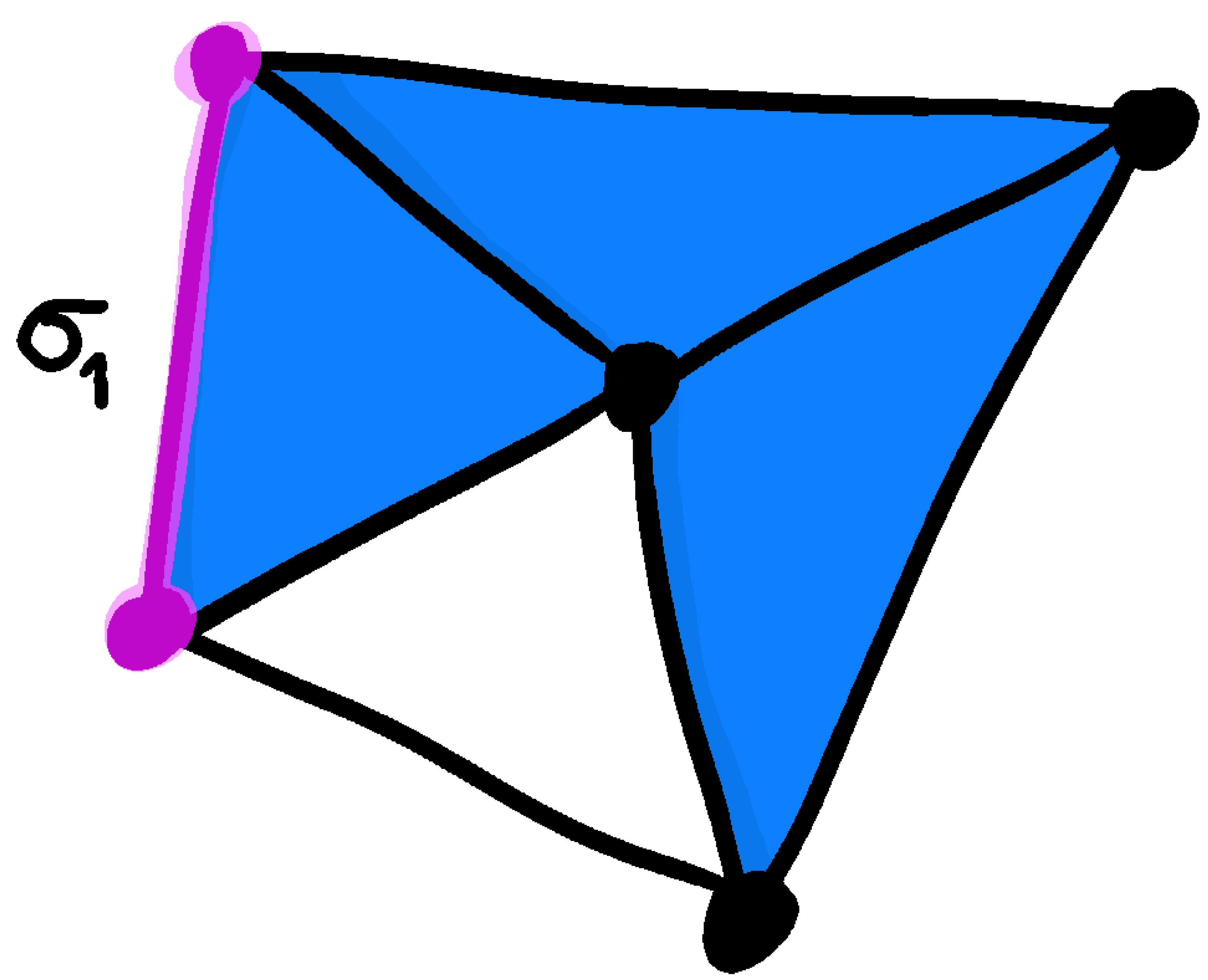
$K =$ 

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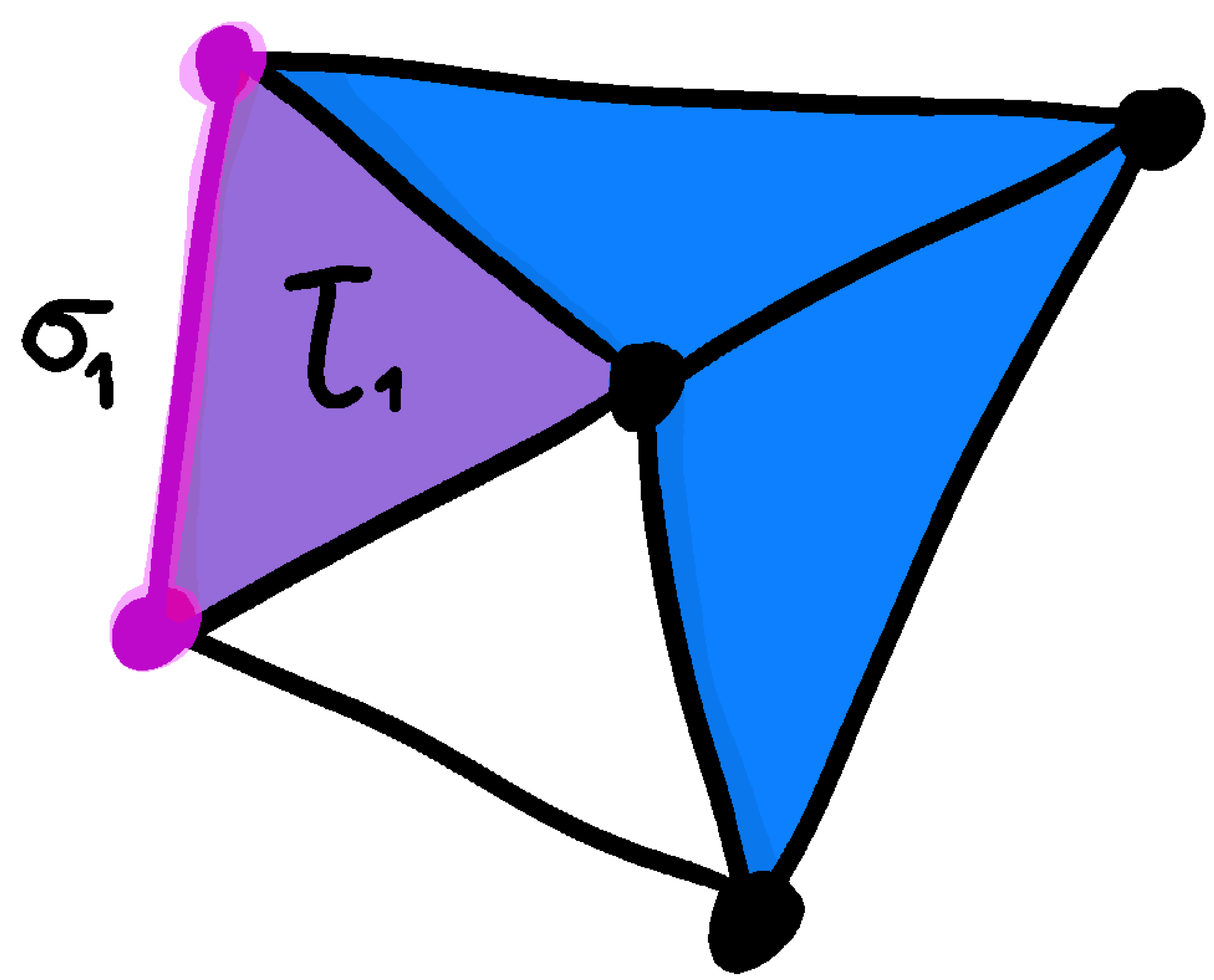

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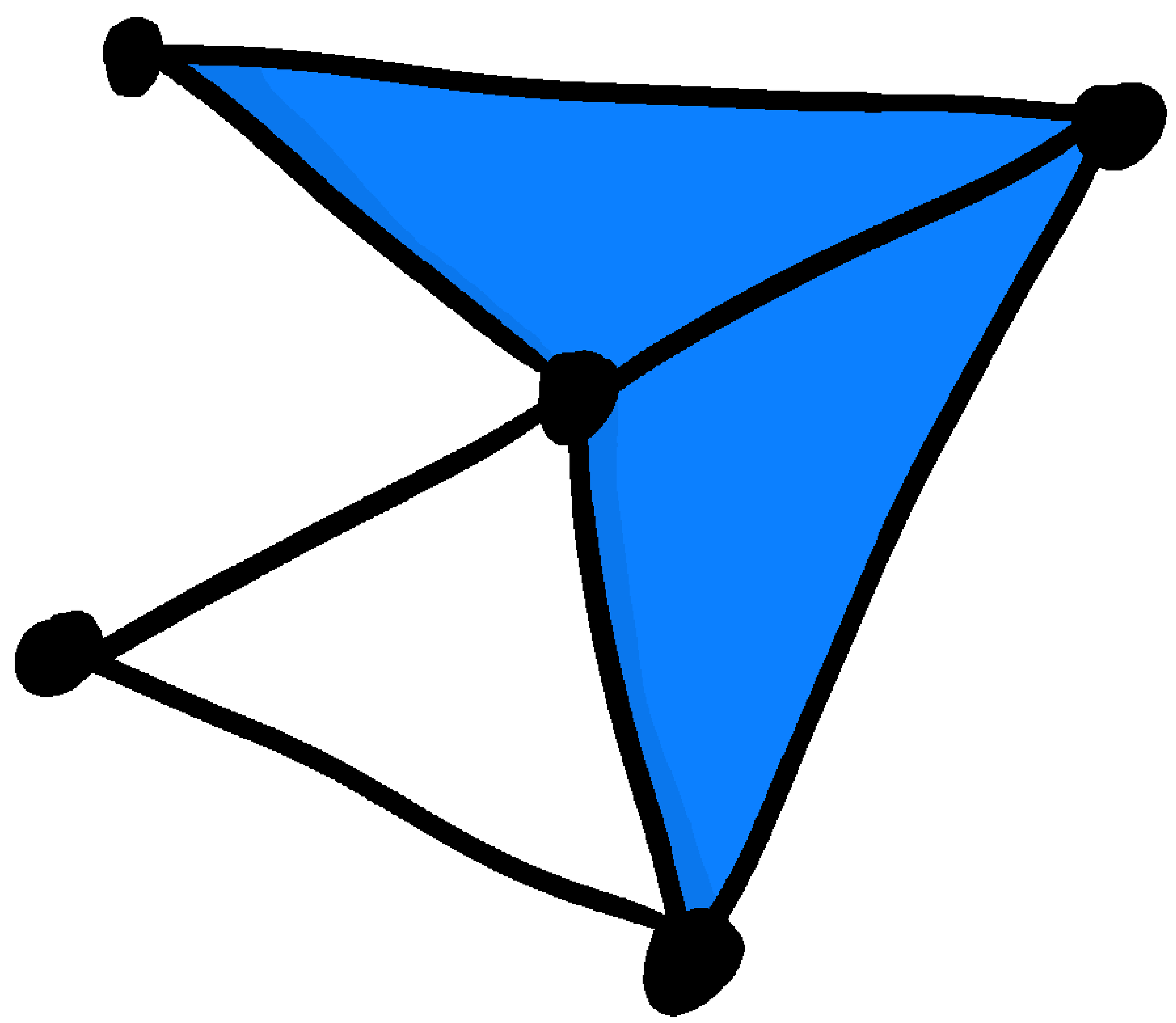
$K_1 =$ 

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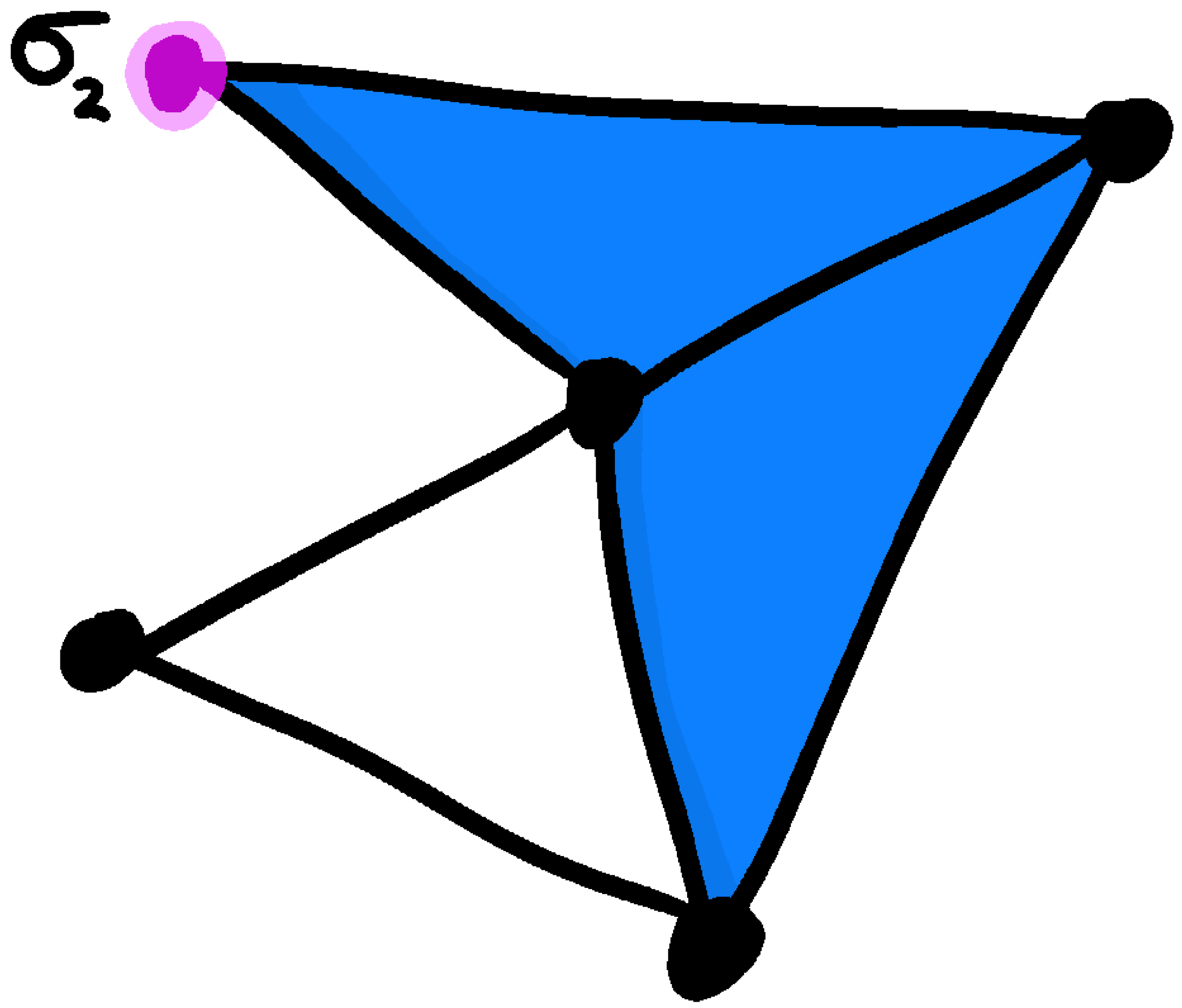
$K_1 =$ 

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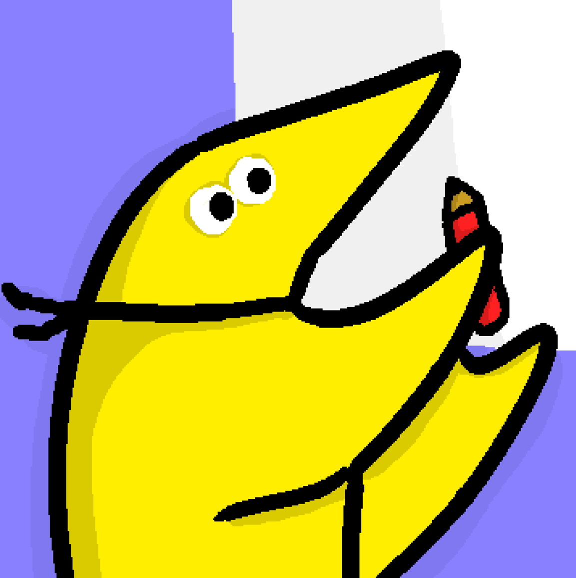

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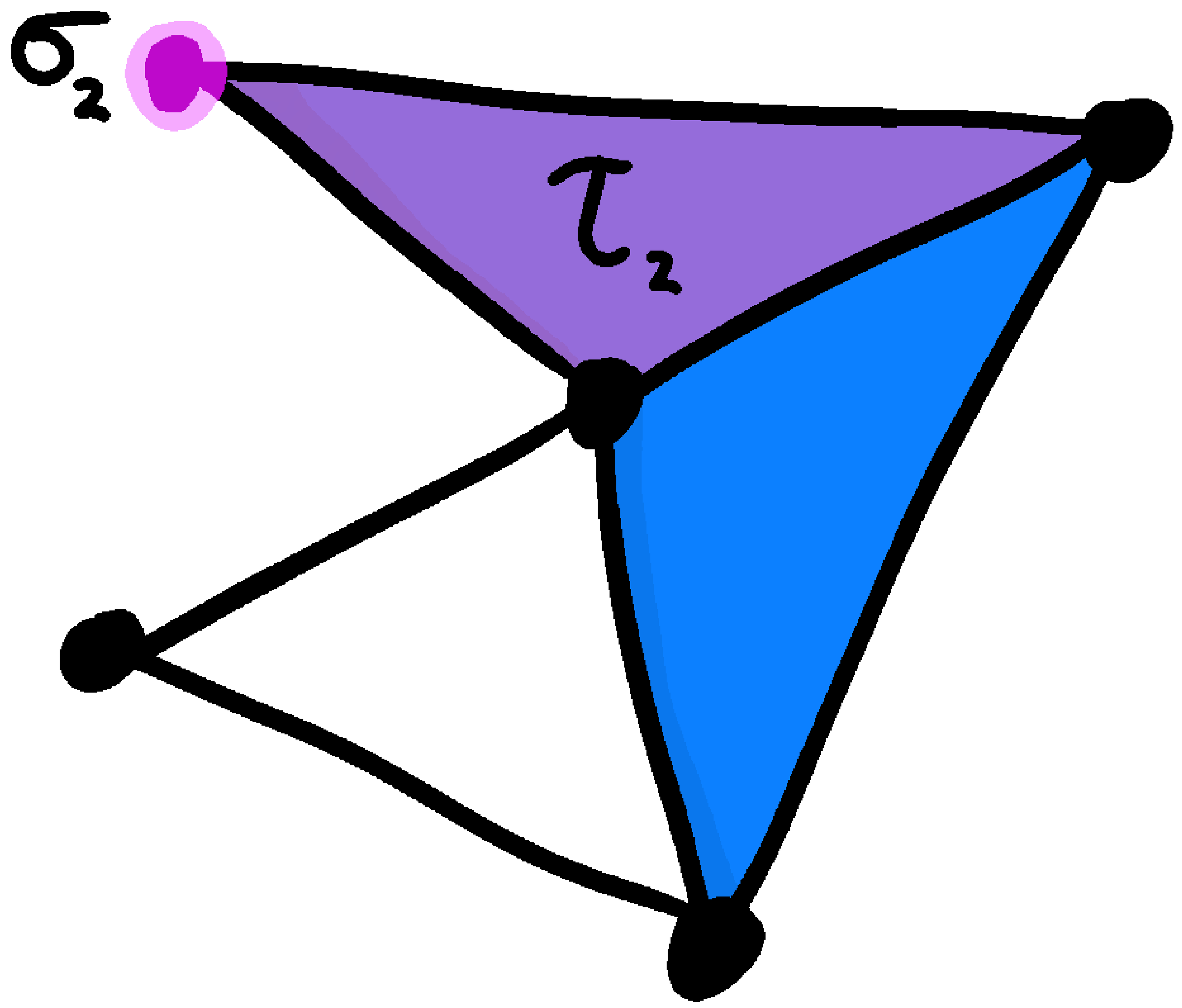
$K_1 =$ 

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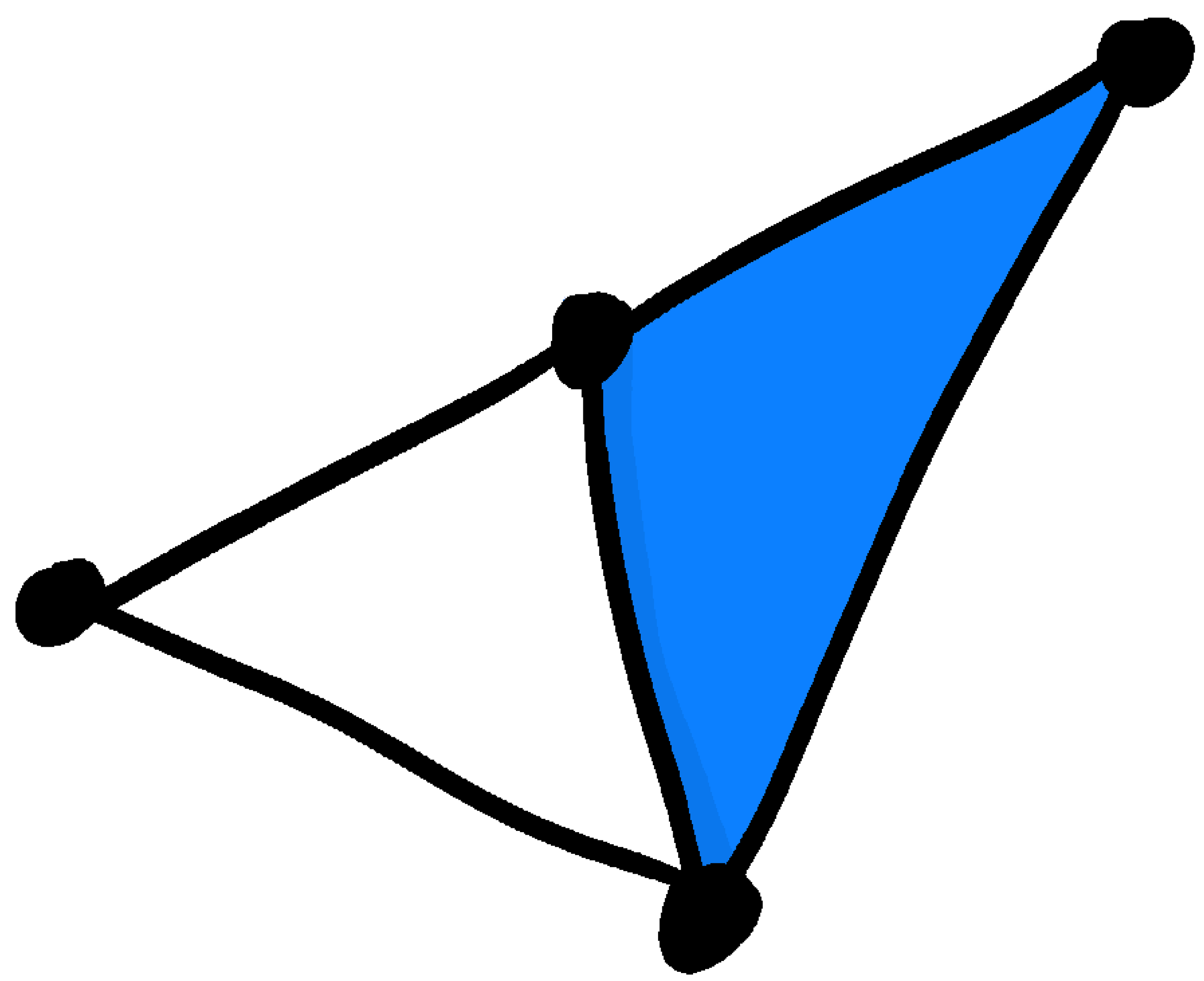

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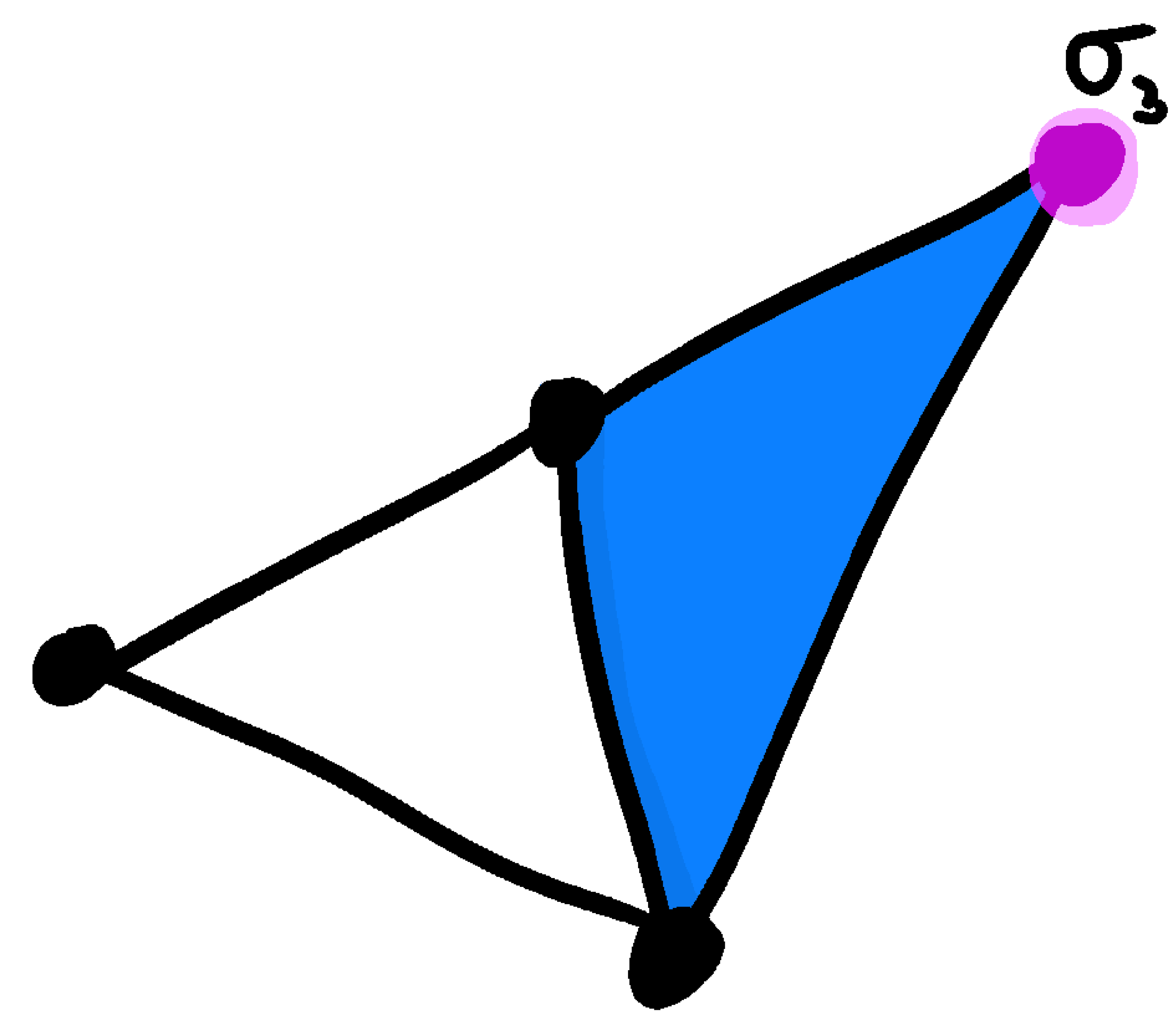

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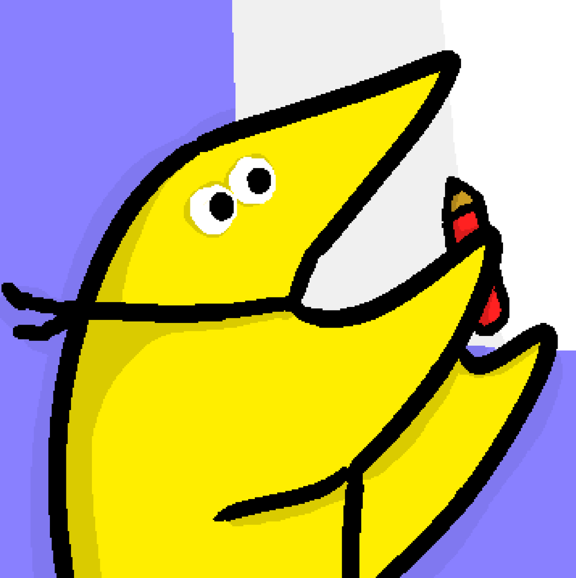

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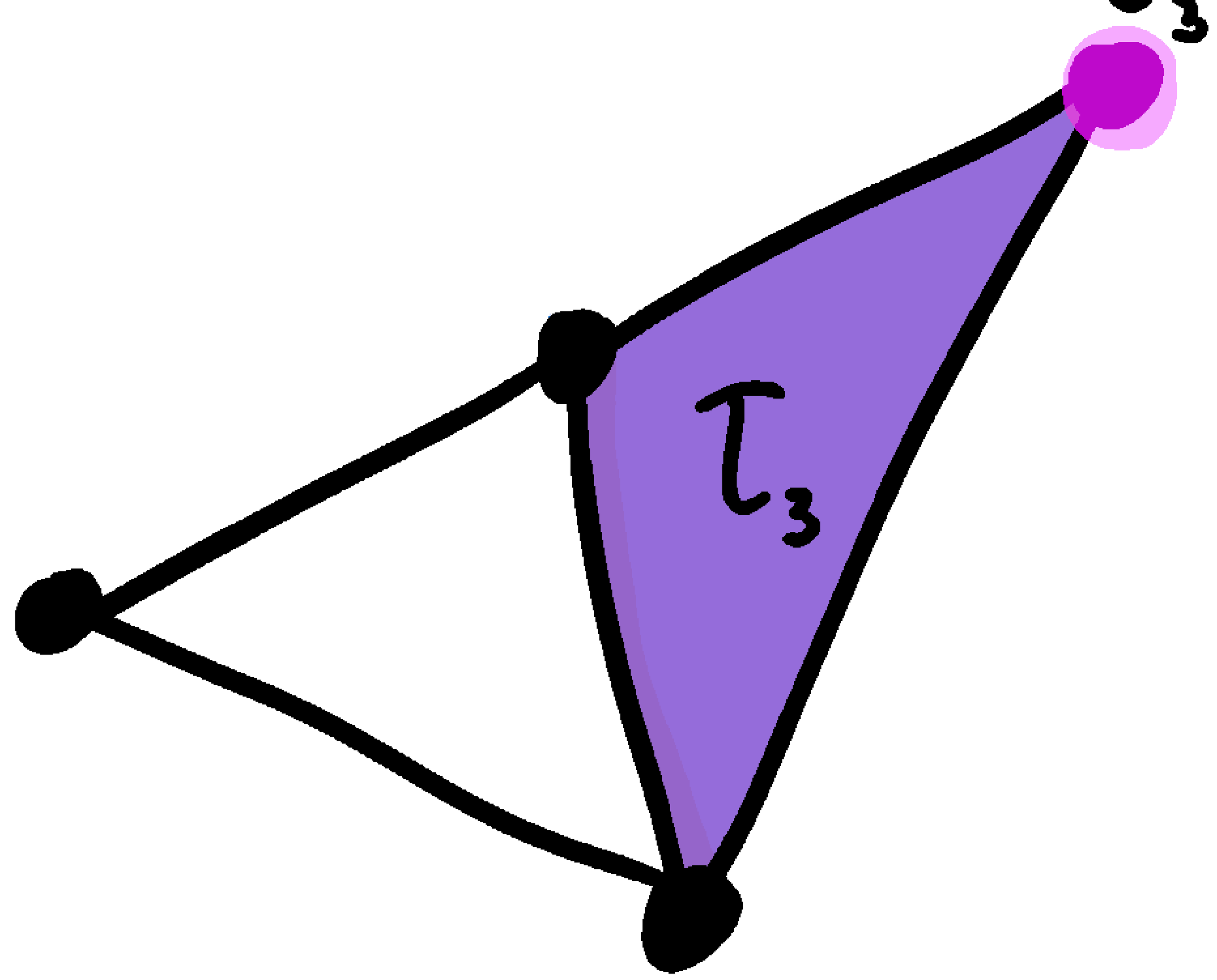

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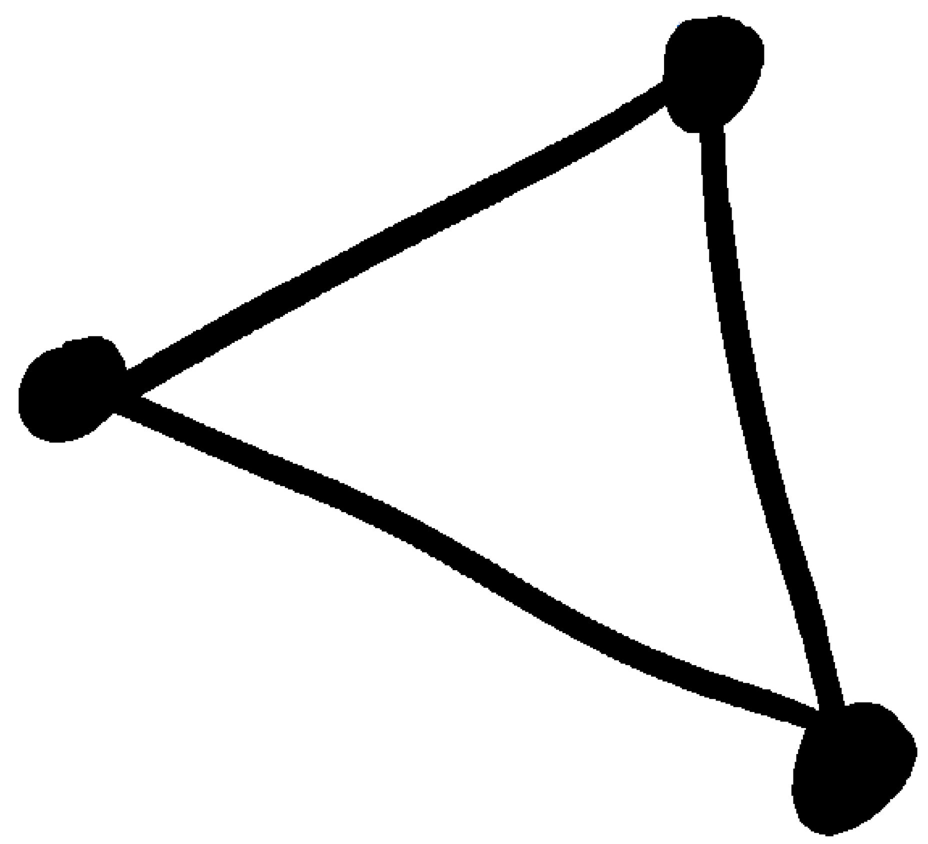
$K_3 =$ 

We will show that K is 2-collapsible.

Collapsibility

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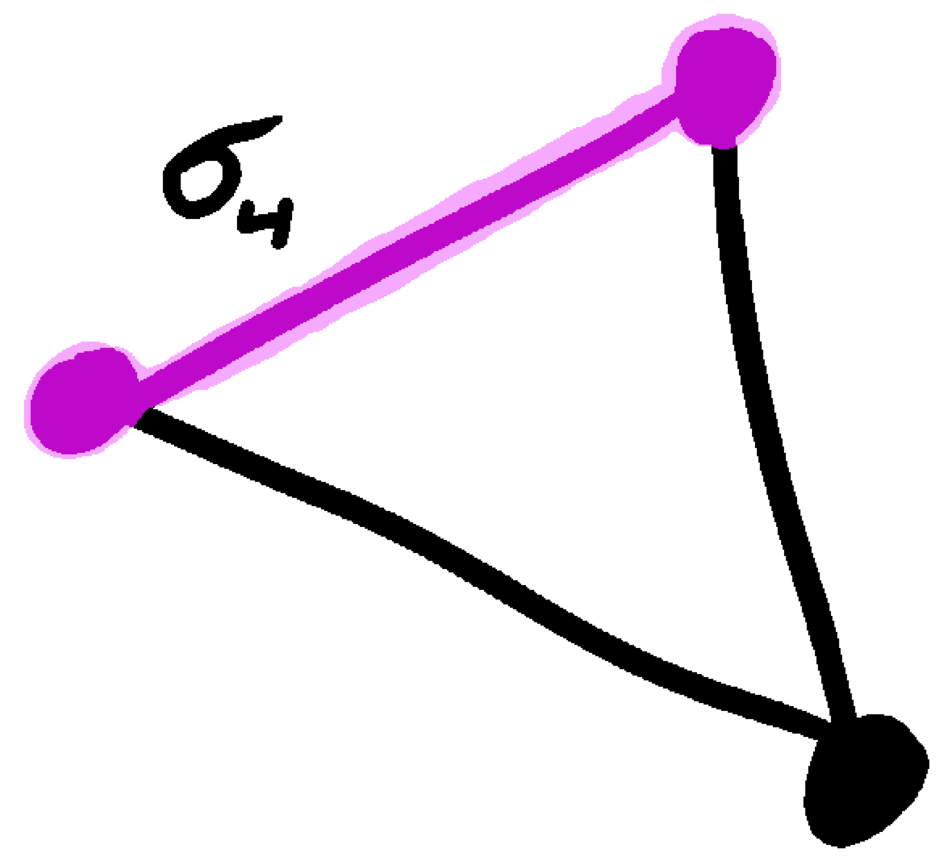
$K_3 =$



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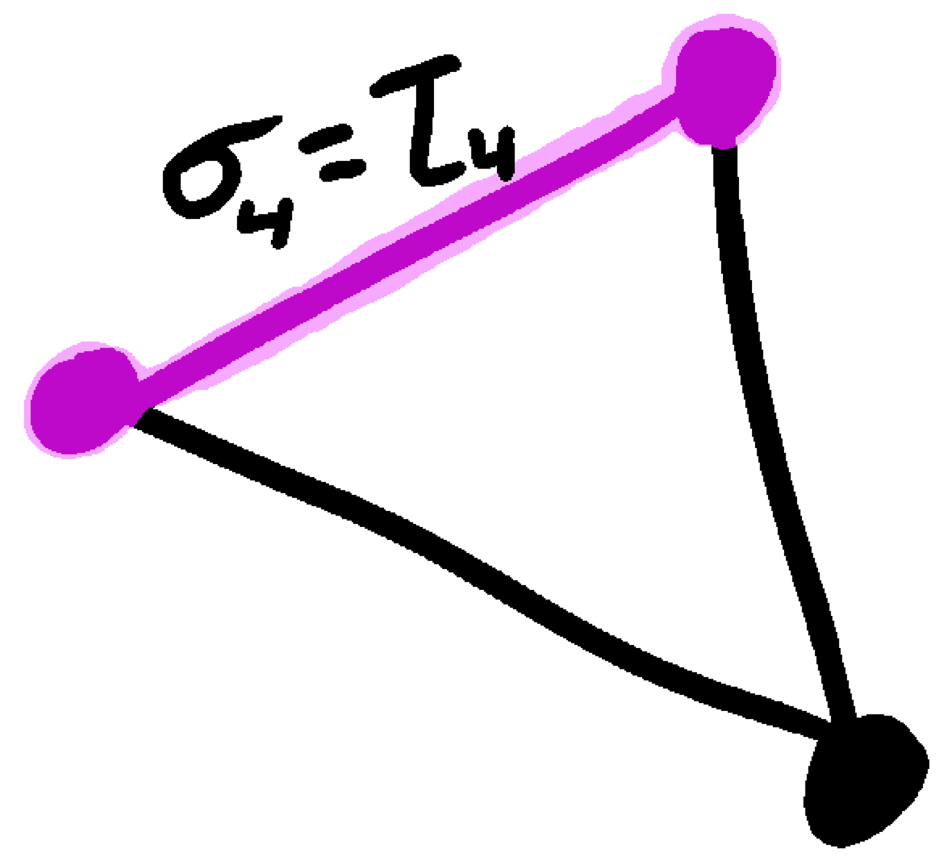
$K_3 =$ 

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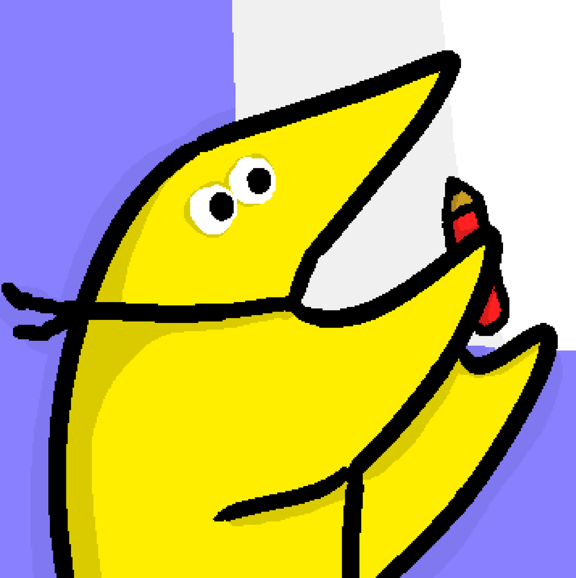
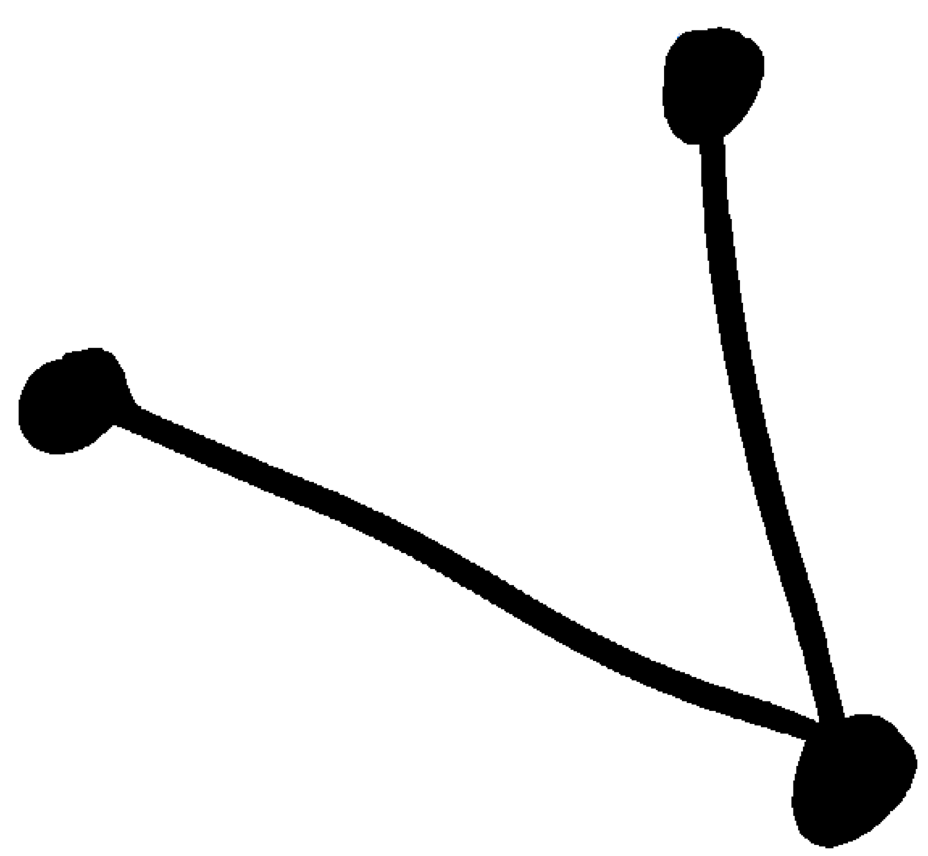


Collapsibility

E.g.

$K_4 =$ 

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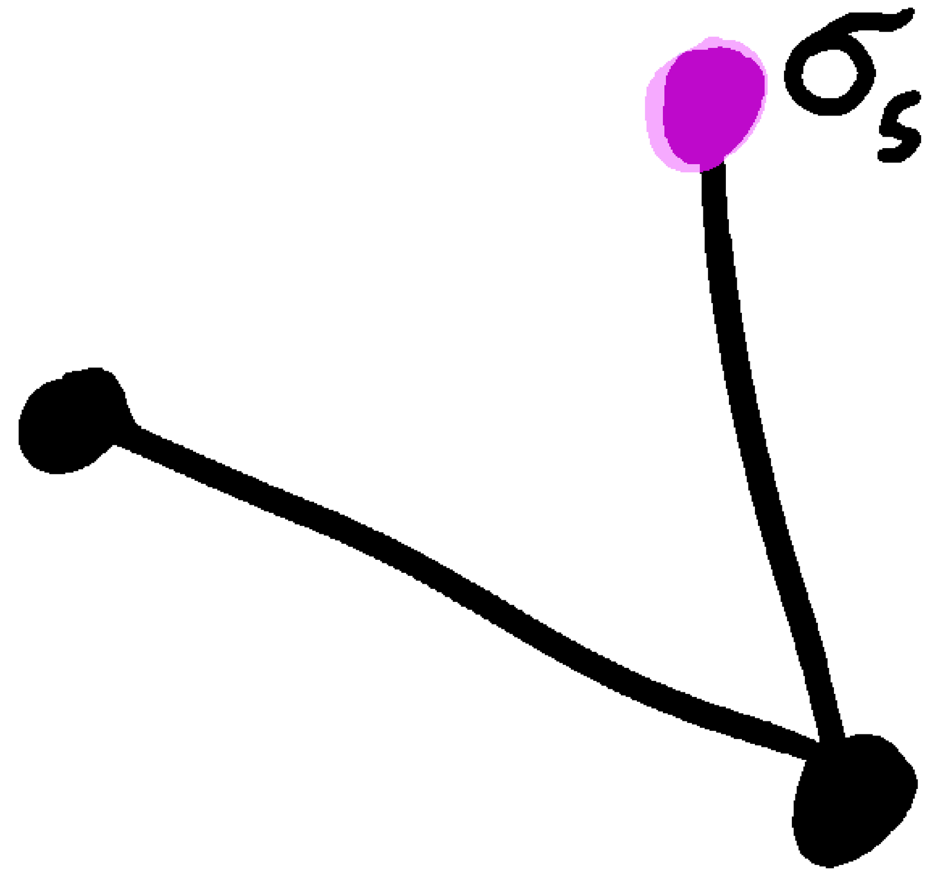
$K_4 =$ 

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E.g.

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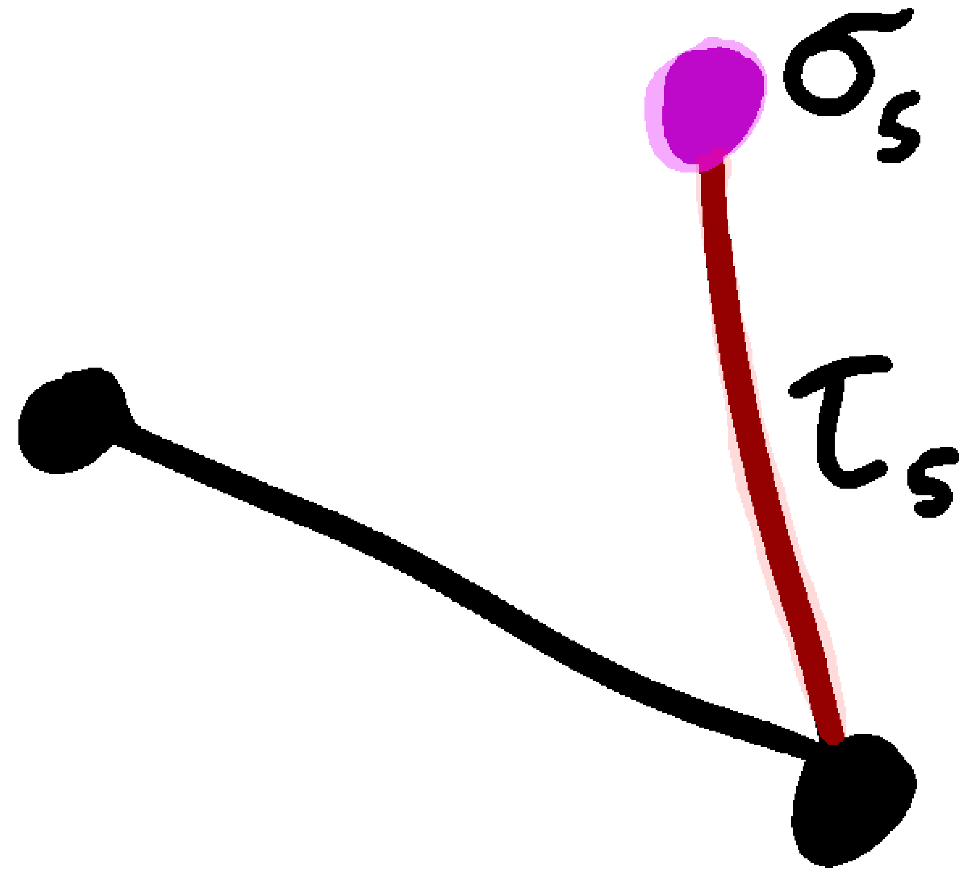

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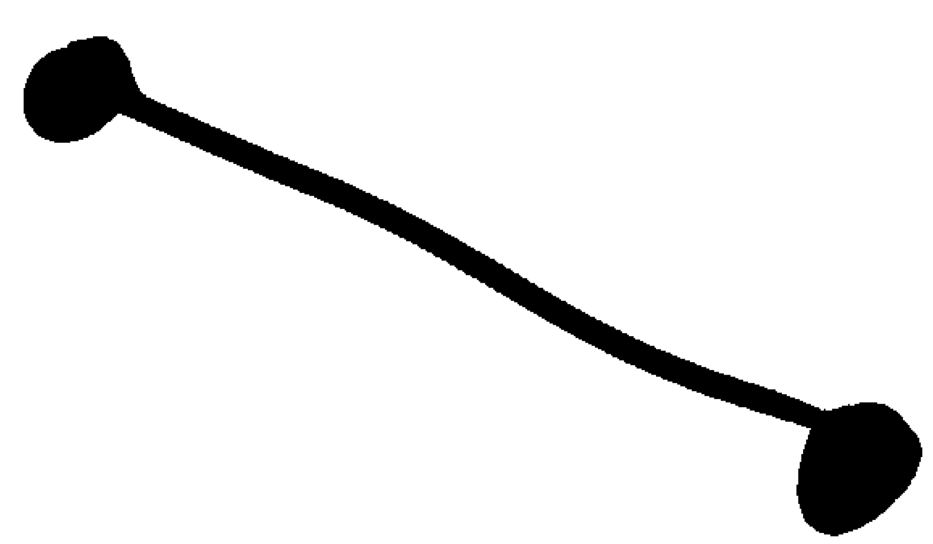
Collapsibility

E.g.

$K_5 =$



We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_5 =$

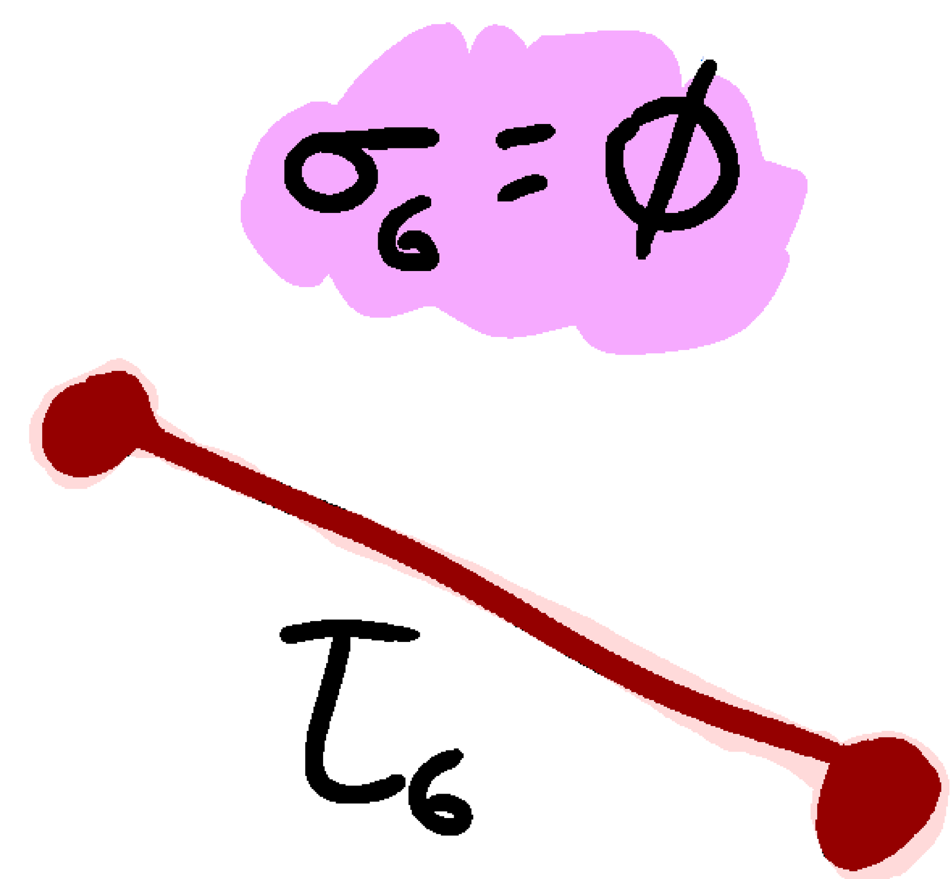
$\sigma_6 = \emptyset$

We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_5 =$



We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_6 =$

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Collapsibility

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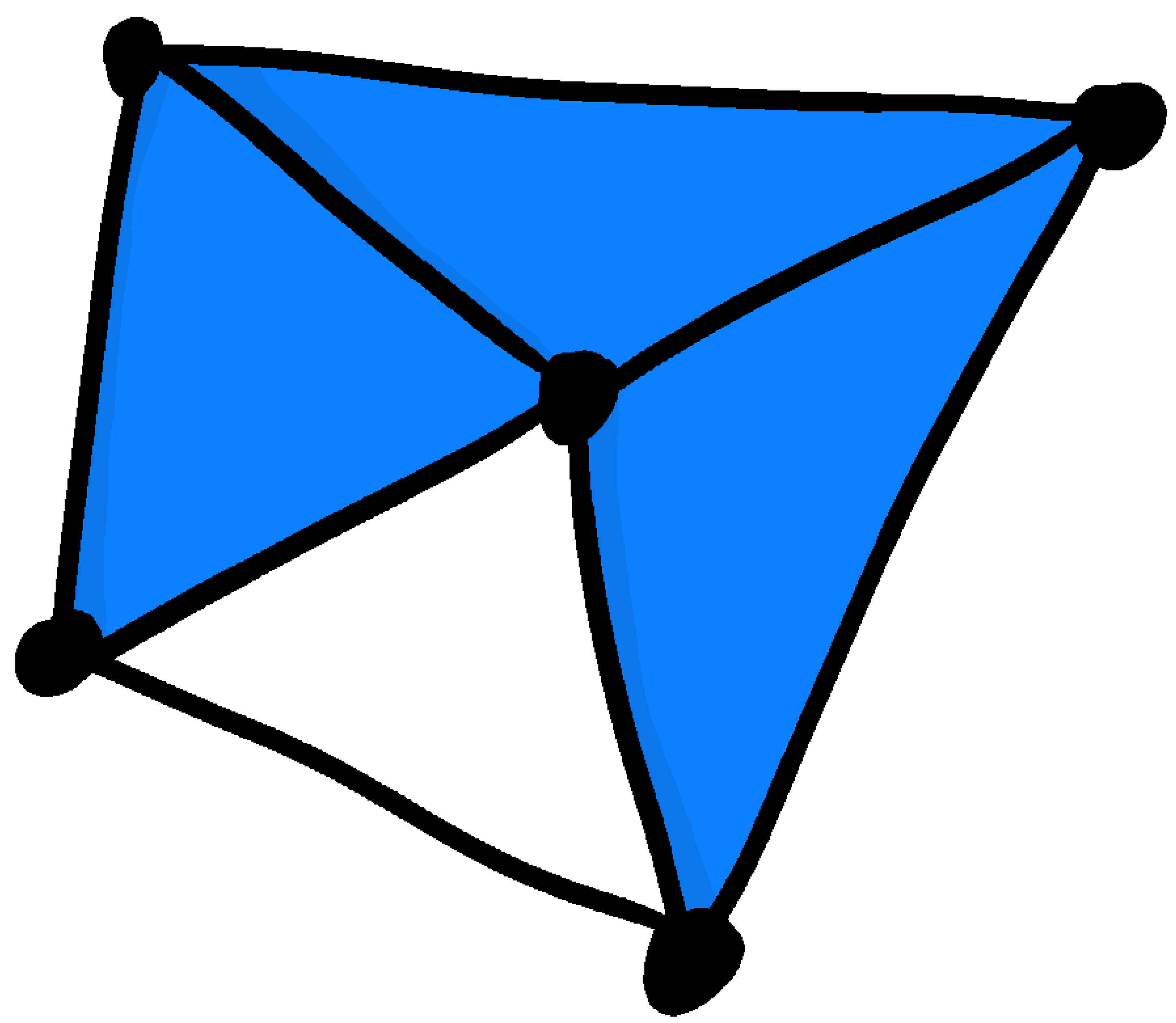
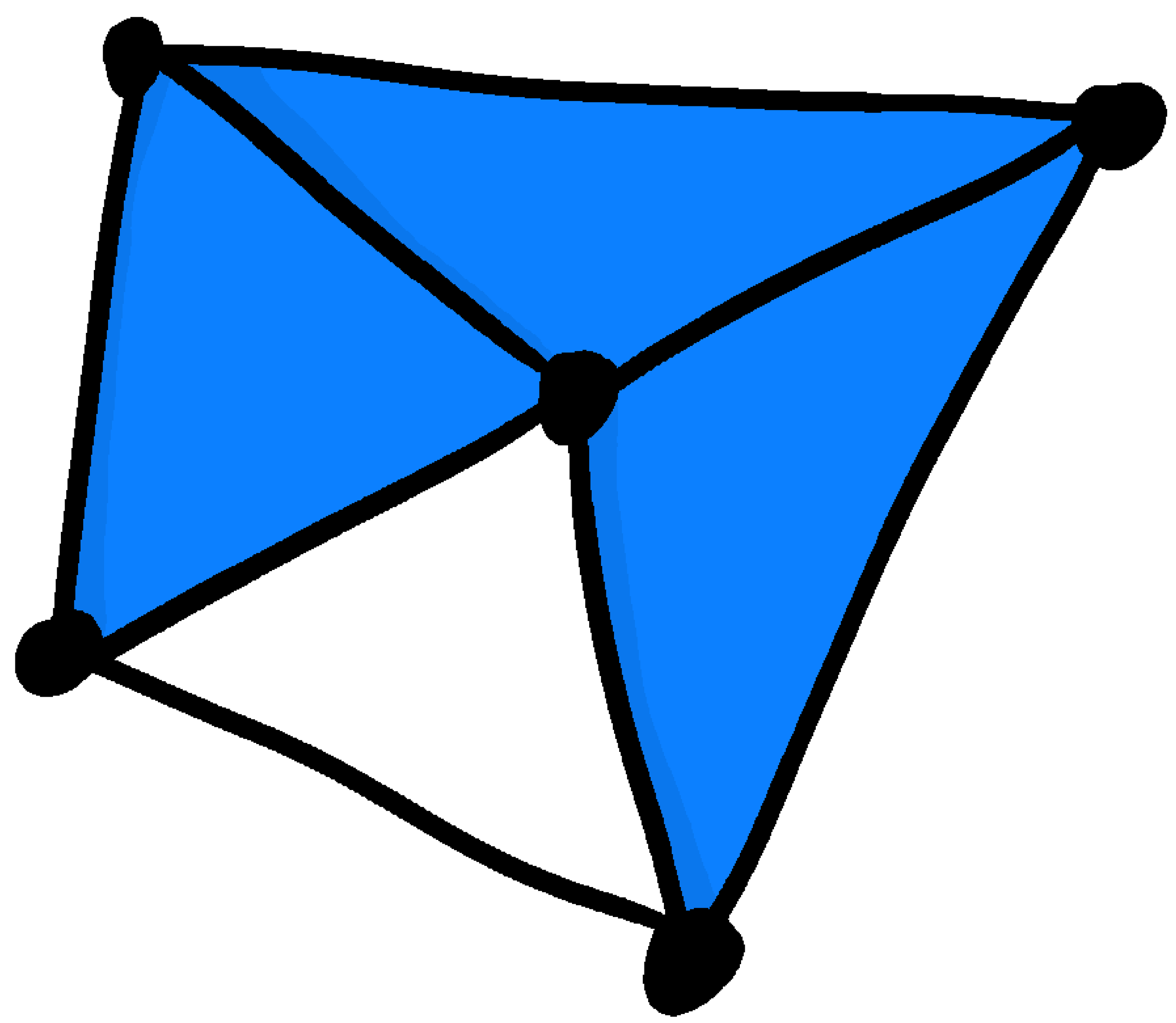
$K =$ 

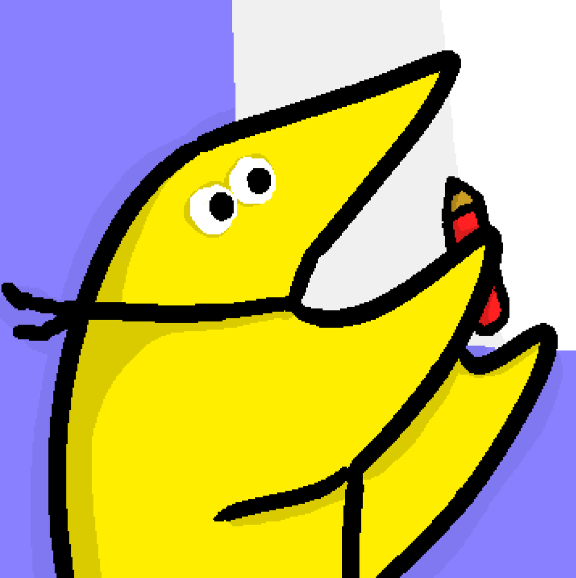
K is 2-collapsible.

Collapsibility

E.g.

$K =$  K is 2-collapsible.

- Collapsibility of K = minimal d s.t. K is d-collapsible.



Representability

- $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ family of sets.

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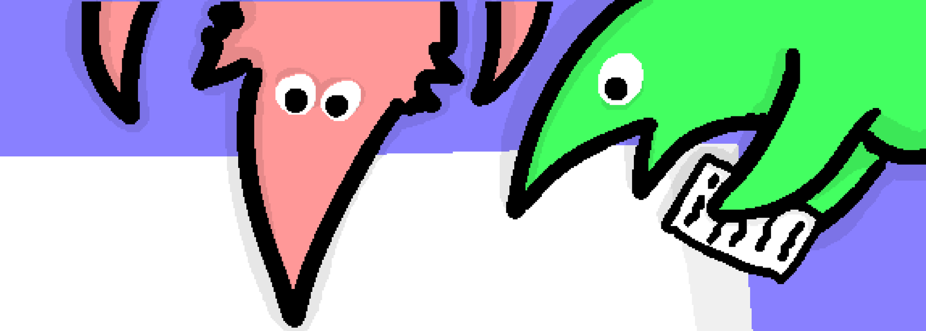
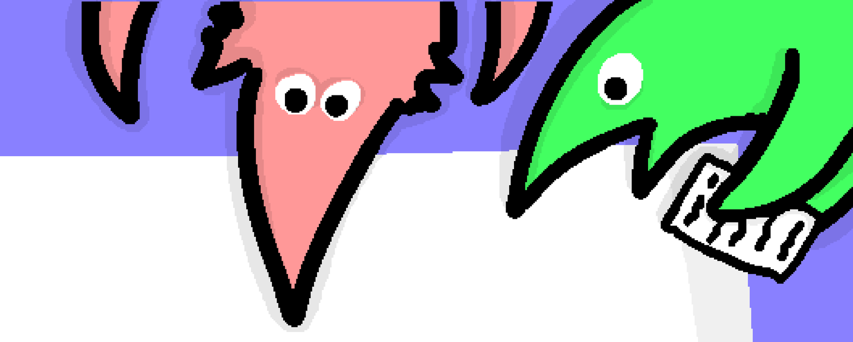


Representability

- $F = \{F_1, F_2, \ldots, F_n\}$ family of sets.
- $N(F)$ = nerve of F:
  - A vertex is assigned to each set in the family.



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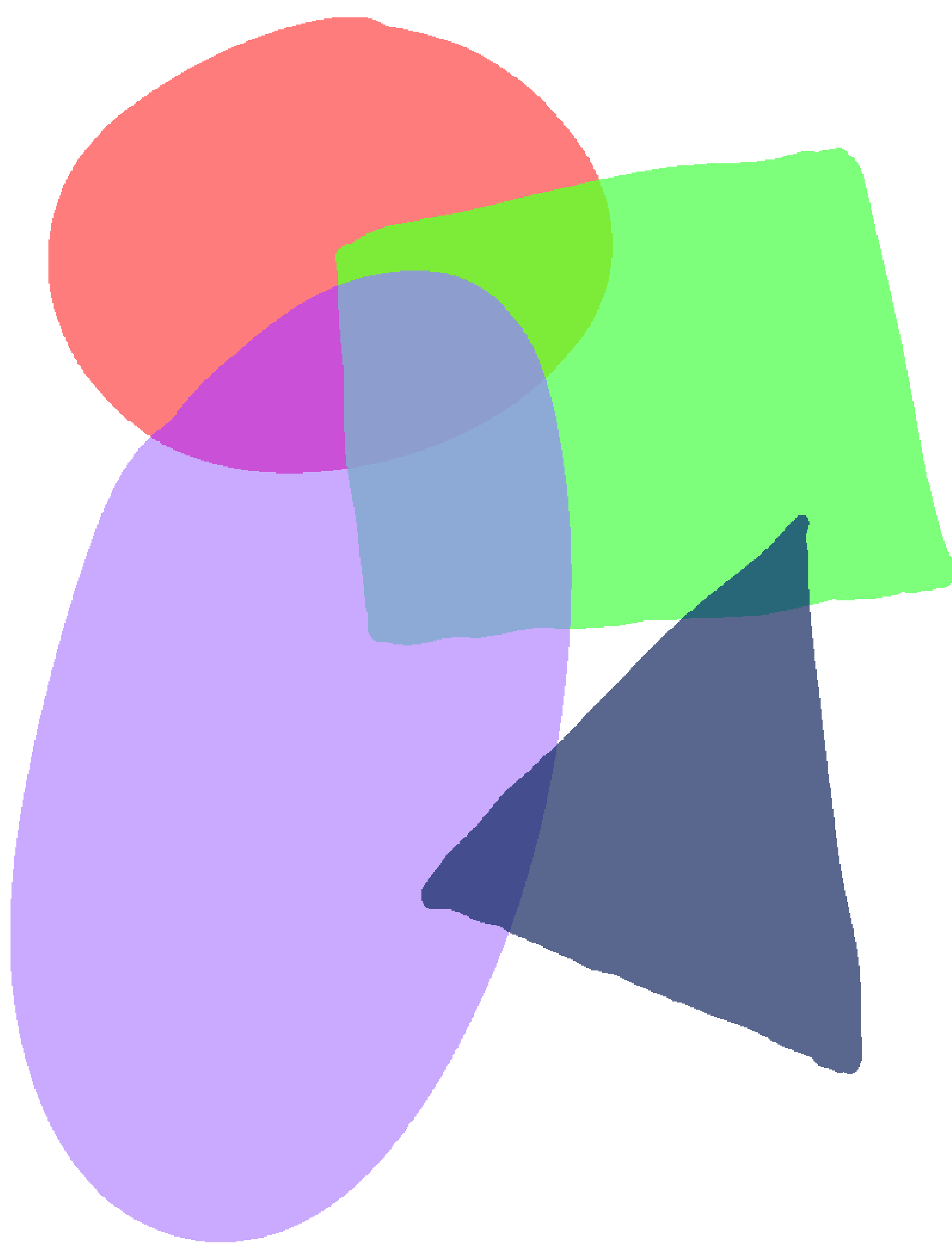
  - A **vertex** is assigned to each **set** in the family.
  - **Simplices** correspond to subfamilies with **non empty intersection**.

Representability

E.g.

$F =$ 

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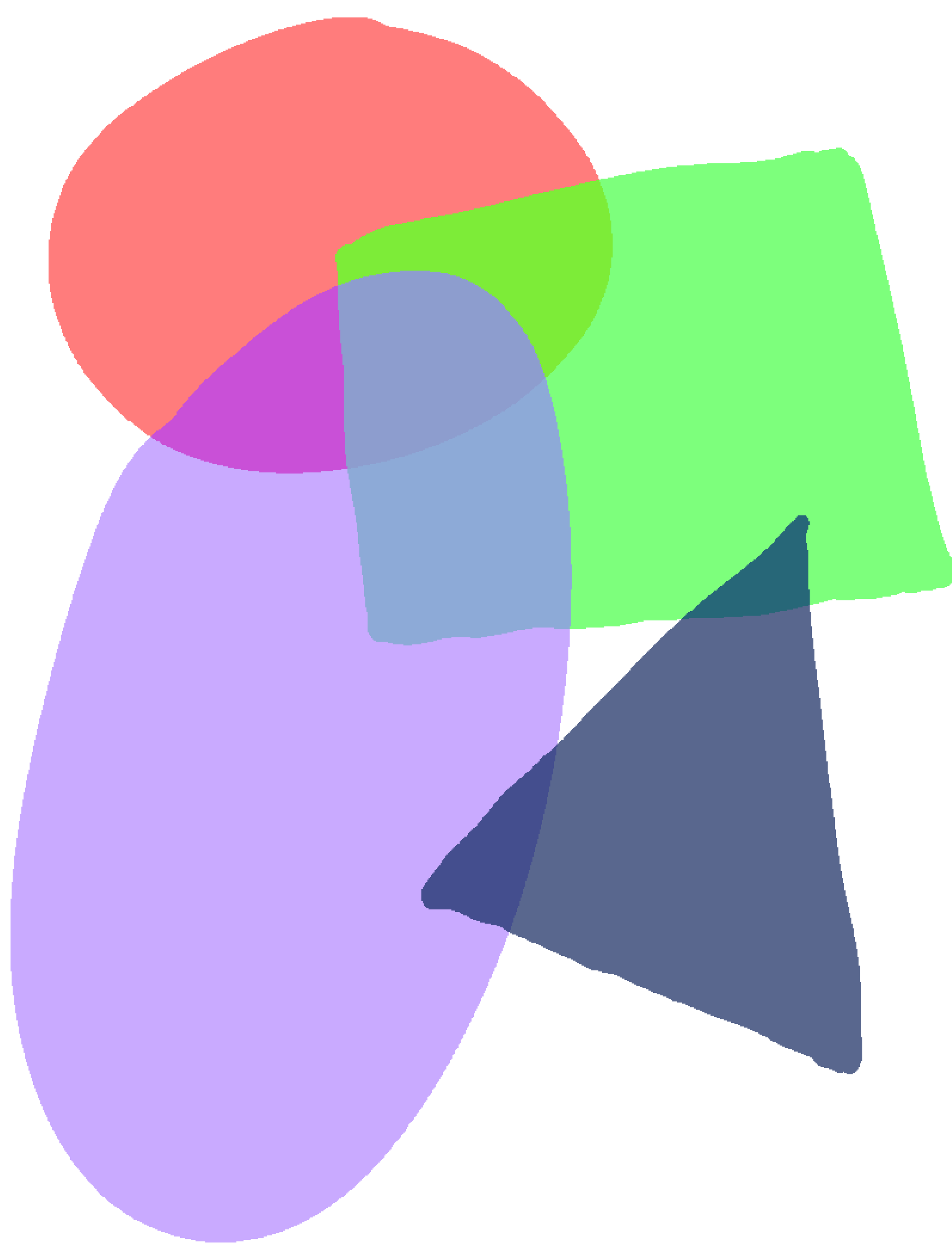
$F =$ 

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Representability

E.g.

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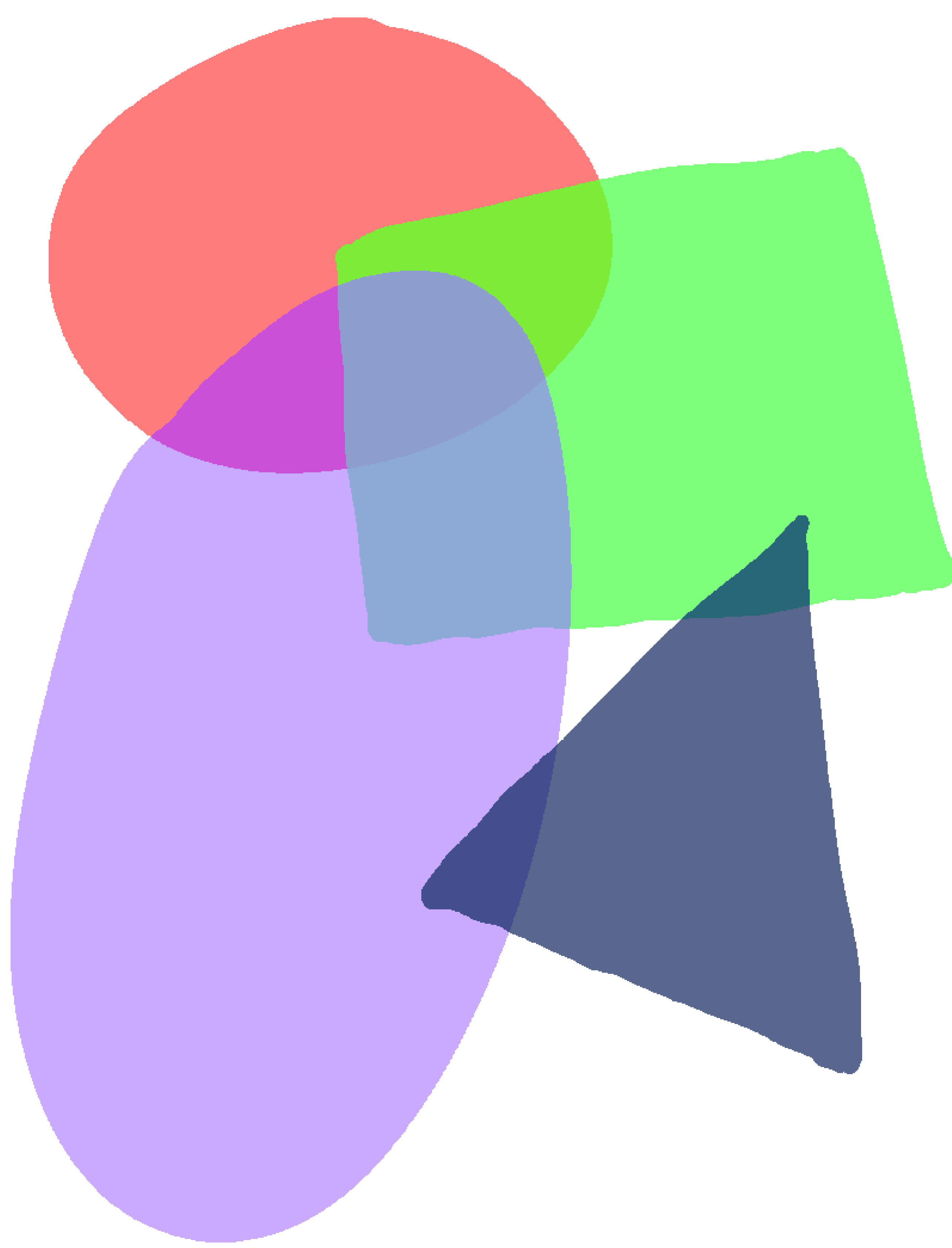

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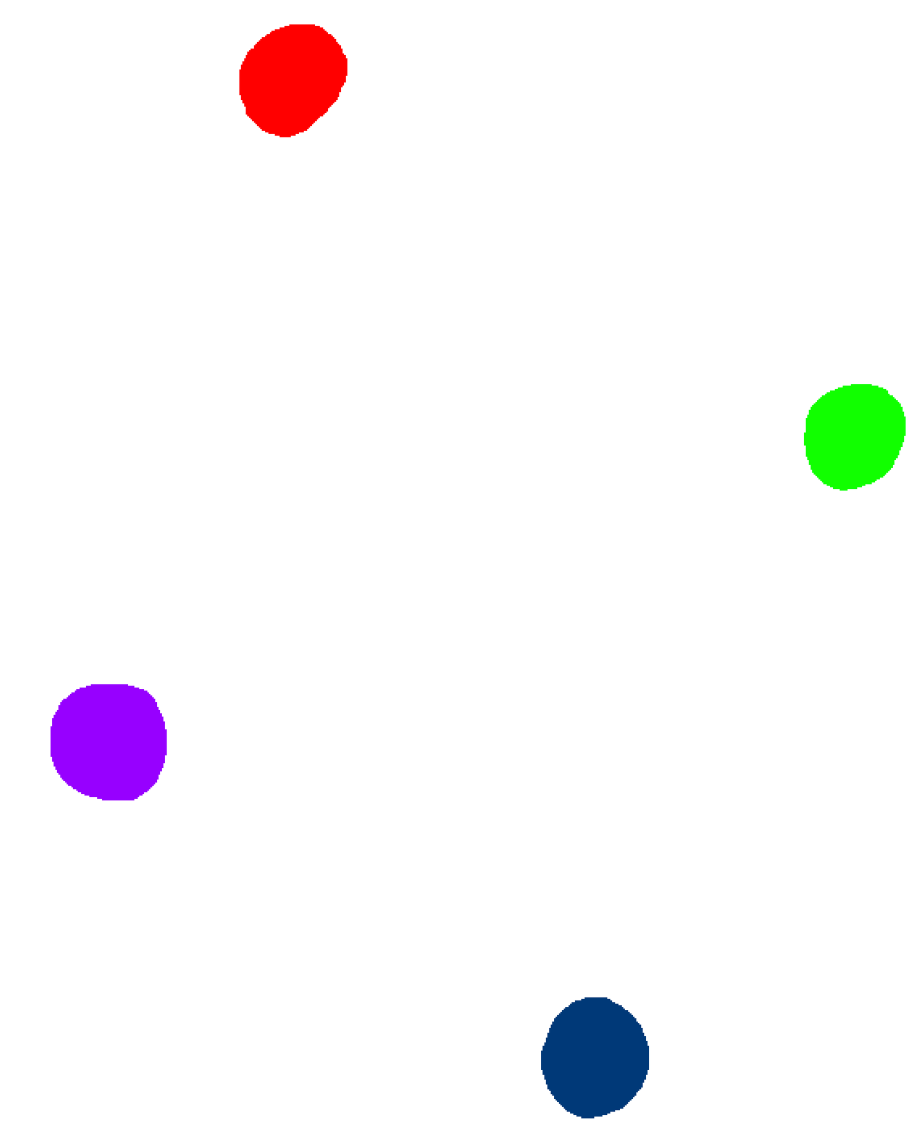

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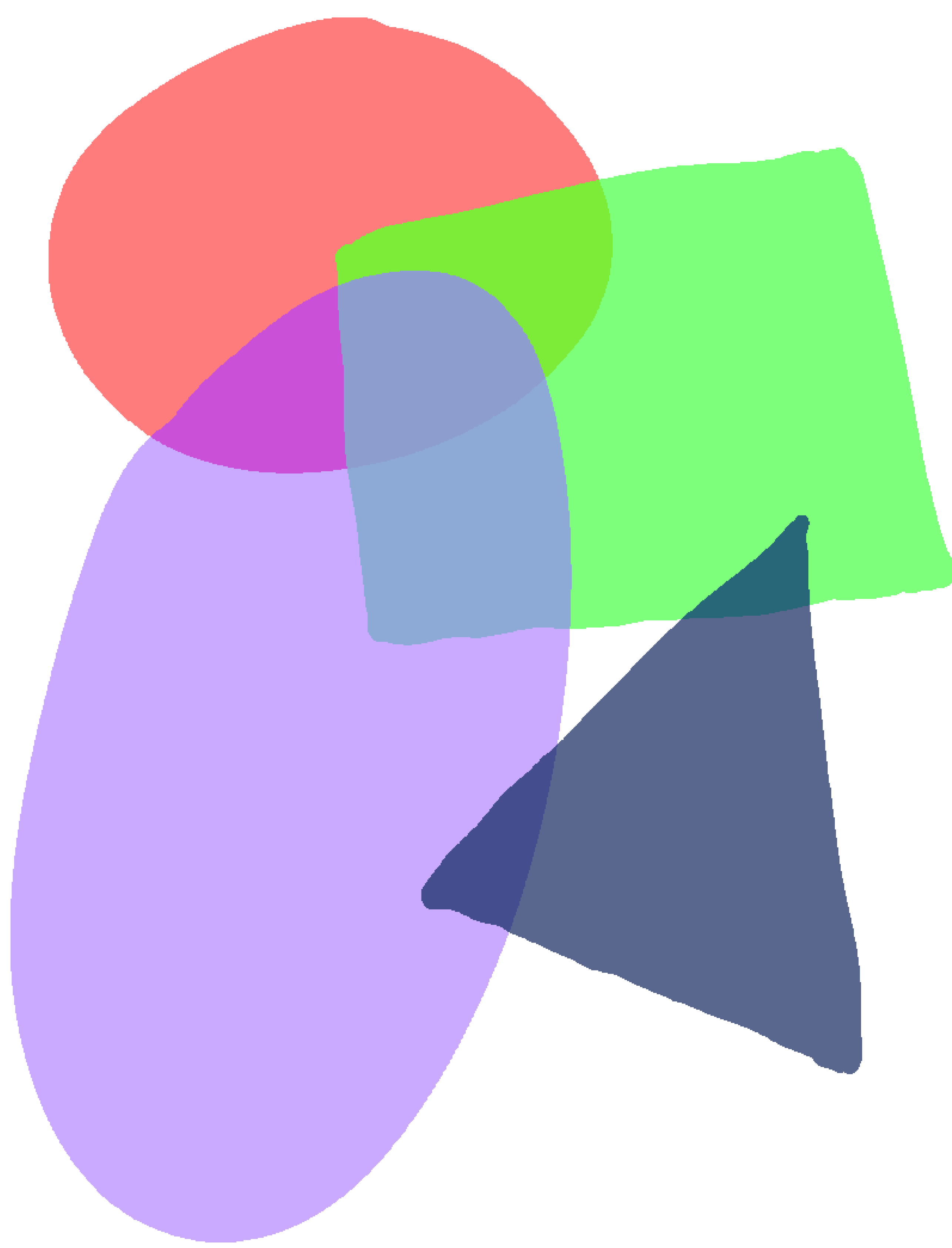

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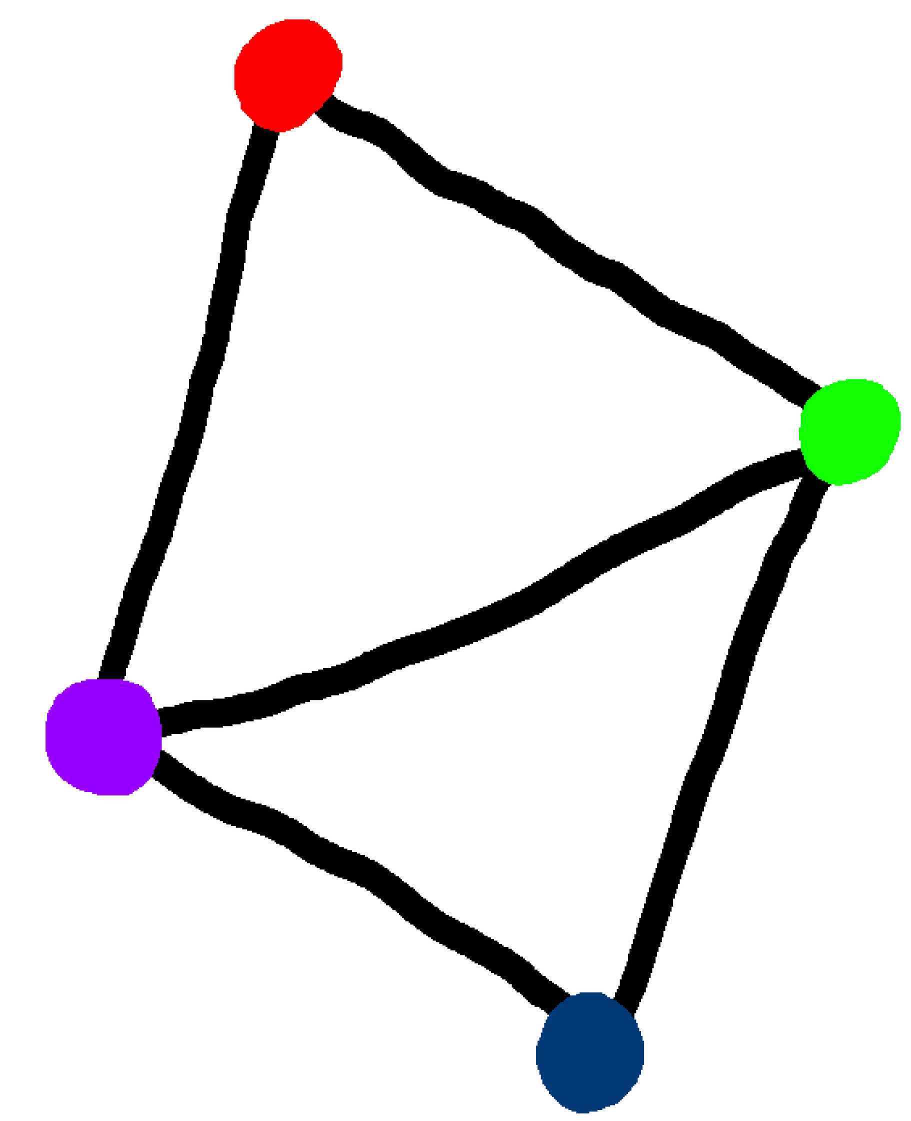

Representability

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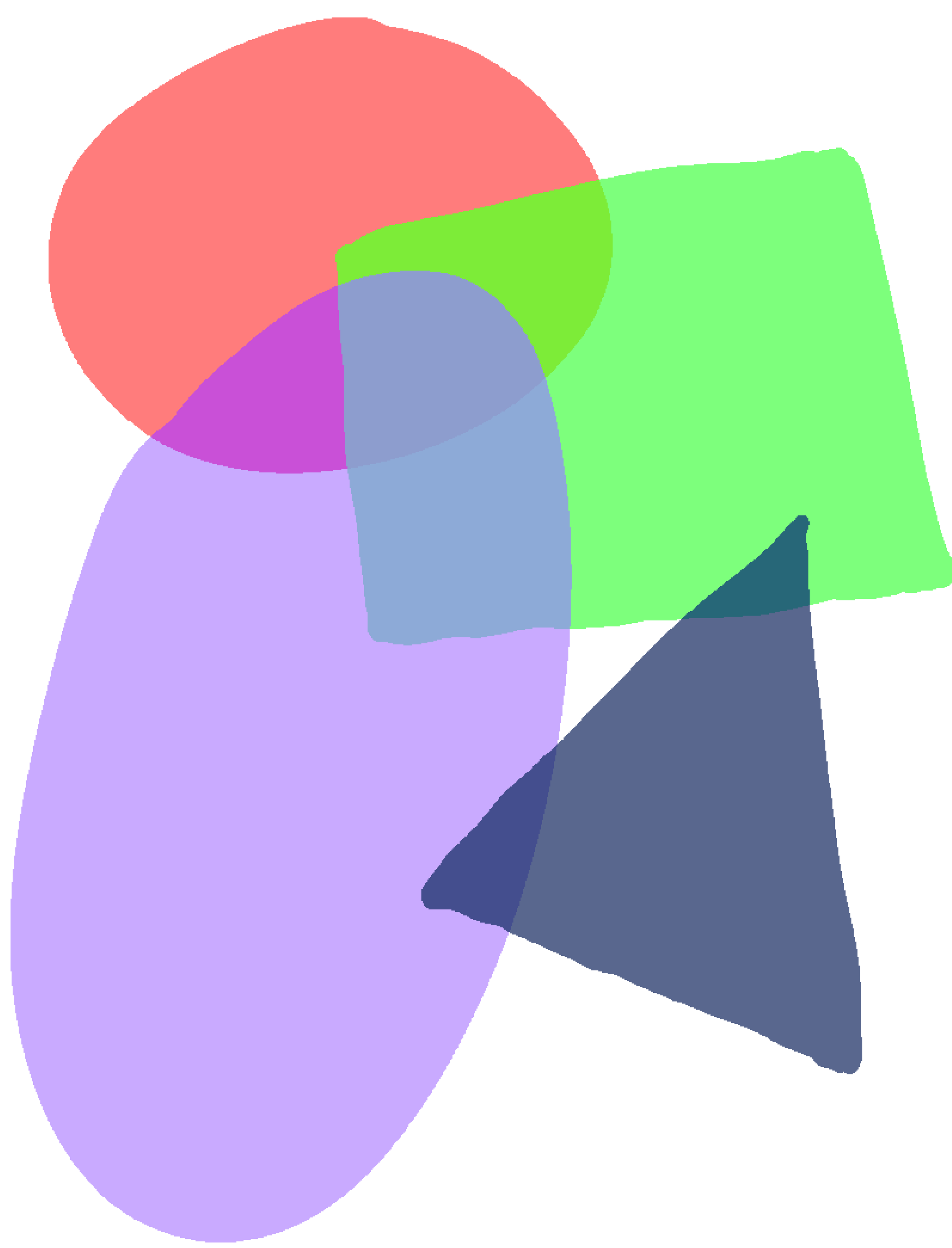
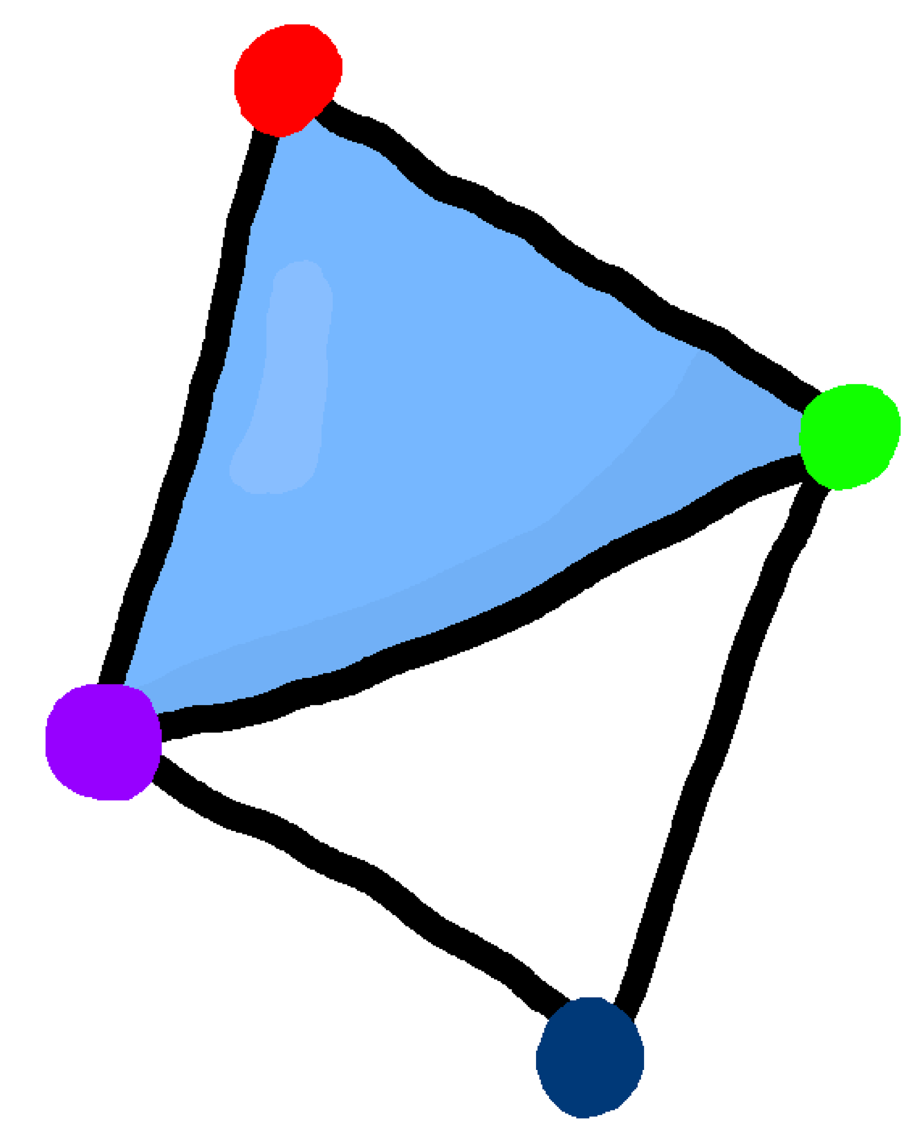
$F =$ 

$N(F) =$ 

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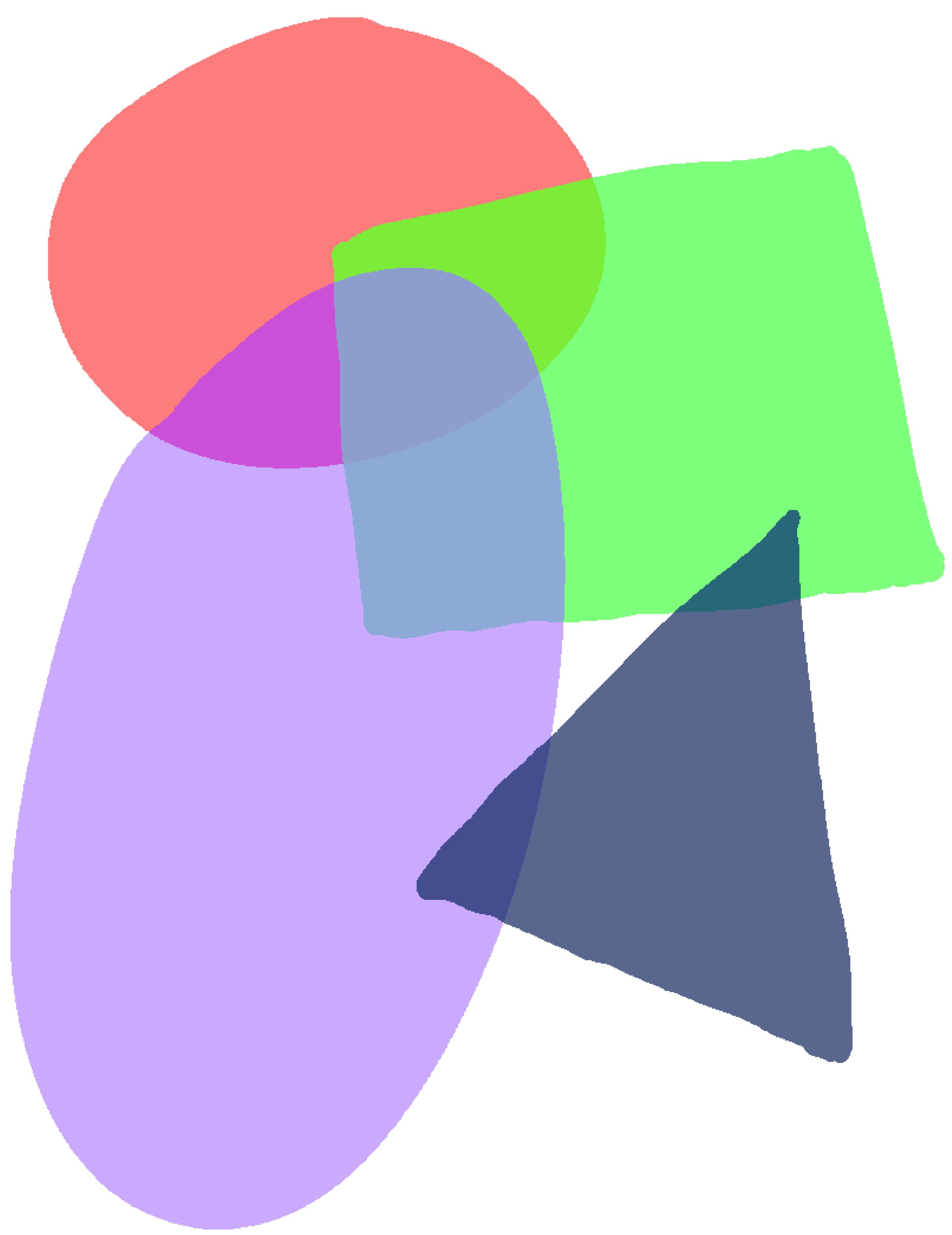
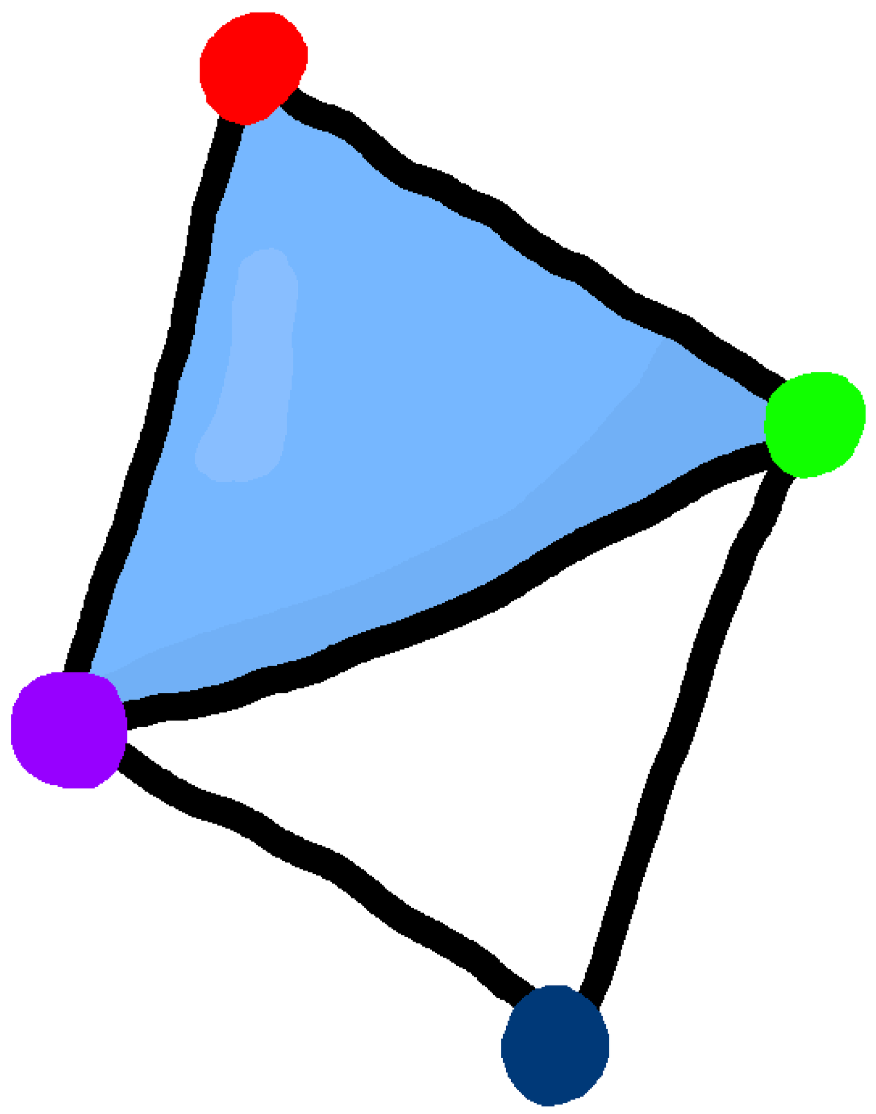
$F =$  $N(F) =$ 

- **d-representable** complex = nerve of a family of **convex** sets in $\mathbb{R}^d$

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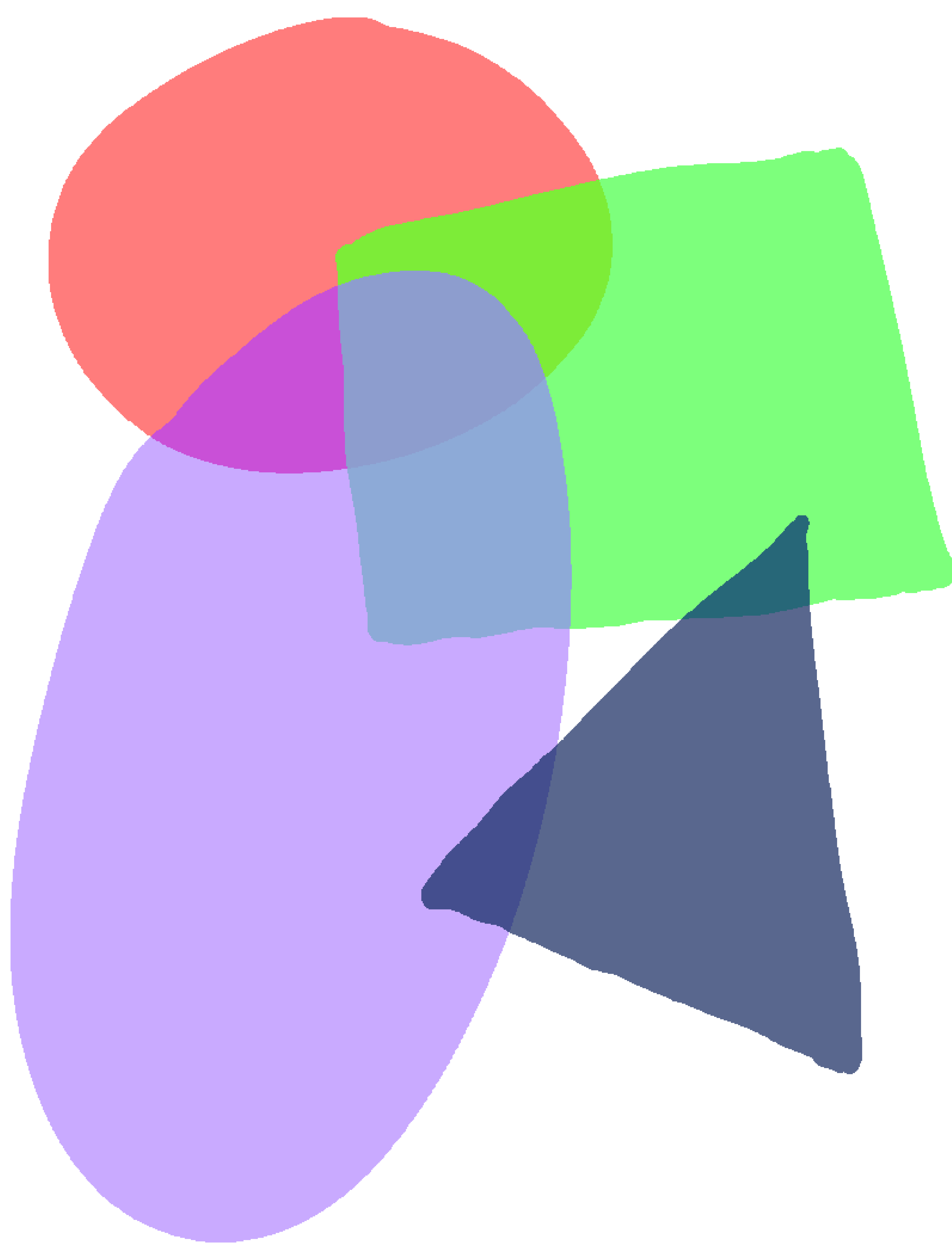
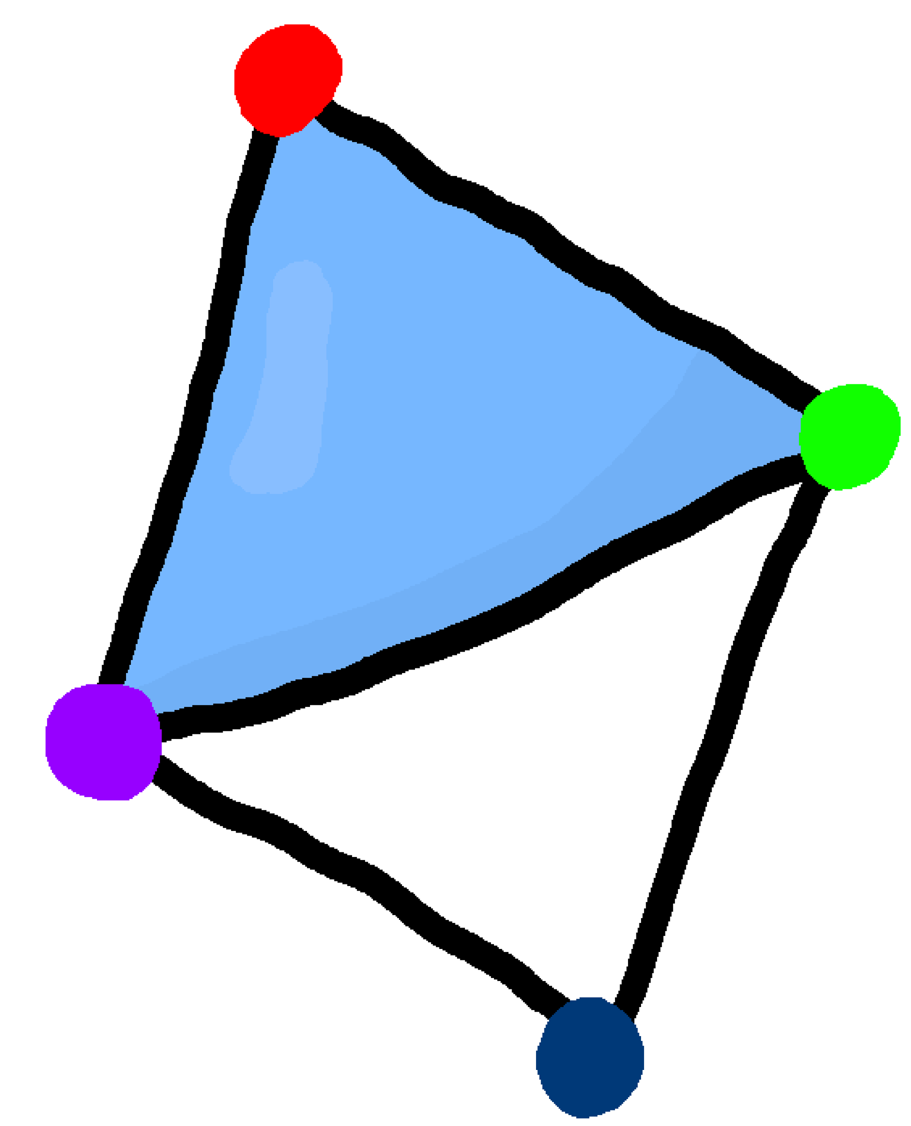
E.g.

$F =$ 

2-representable

$N(F) =$ 

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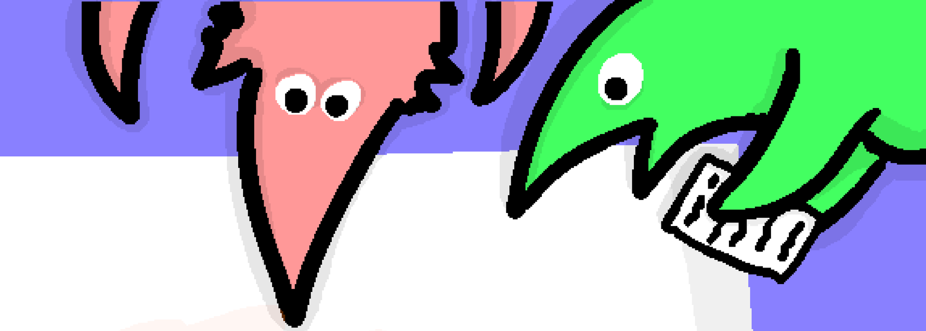


K is

d-representable

K is
d-collapsible

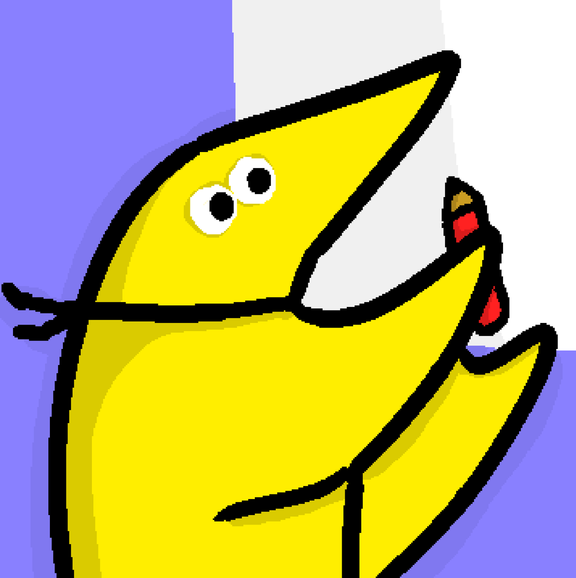
$\Uparrow$

K is
d-representable

K is d-representable $\Longrightarrow$ K is d-collapsible $\Longrightarrow$ K is d-Leray

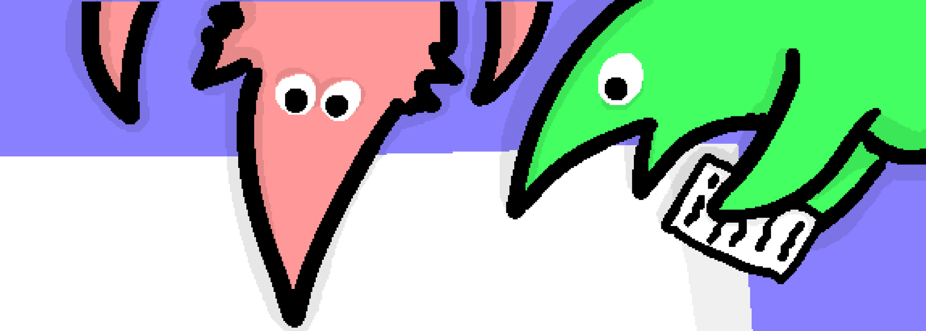
Wegner's Theorem ('75)

K is d-representable $\Rightarrow$ K is d-collapsible $\Rightarrow$ K is d-Leray



Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets.



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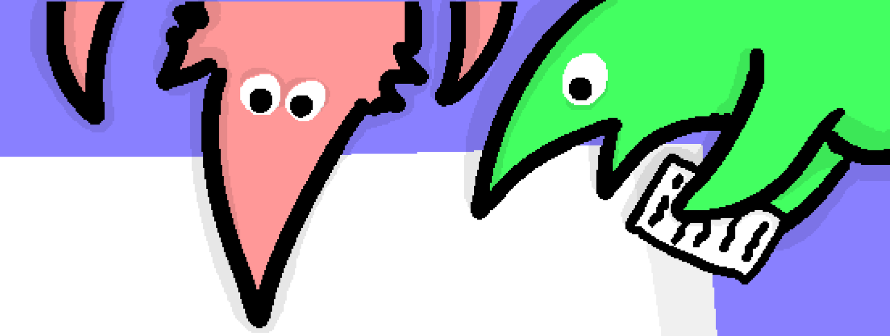

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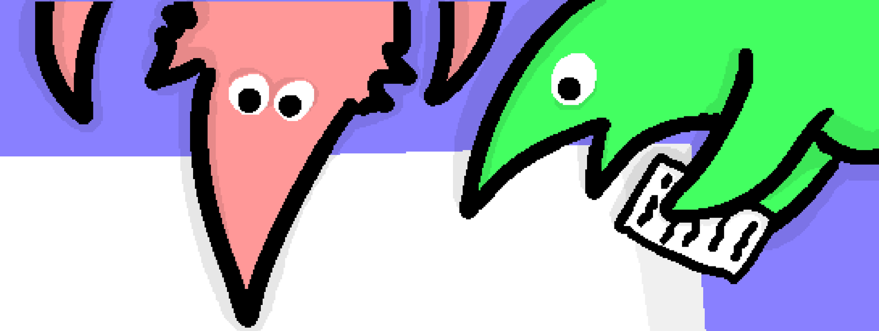

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E.g.

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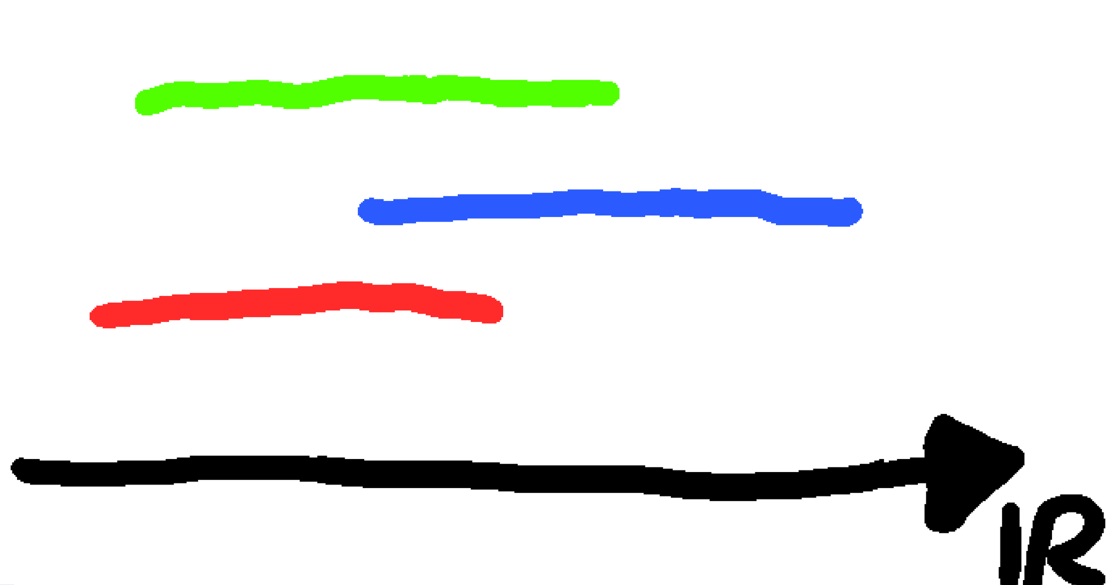

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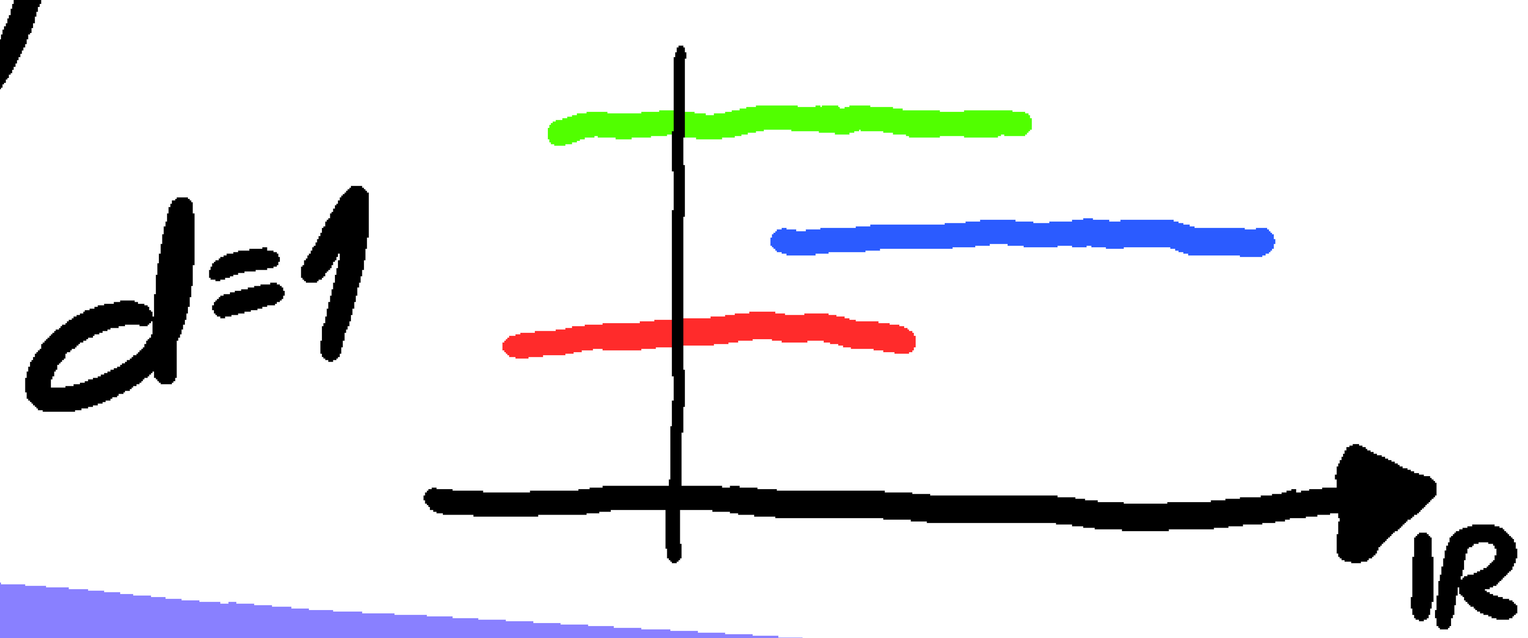

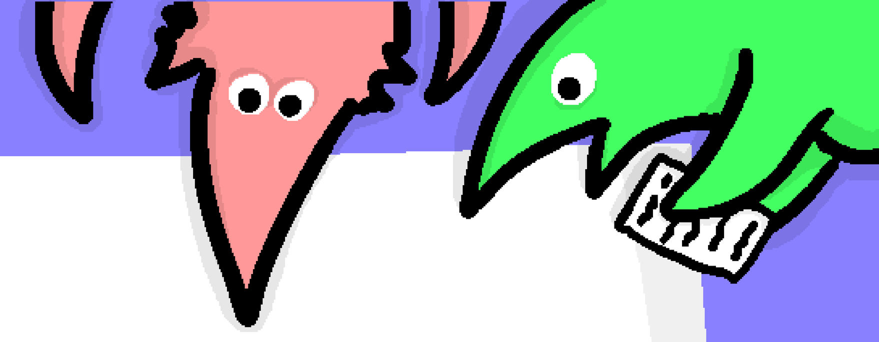

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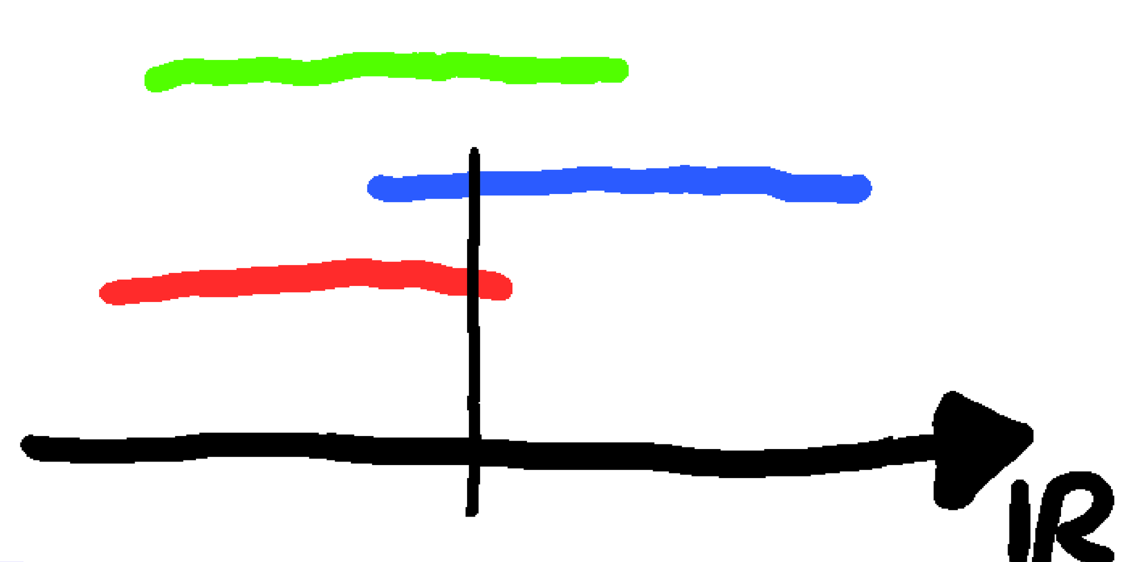

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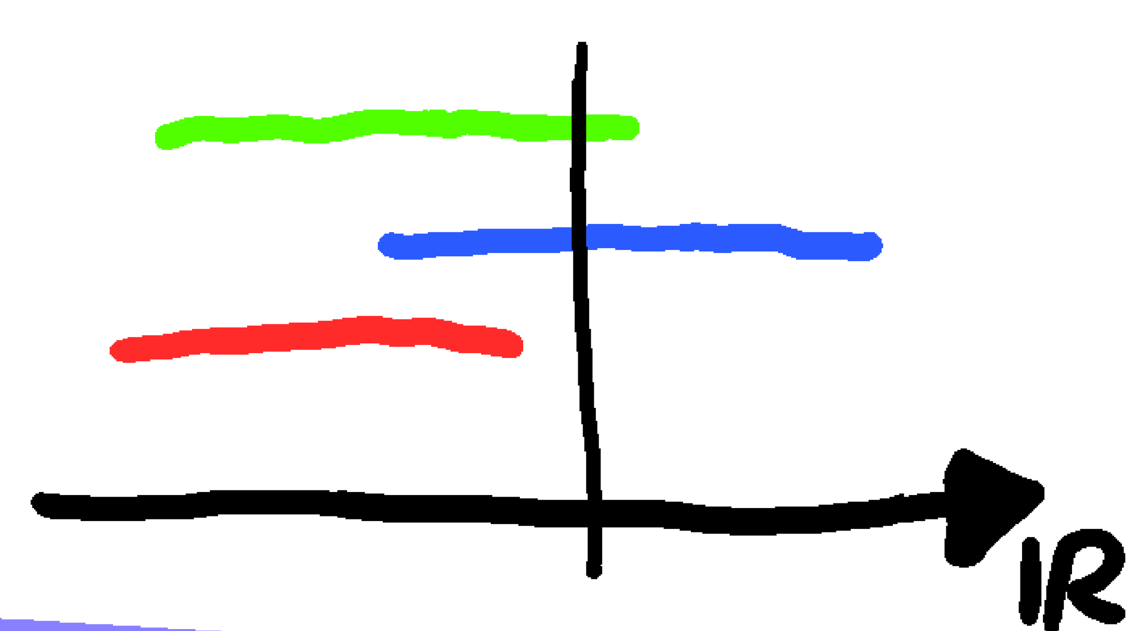

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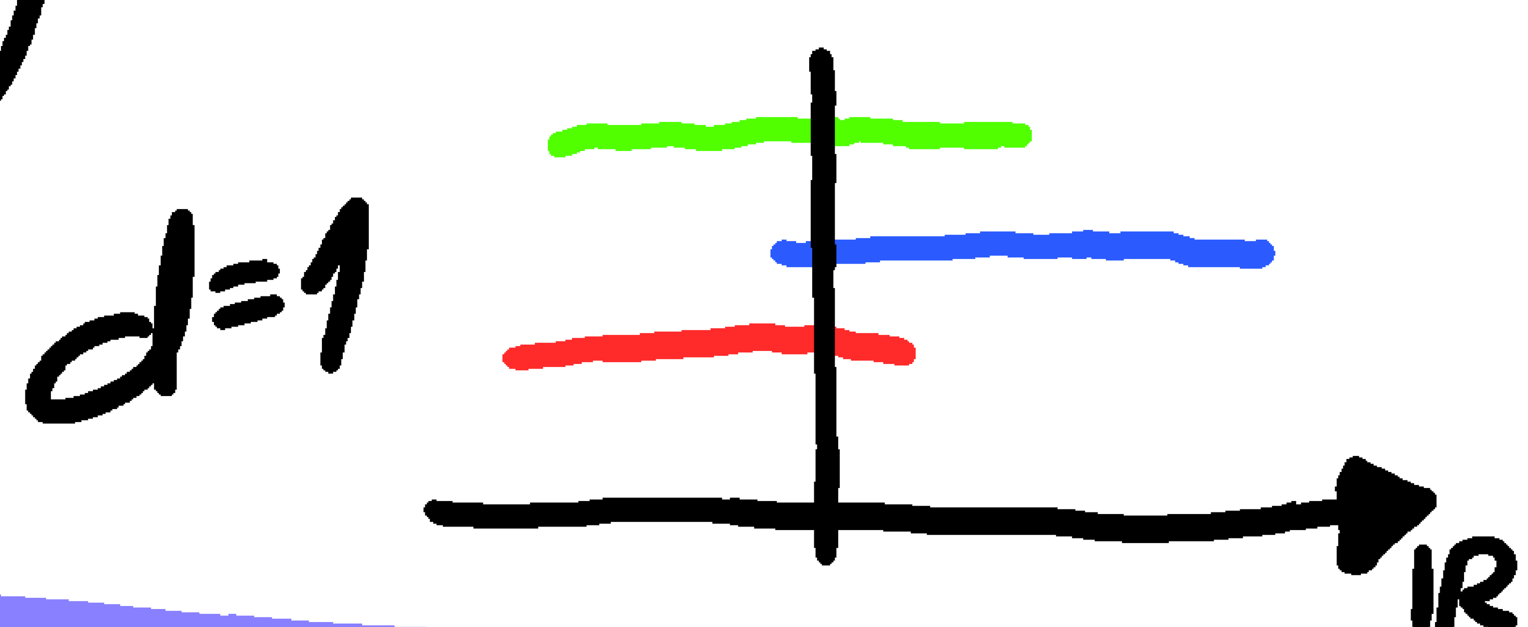

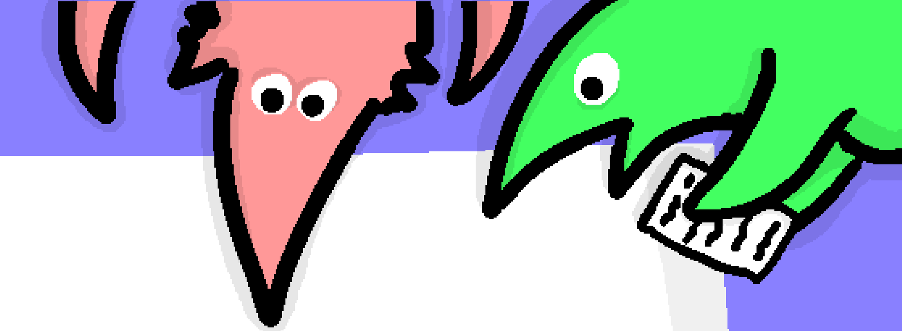

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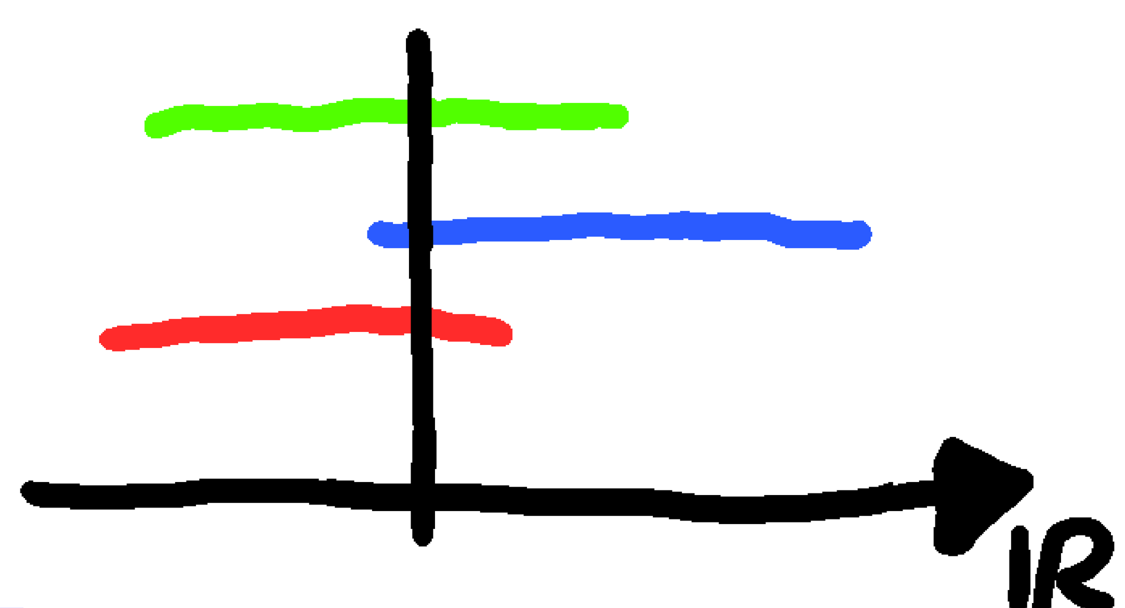

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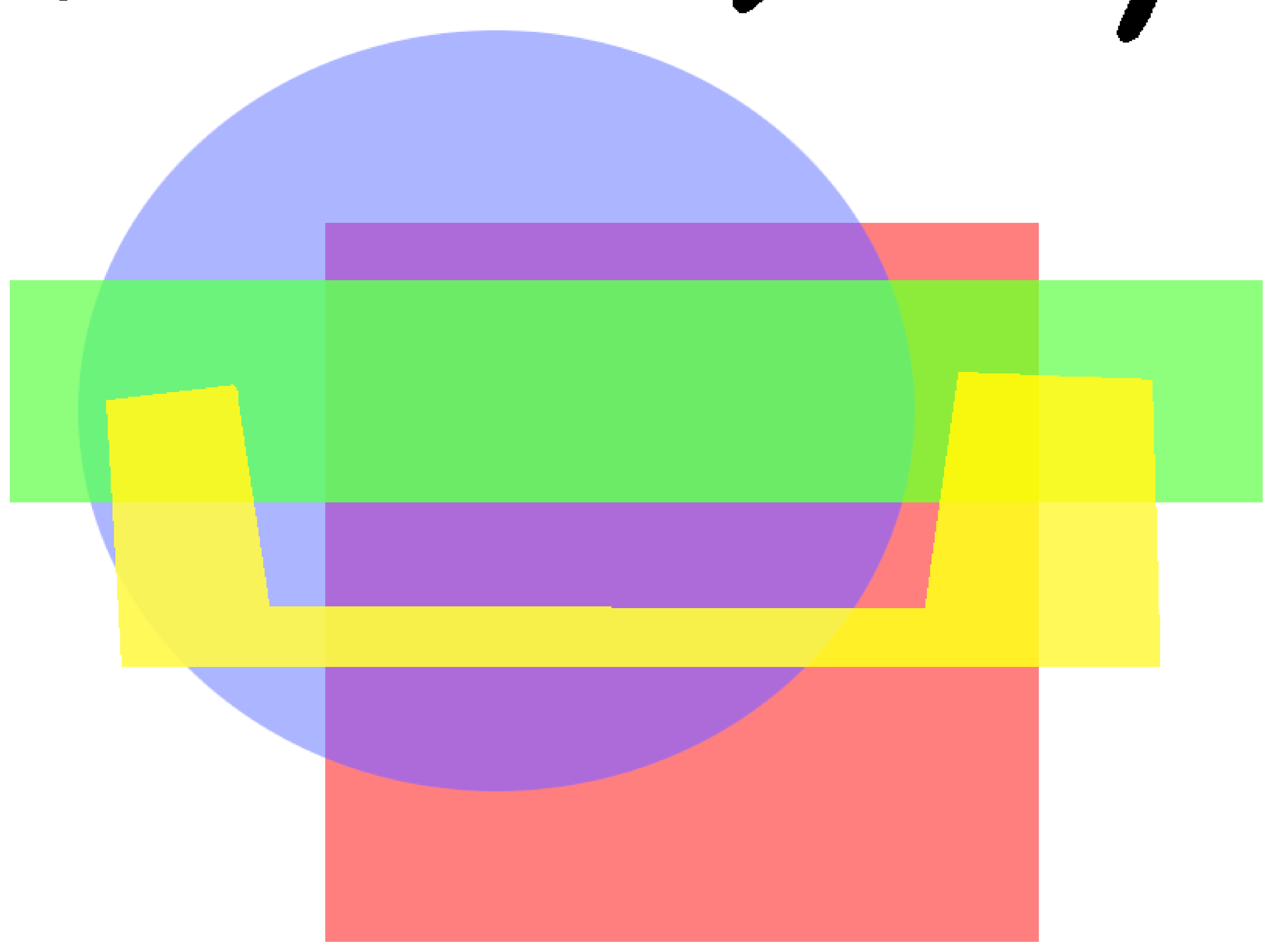

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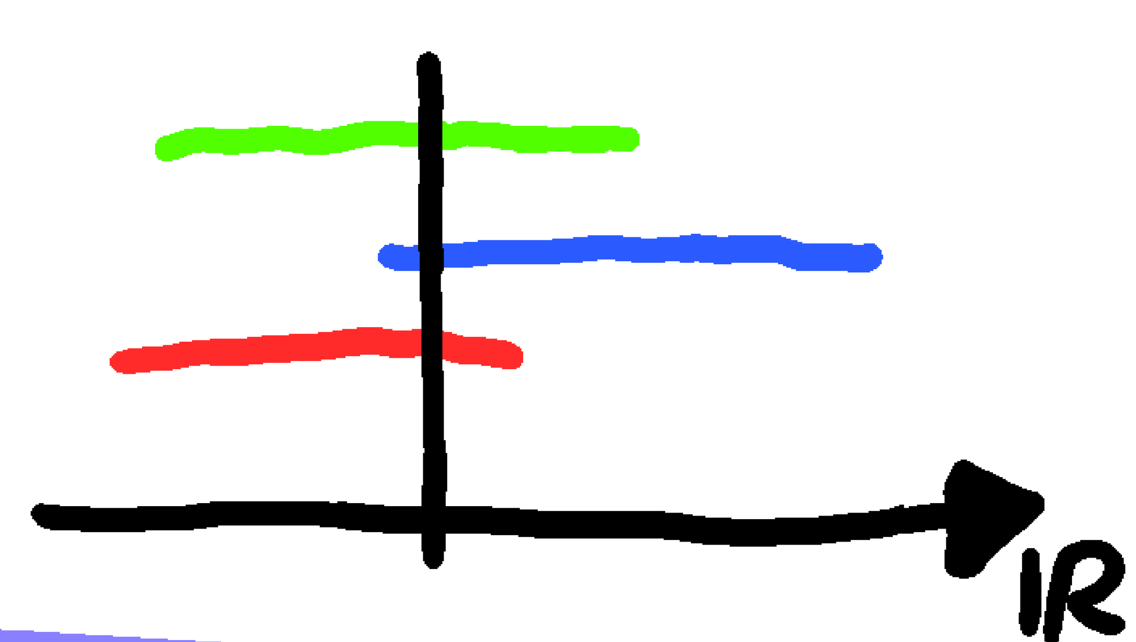

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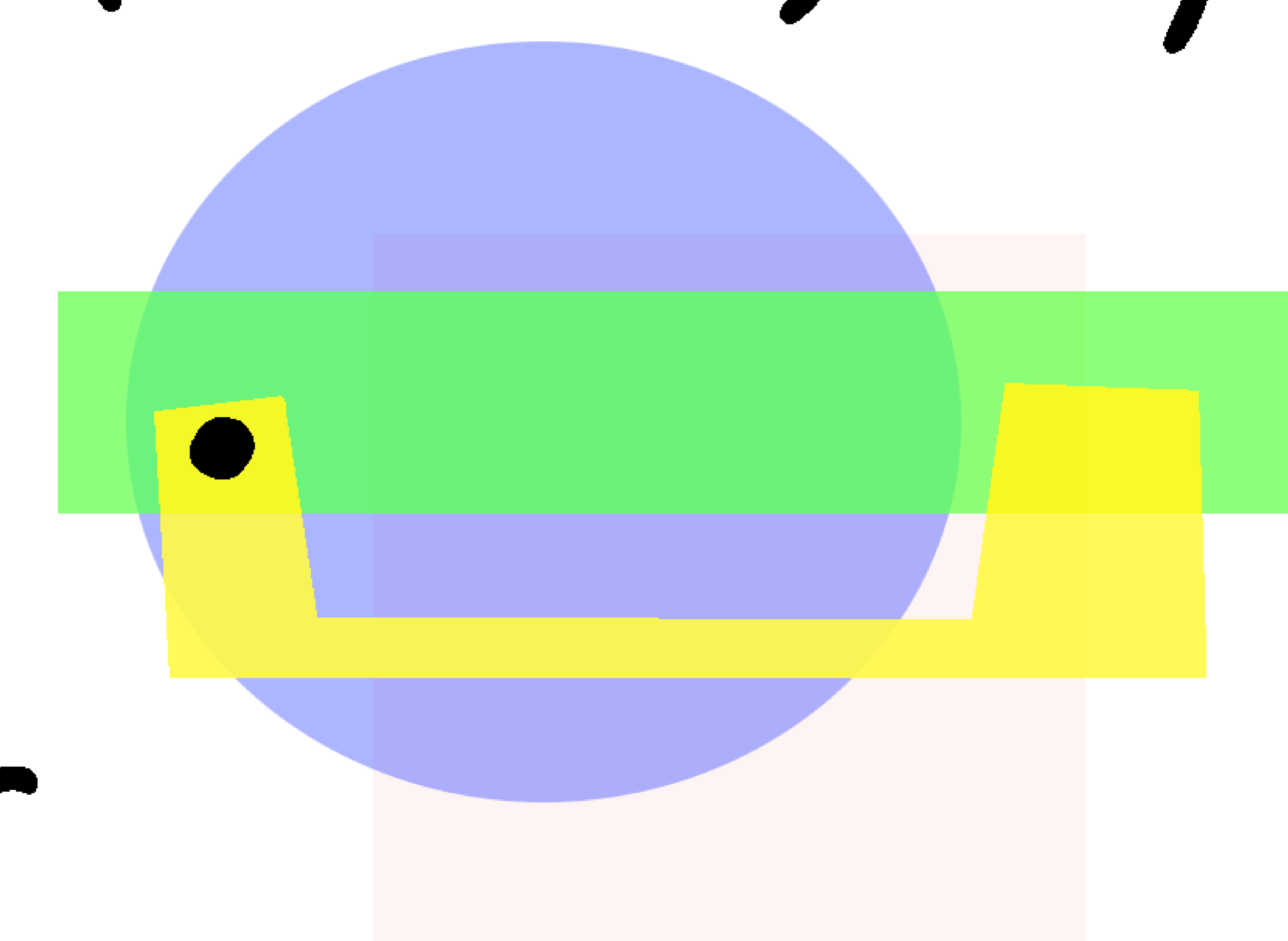

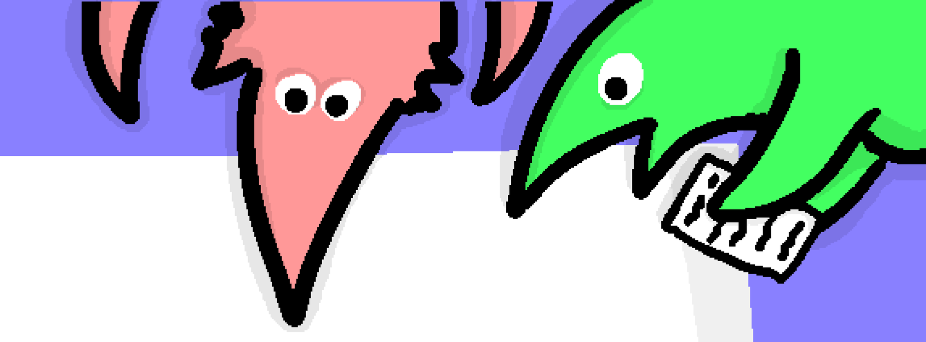

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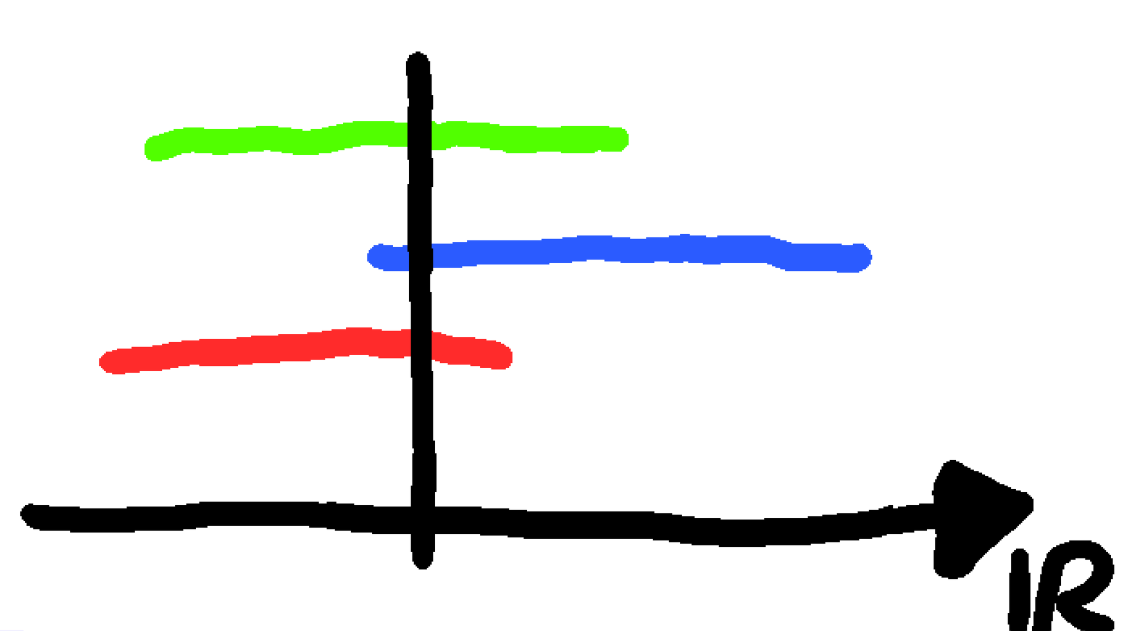

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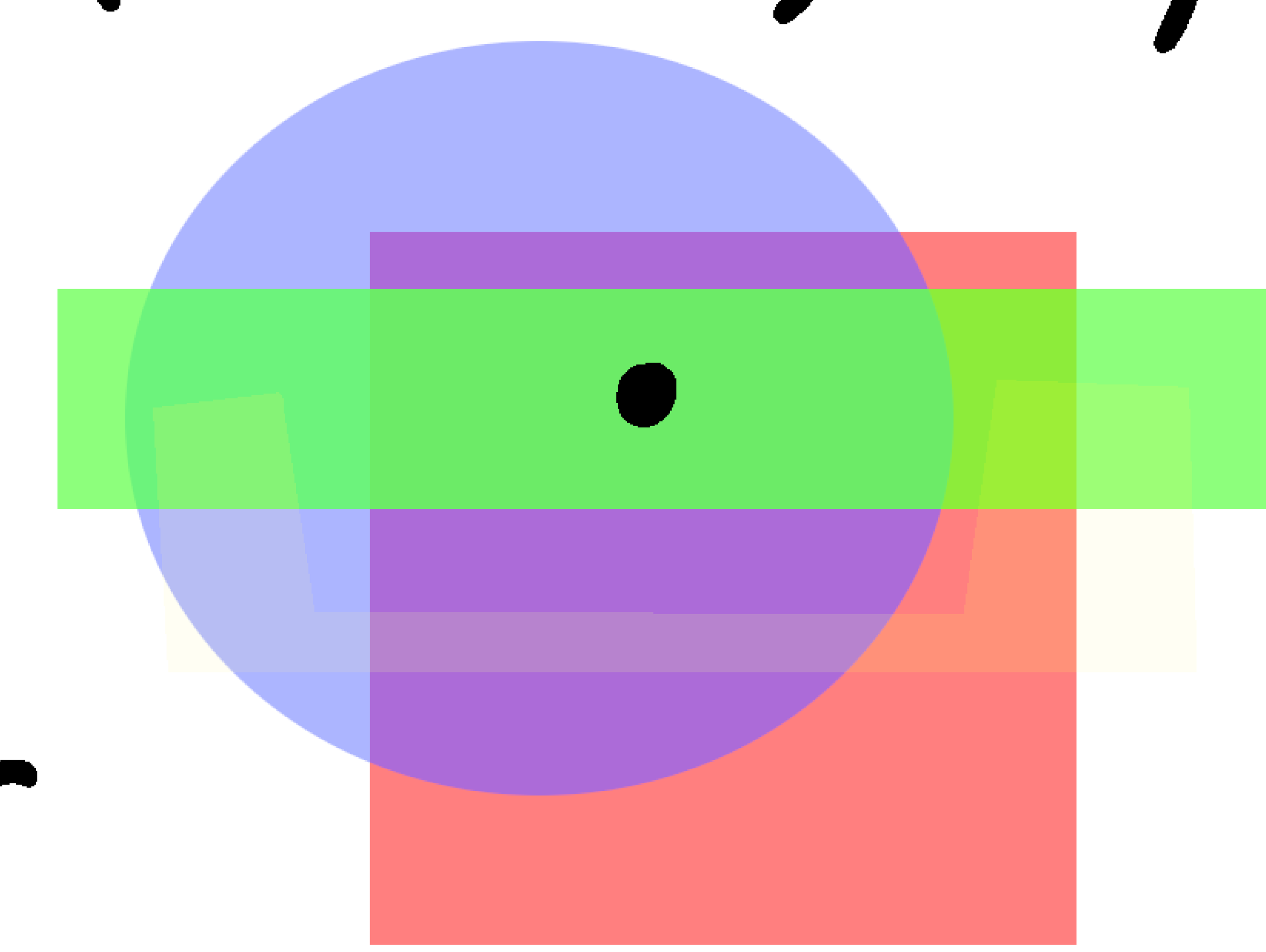

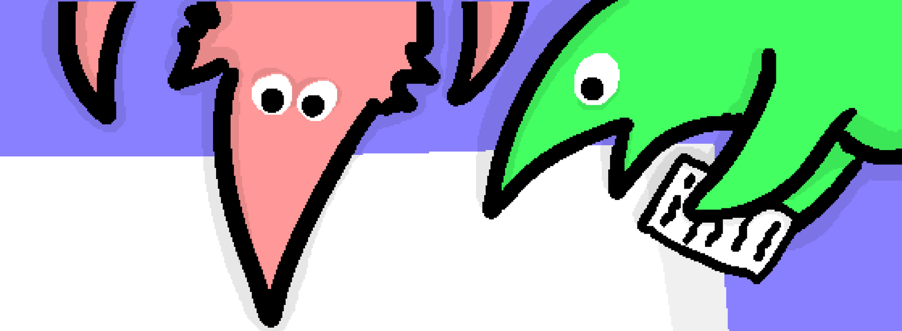

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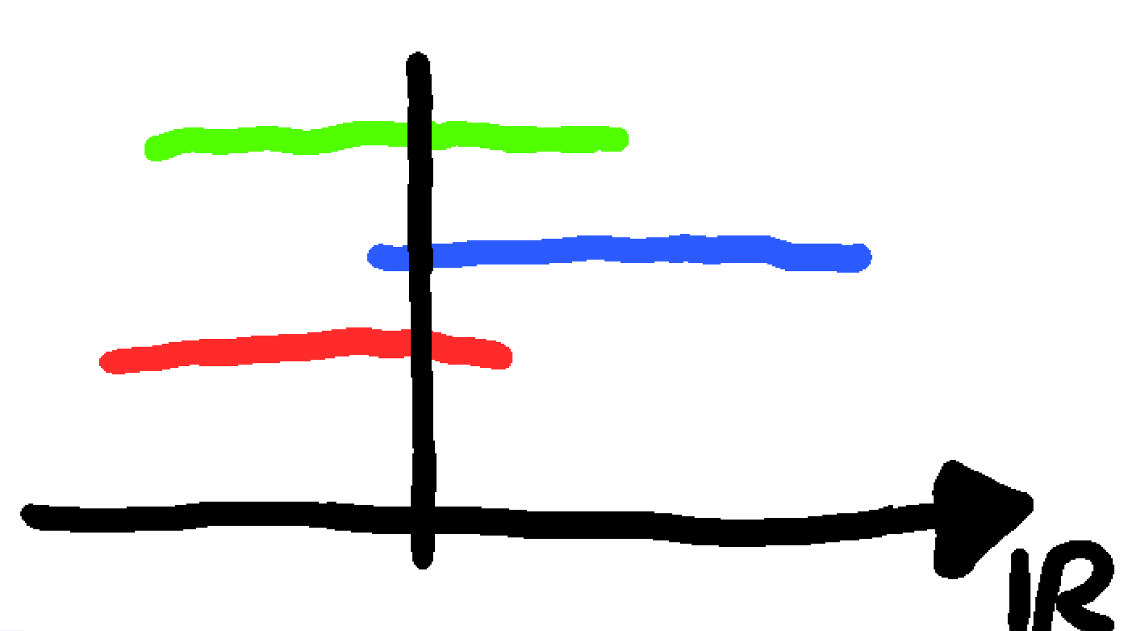

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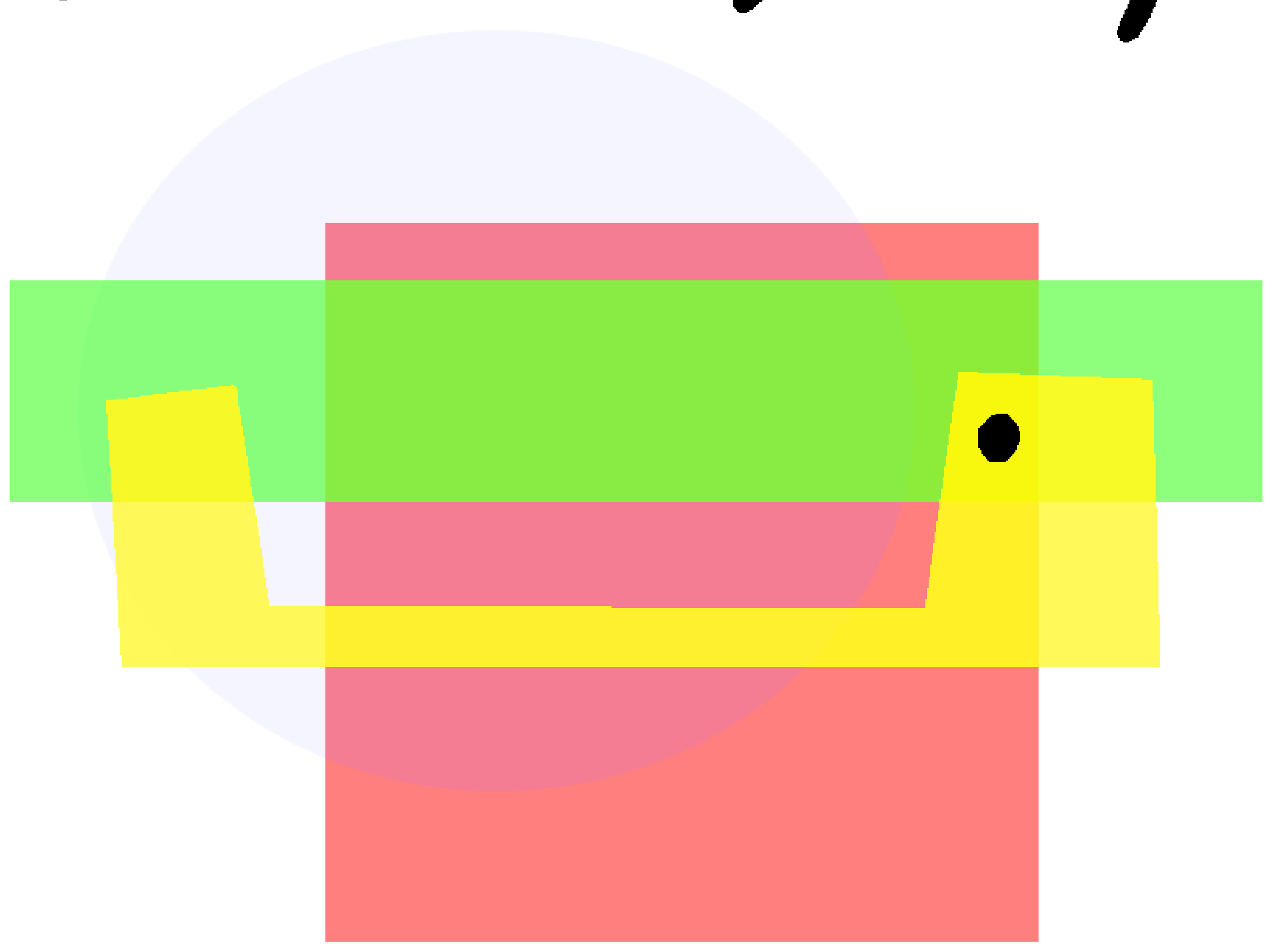

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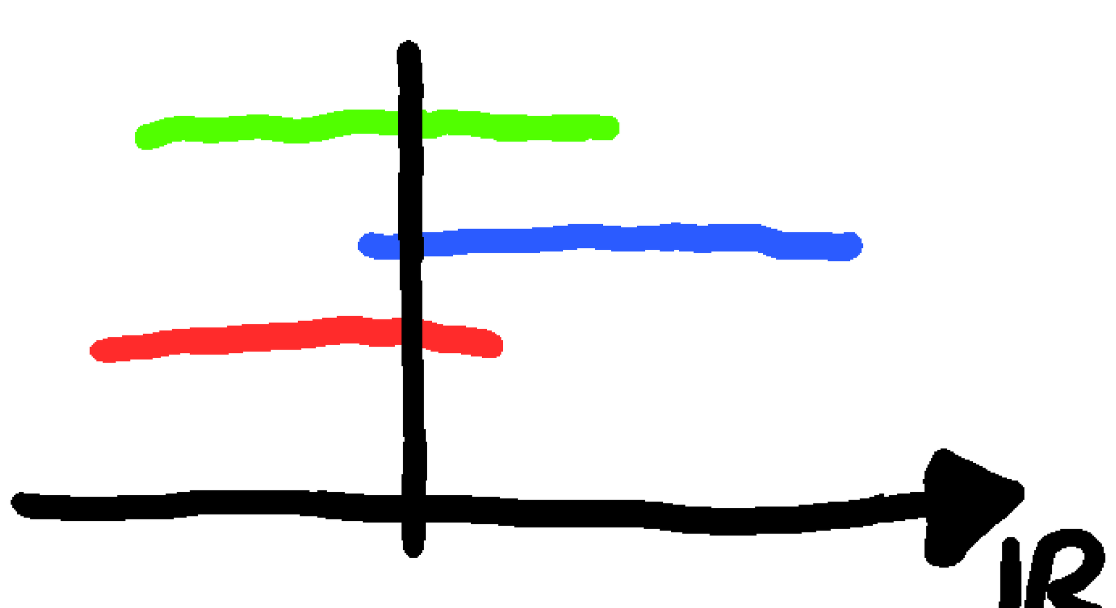

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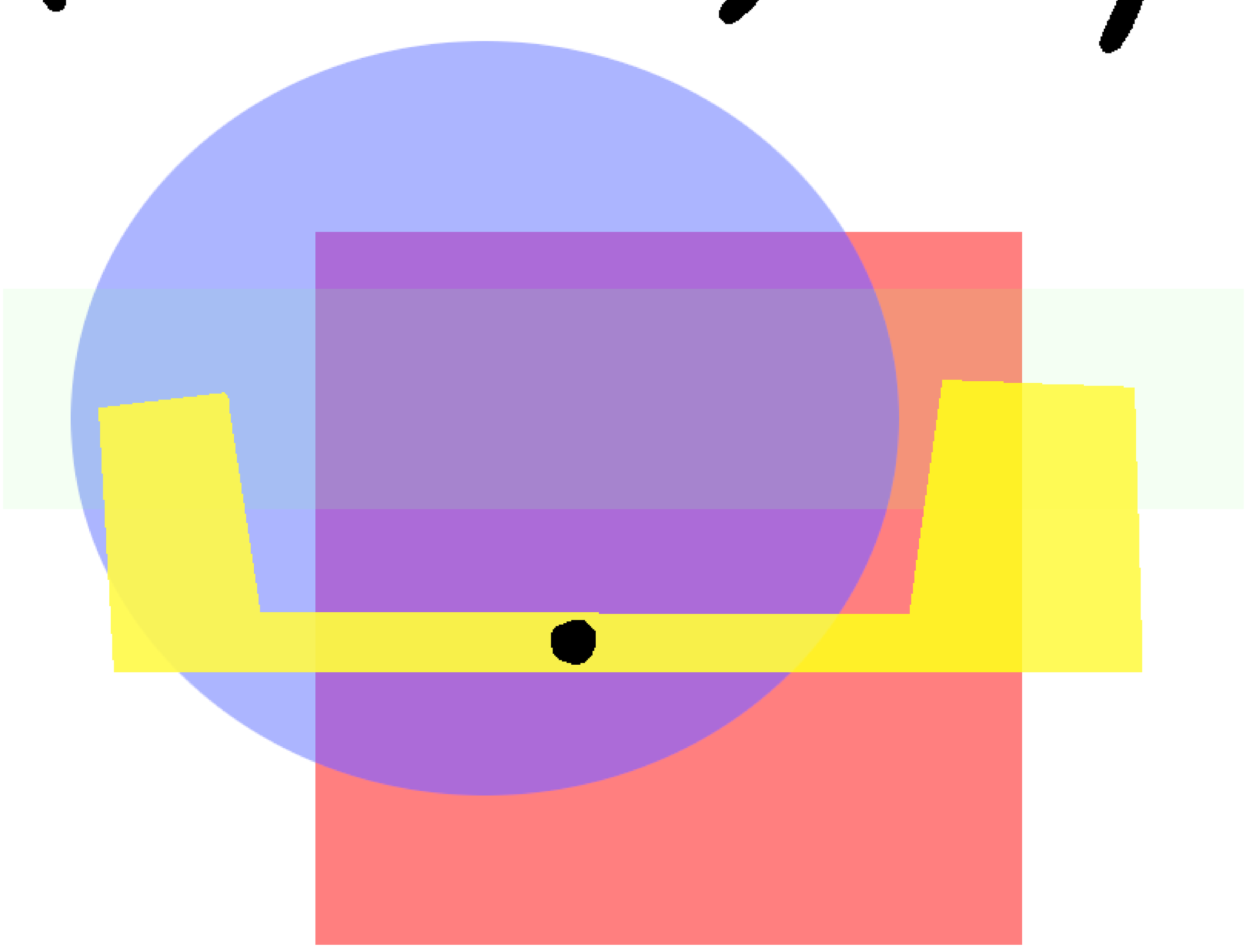

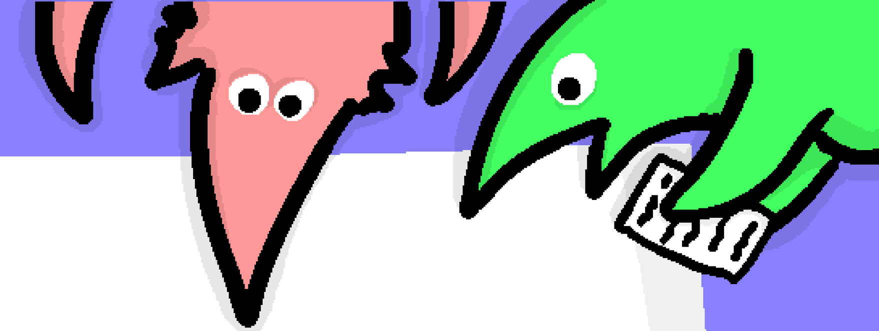

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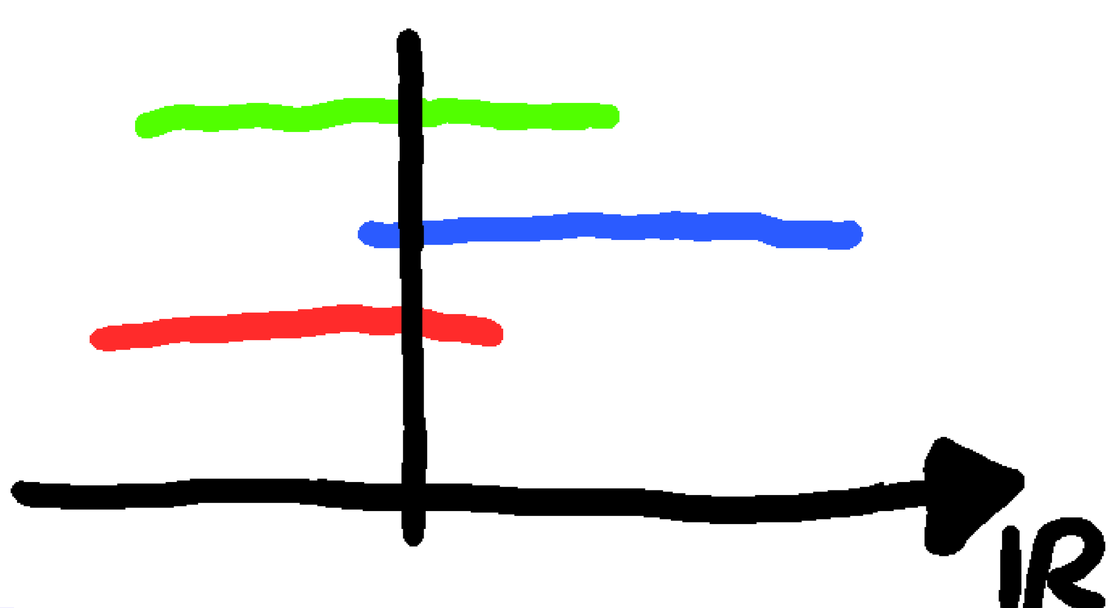

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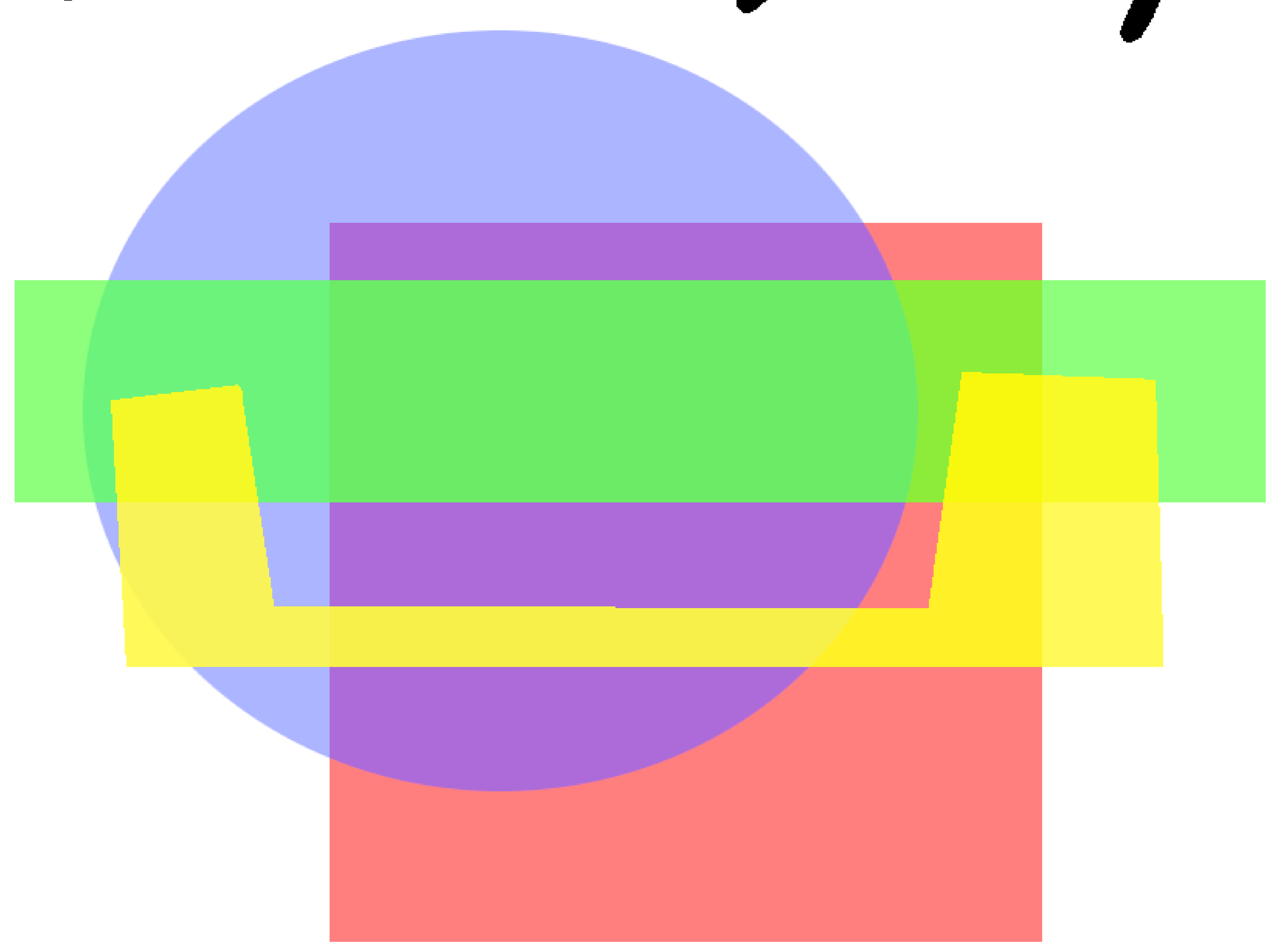

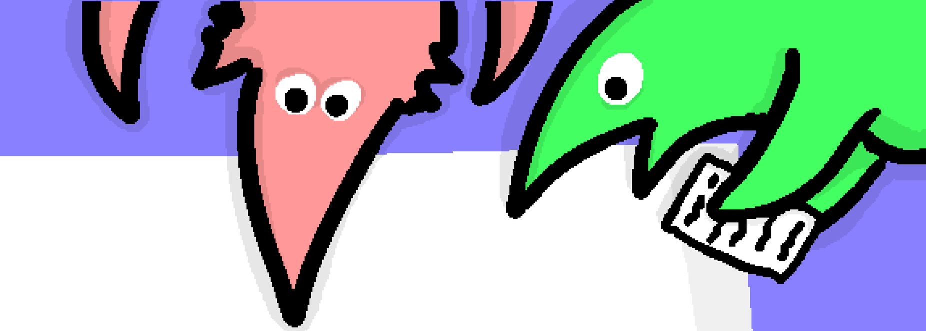

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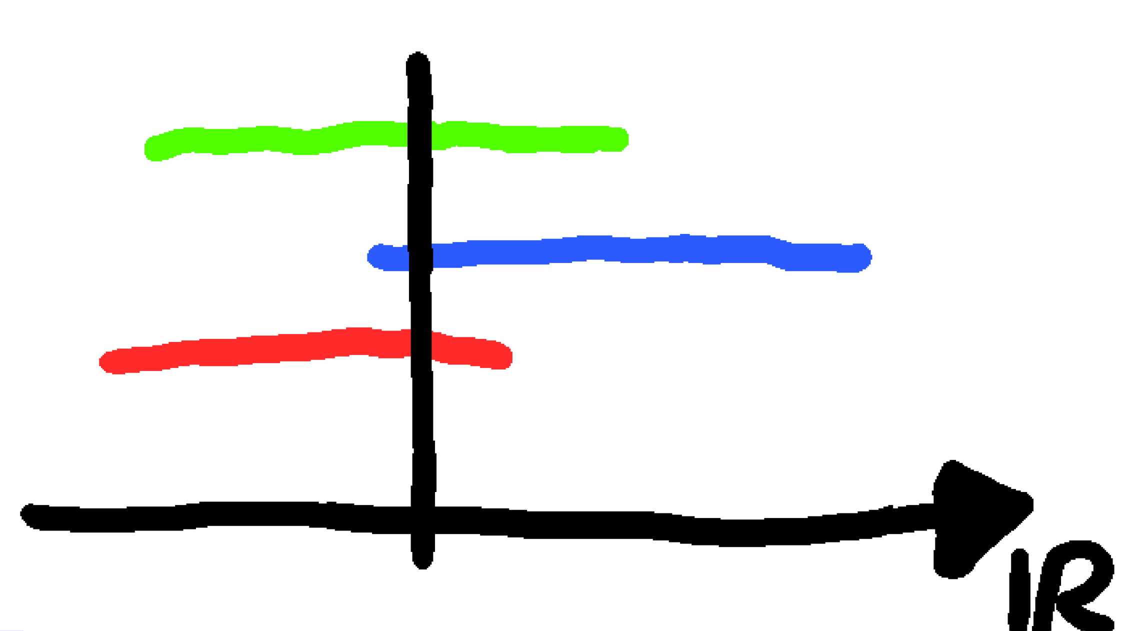

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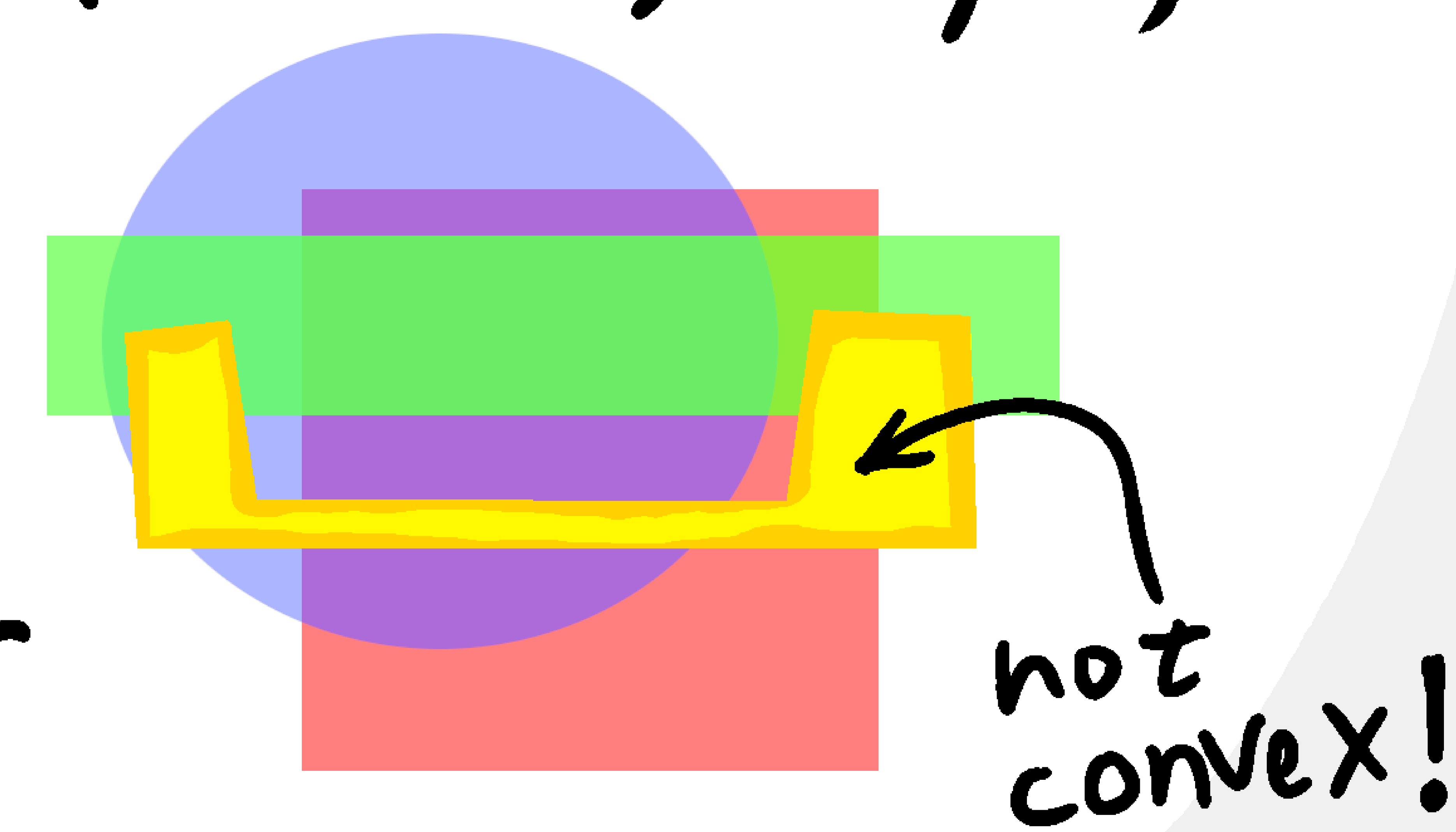

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s.t.:

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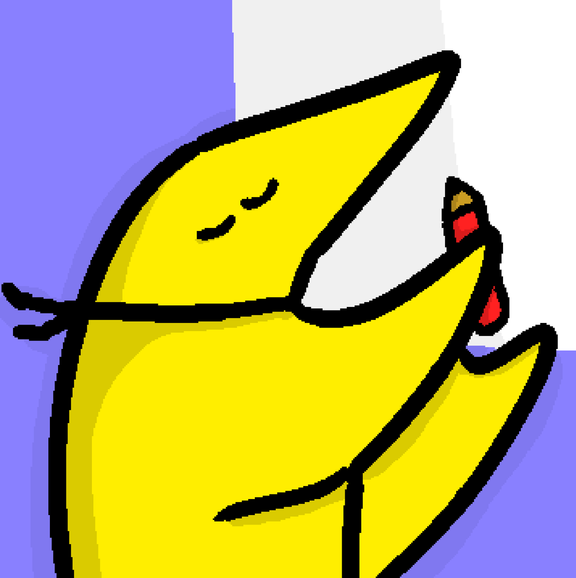
s.t.: 1 $\tau \notin K$



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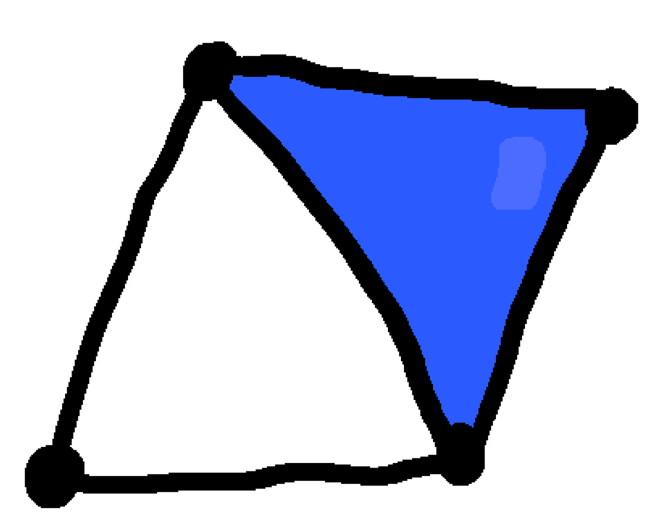
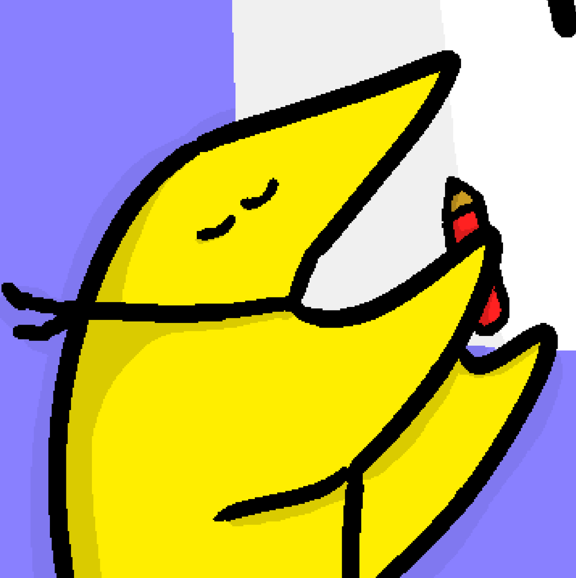
$K =$ 



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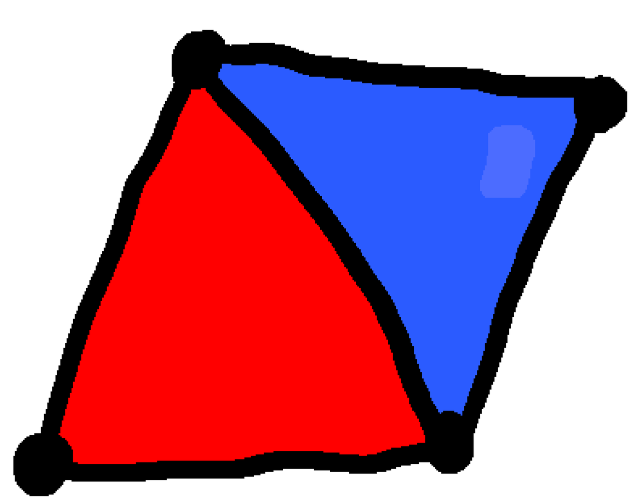
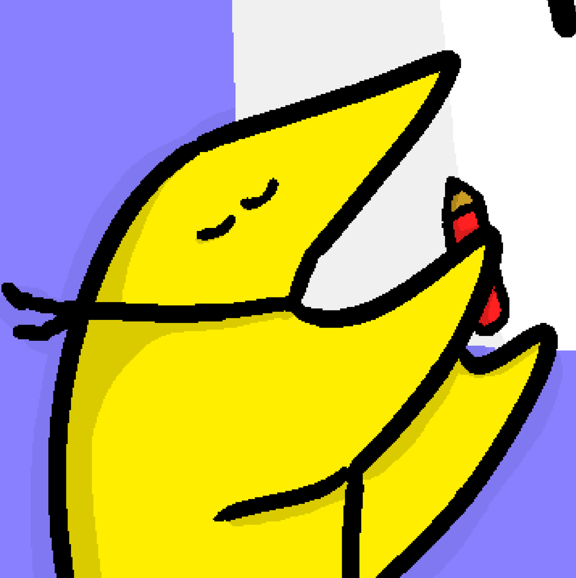
$K =$ 



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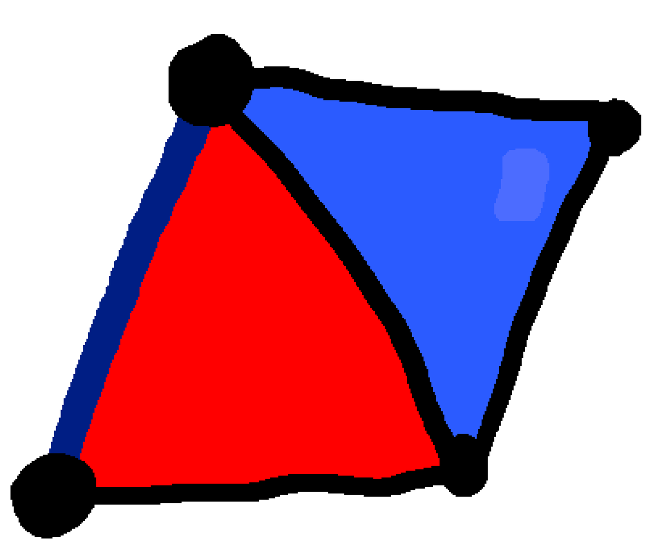
$K =$ 

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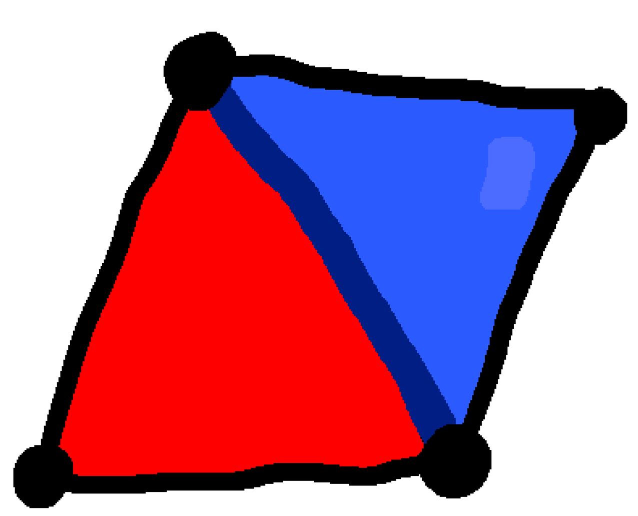
$K =$ 

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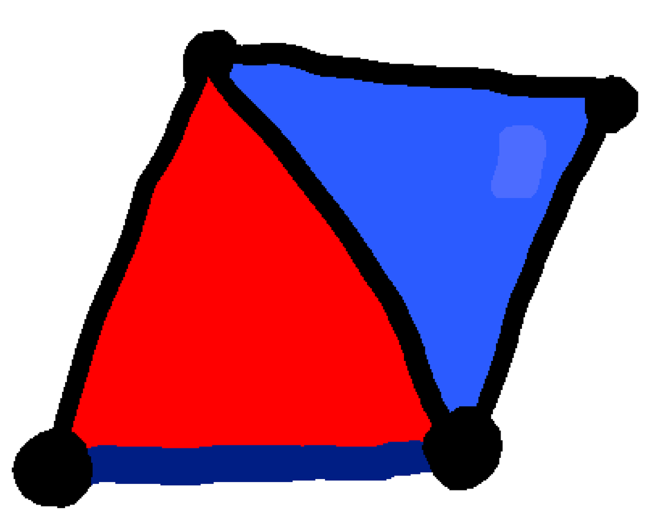
$K =$ 

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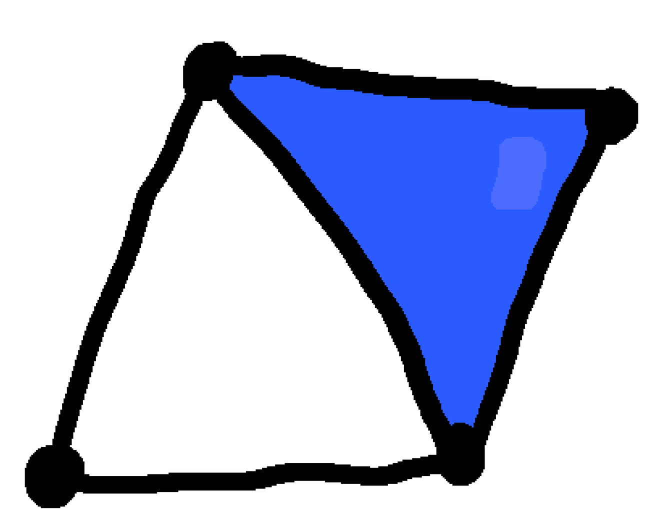
$K =$ 

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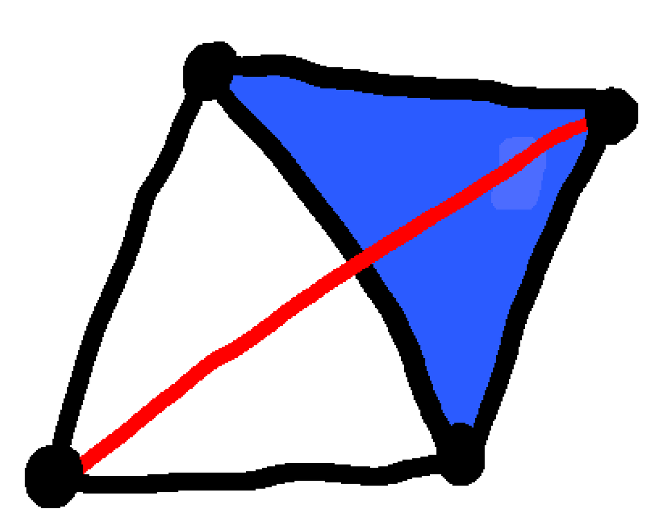
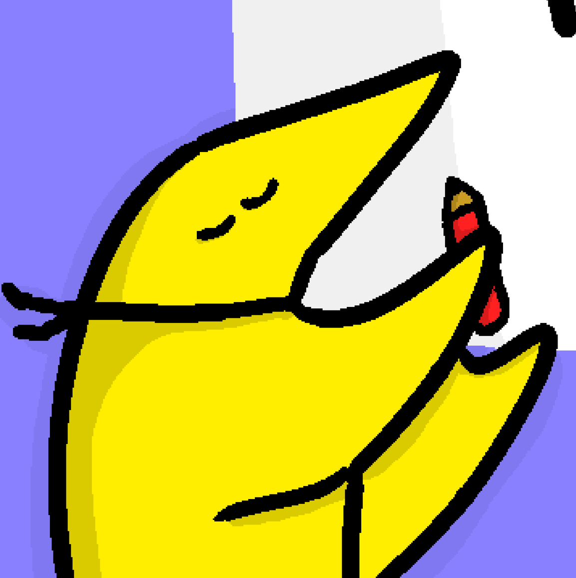
$K =$ 



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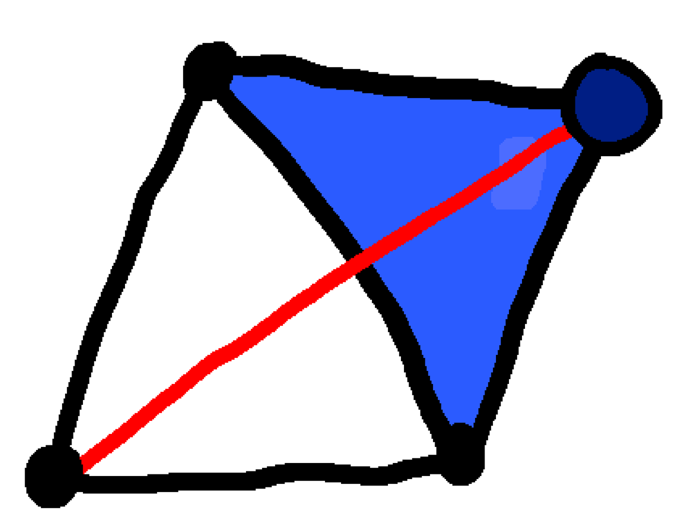
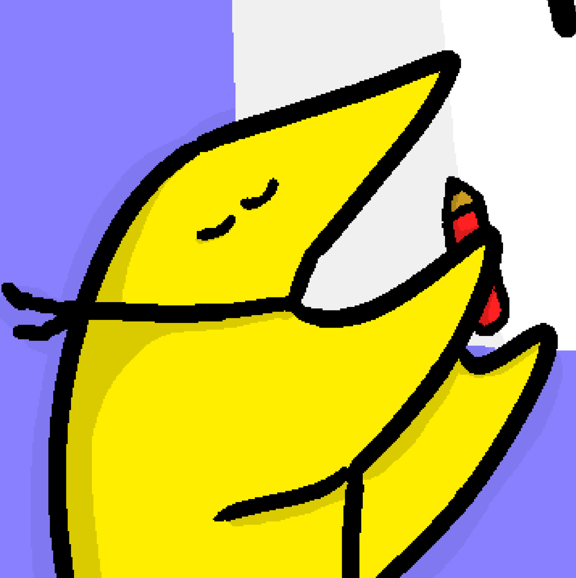
$K =$ 



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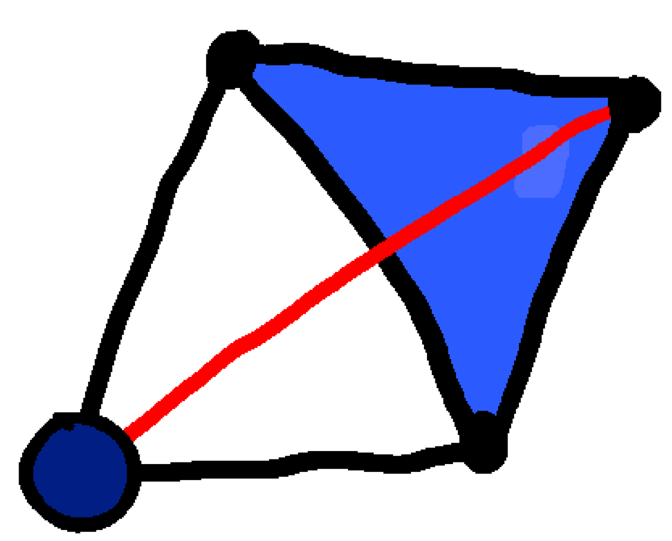
$K =$ 

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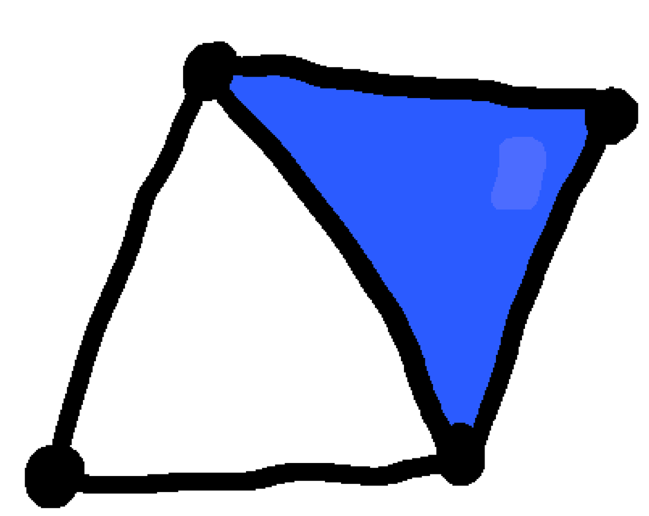
$K =$ 

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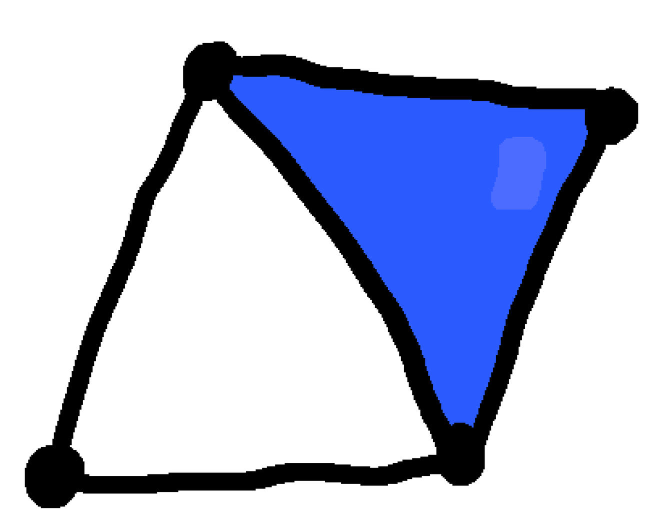
$K =$ 

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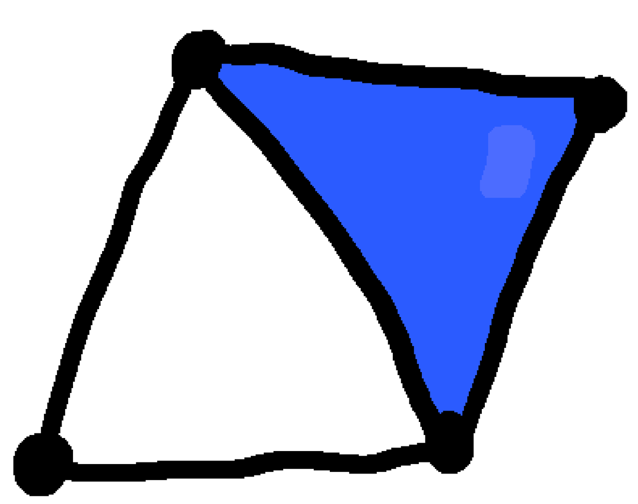
$K =$ 

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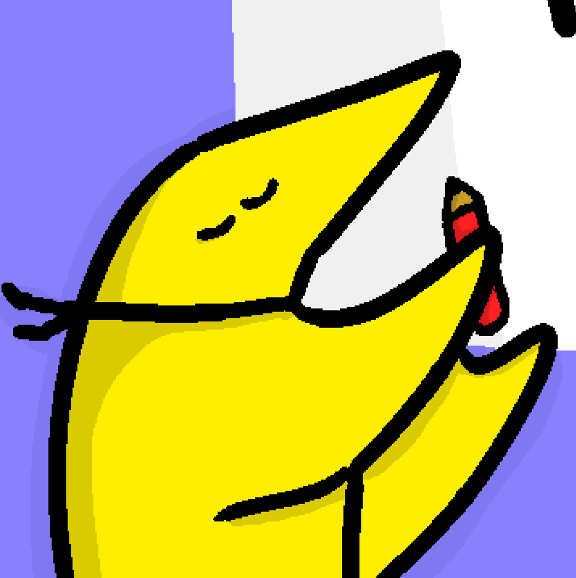

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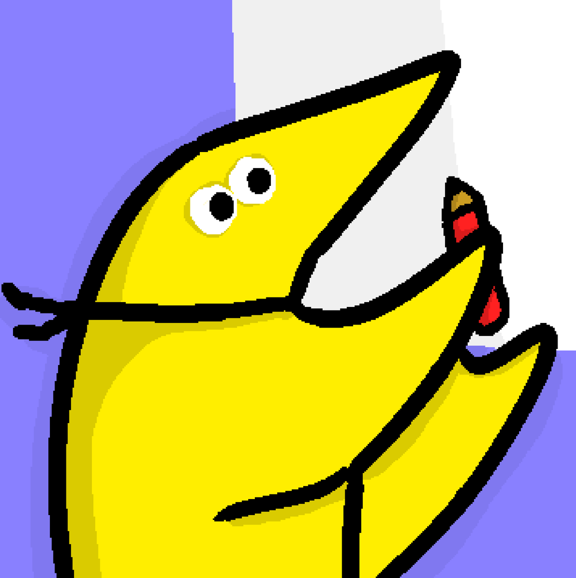

Helly for d-Leray complexes

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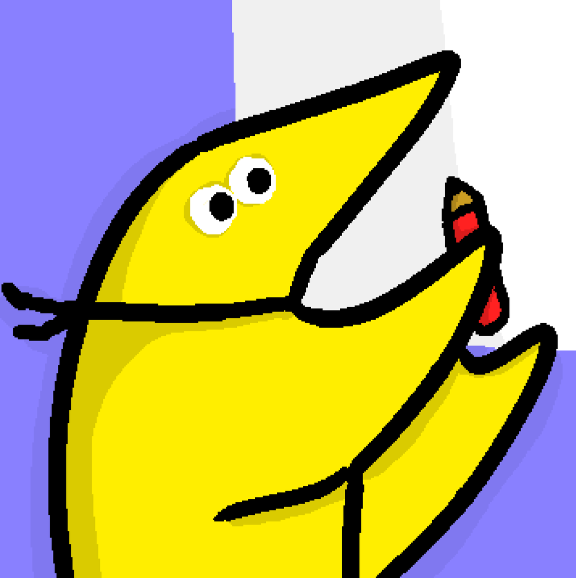

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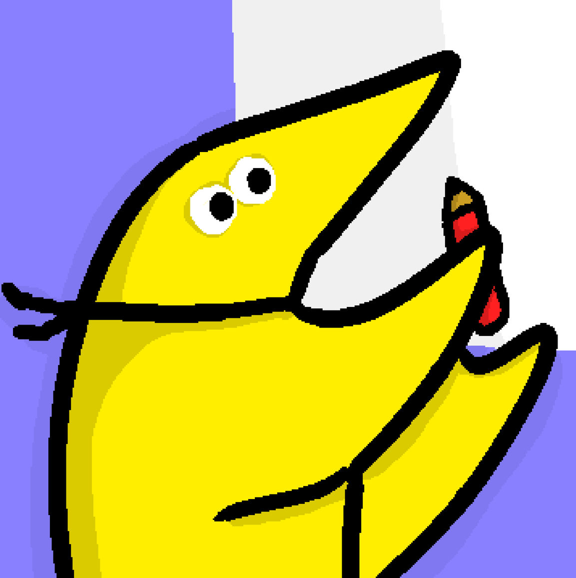

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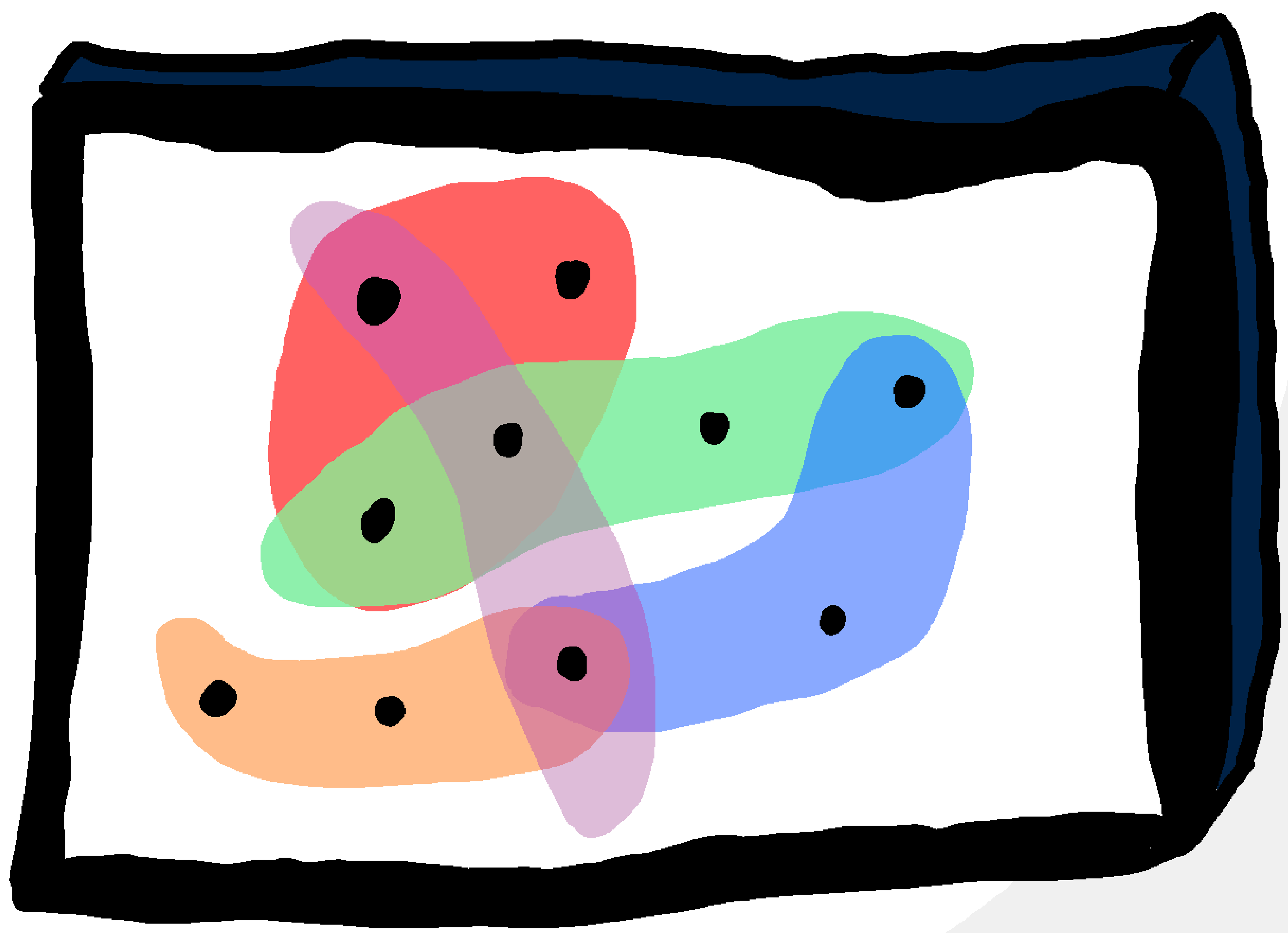

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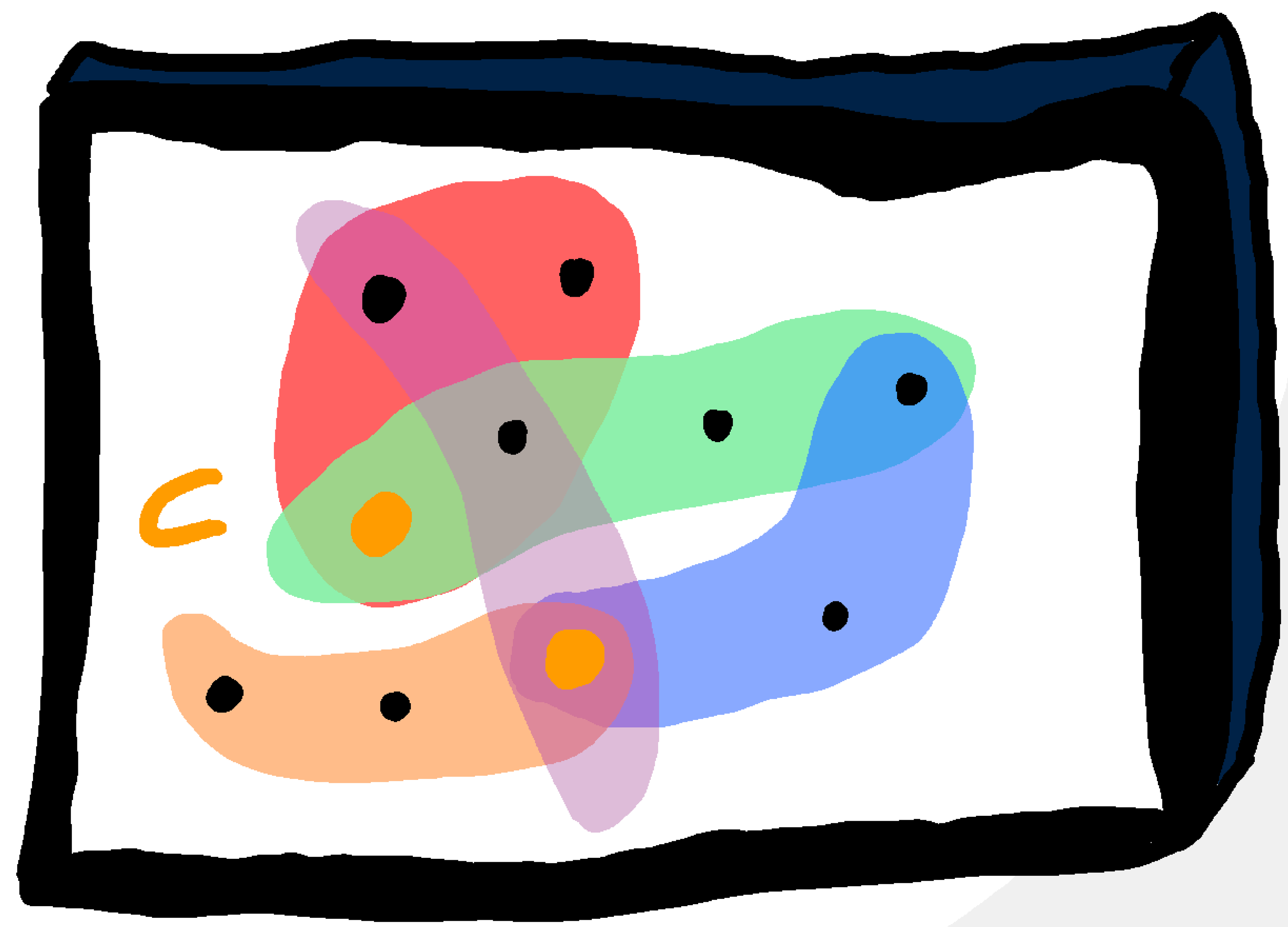

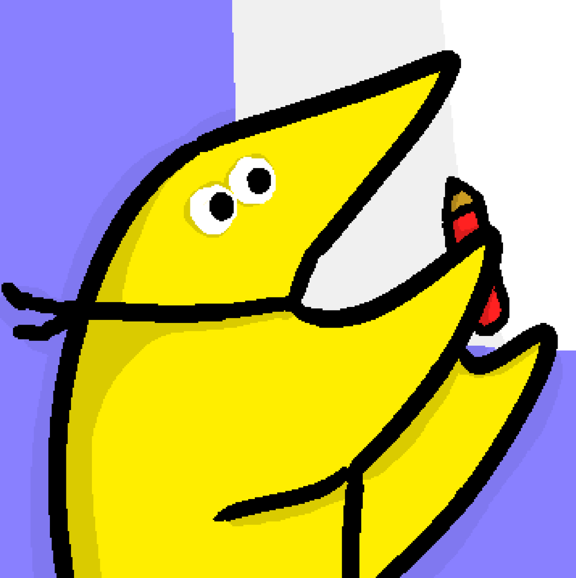

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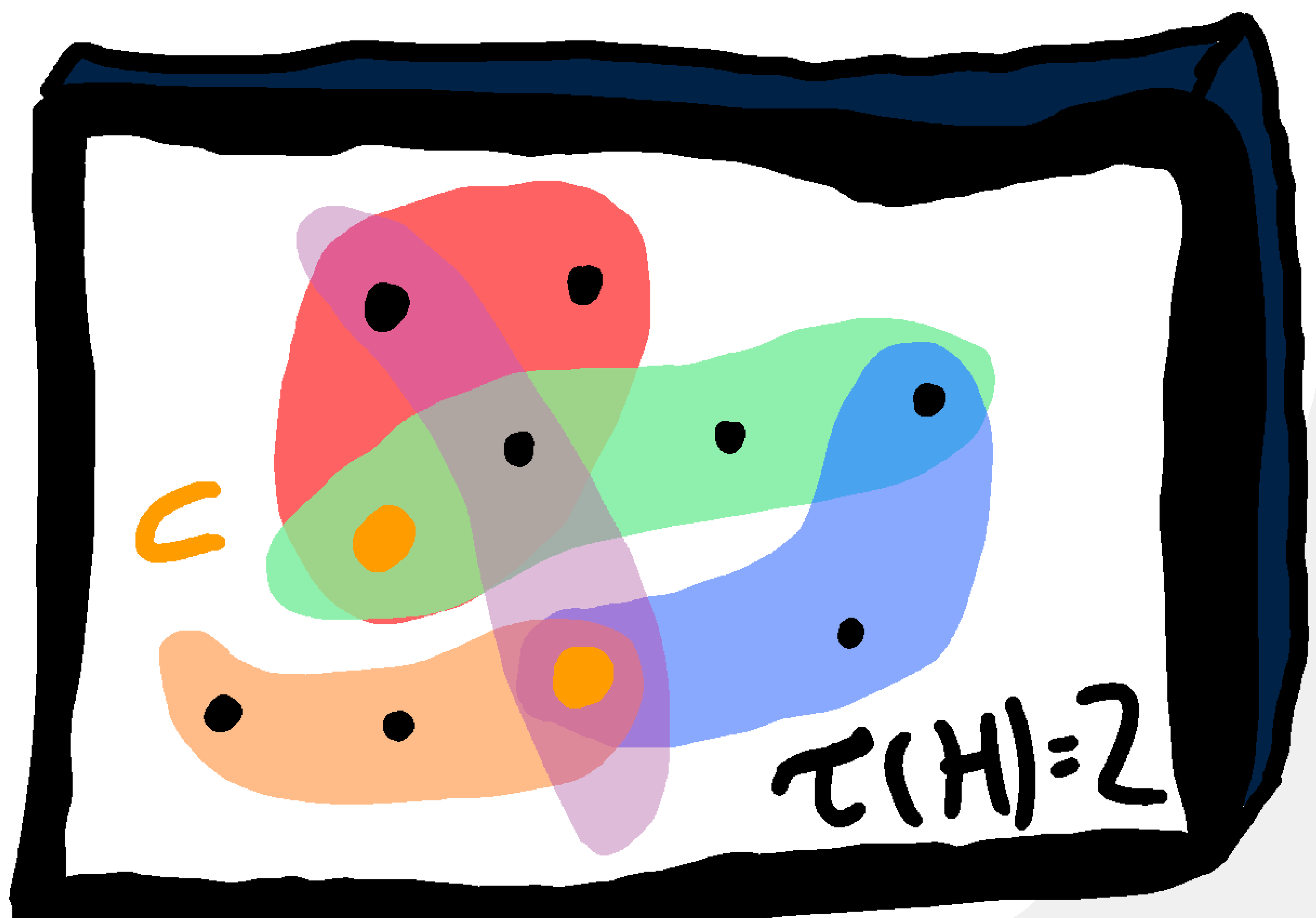

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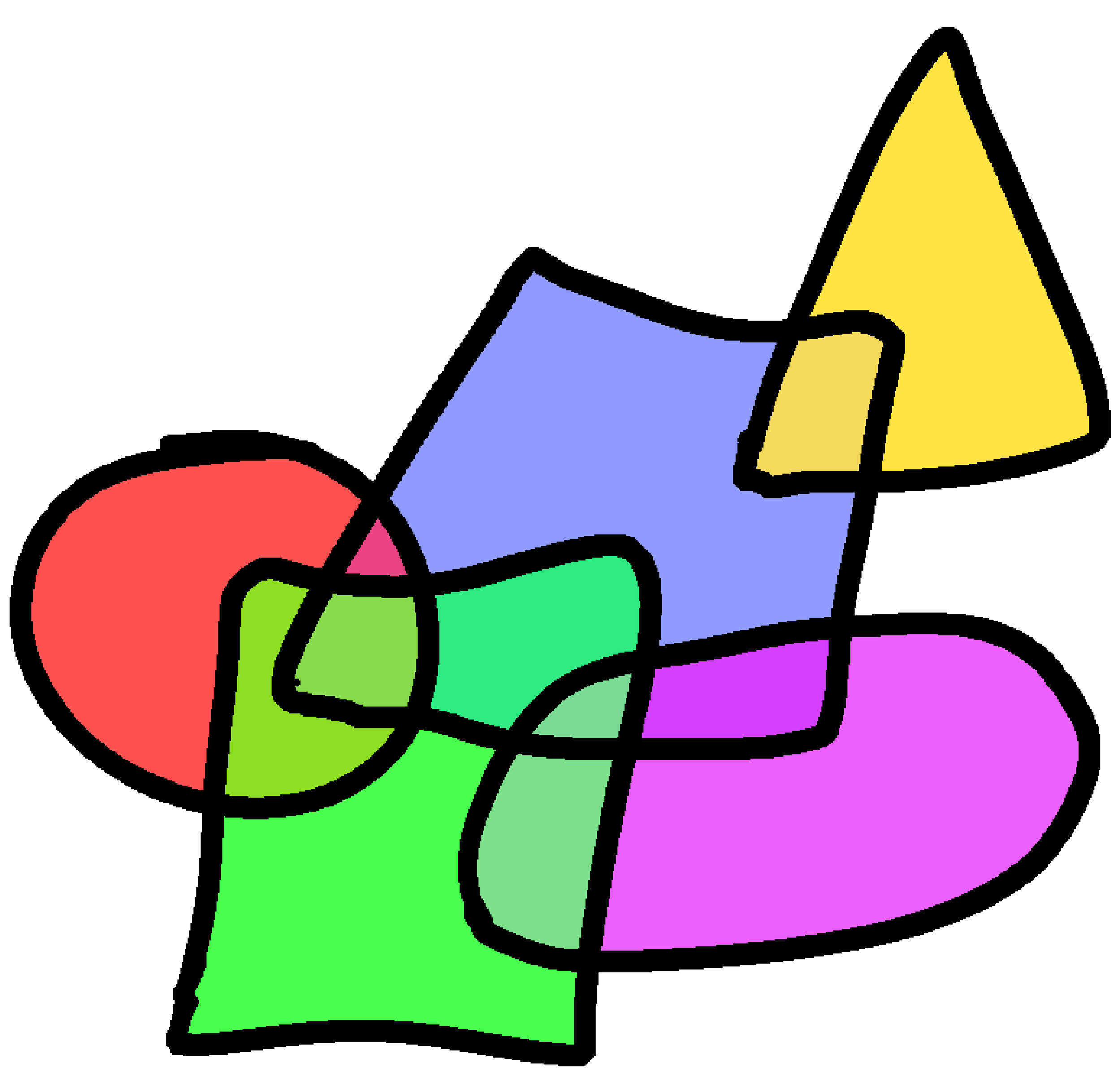
$F =$ 

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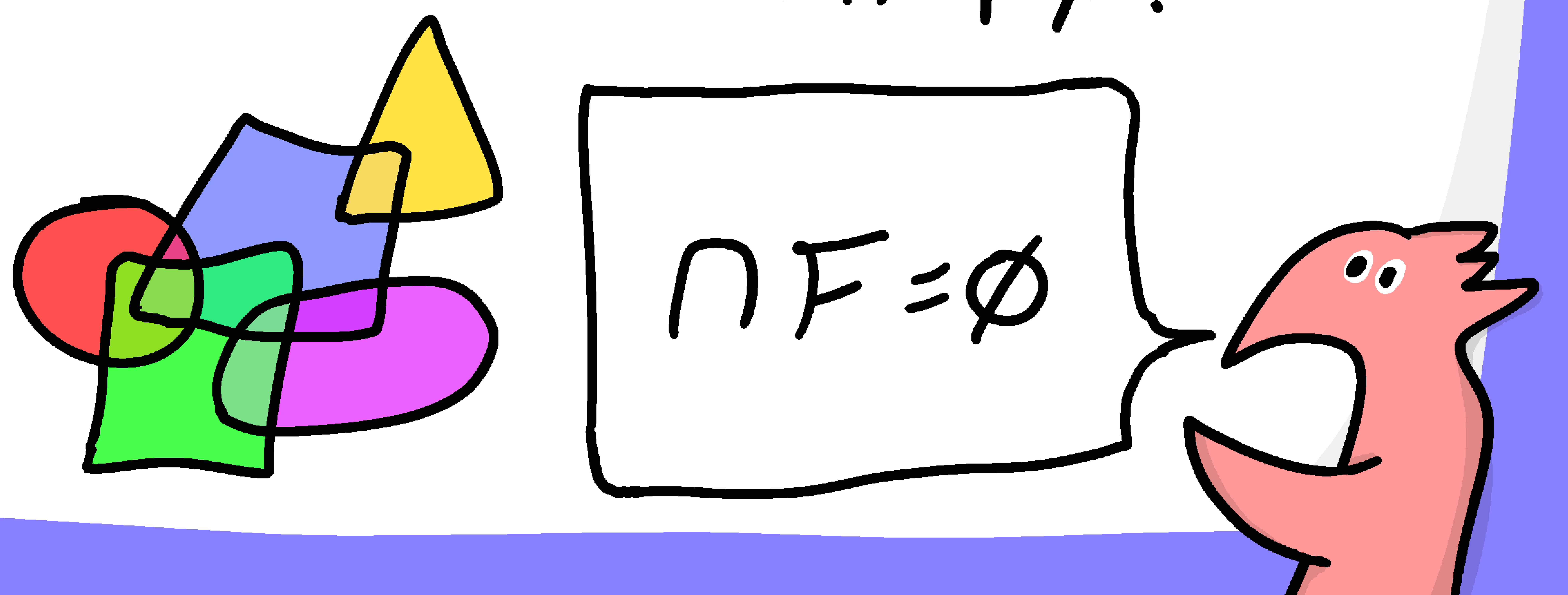

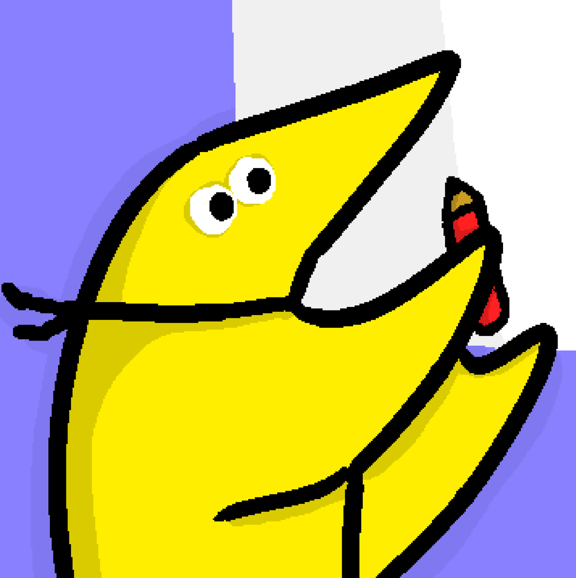

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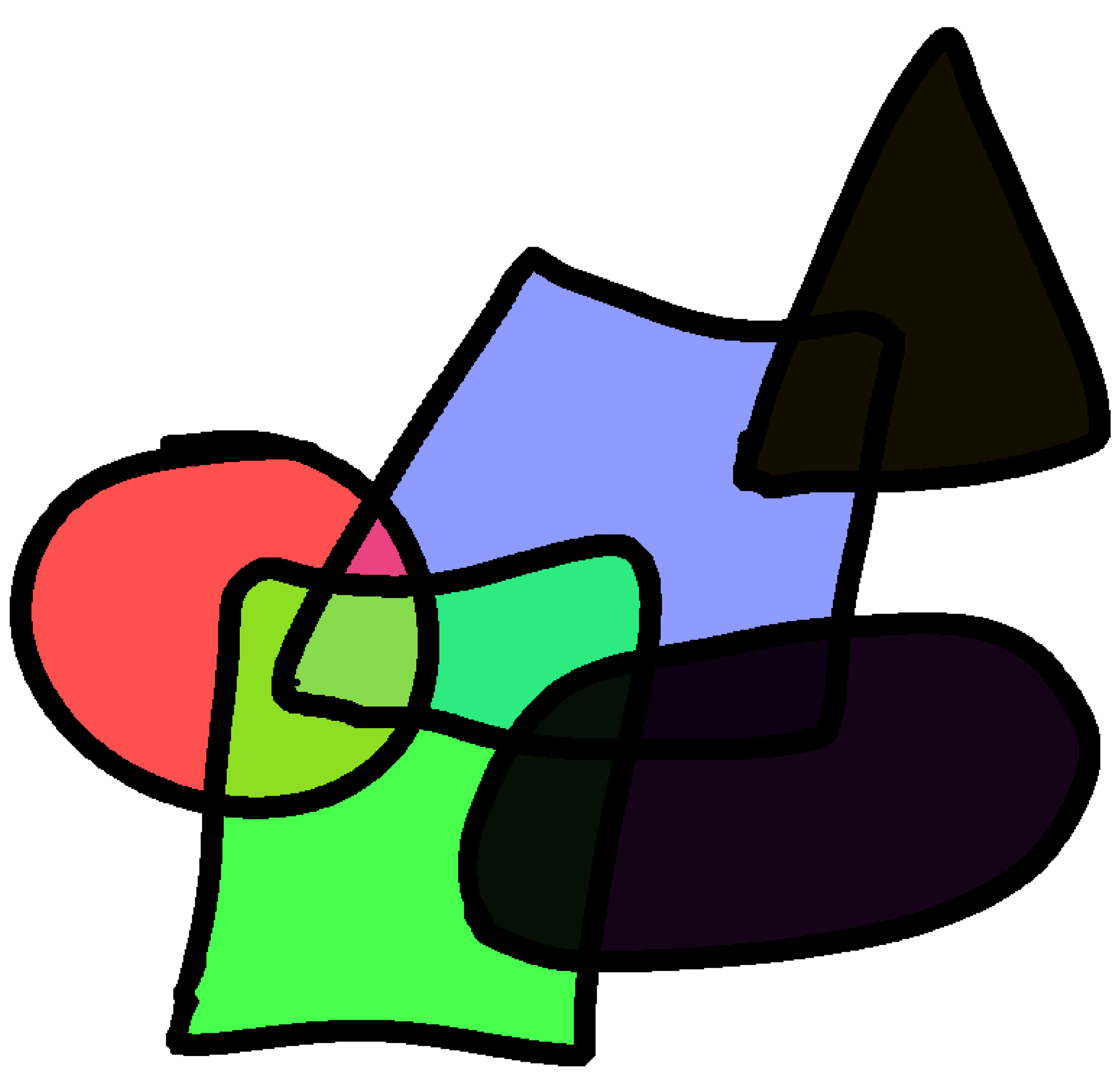
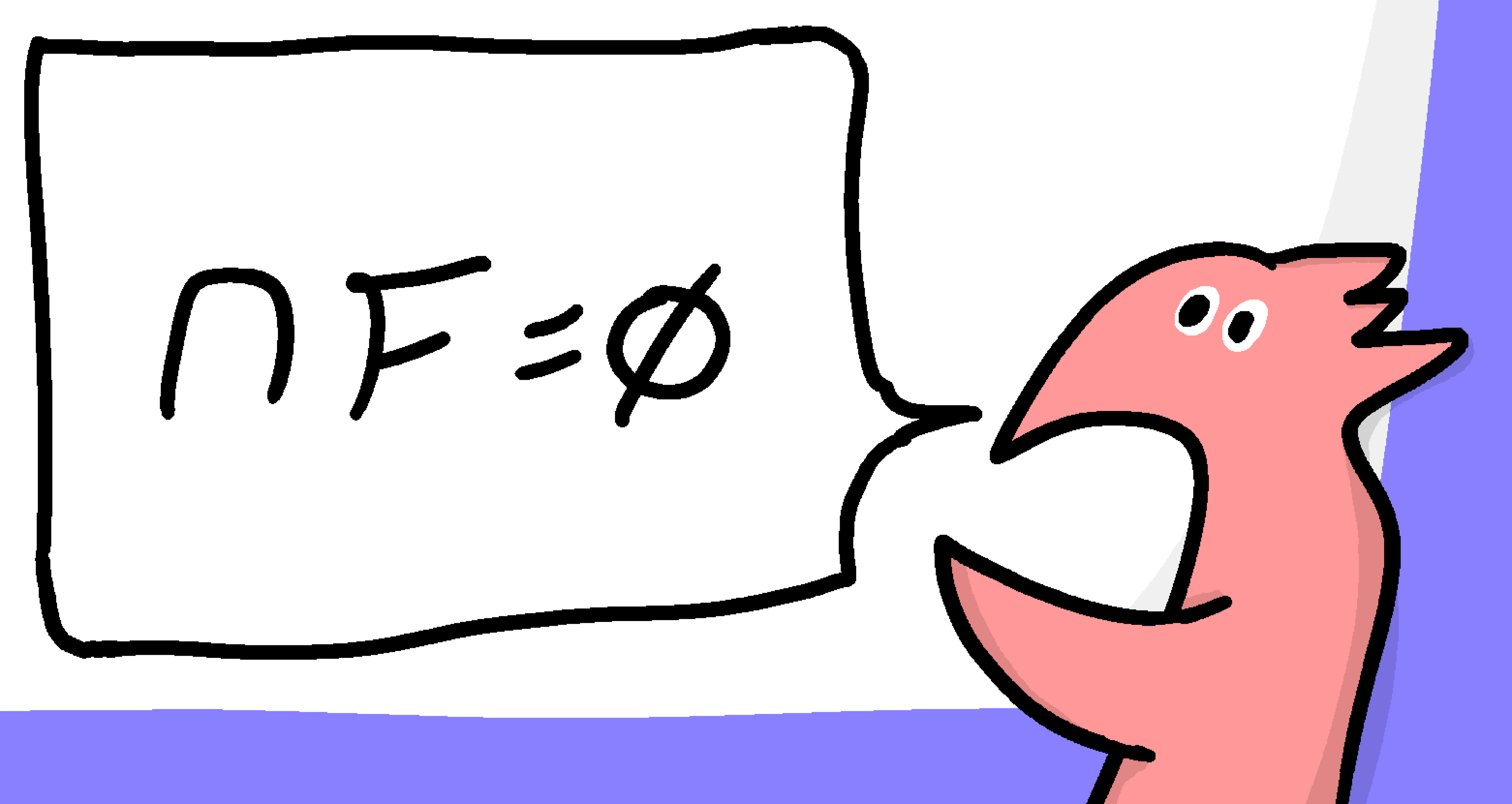
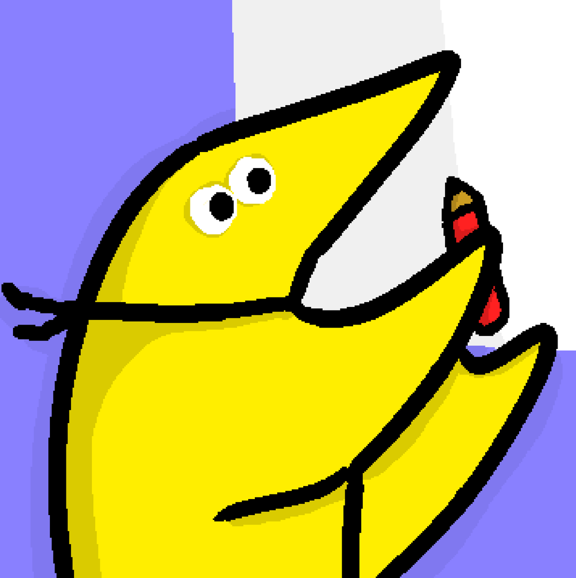
$\mathcal{F} =$ 





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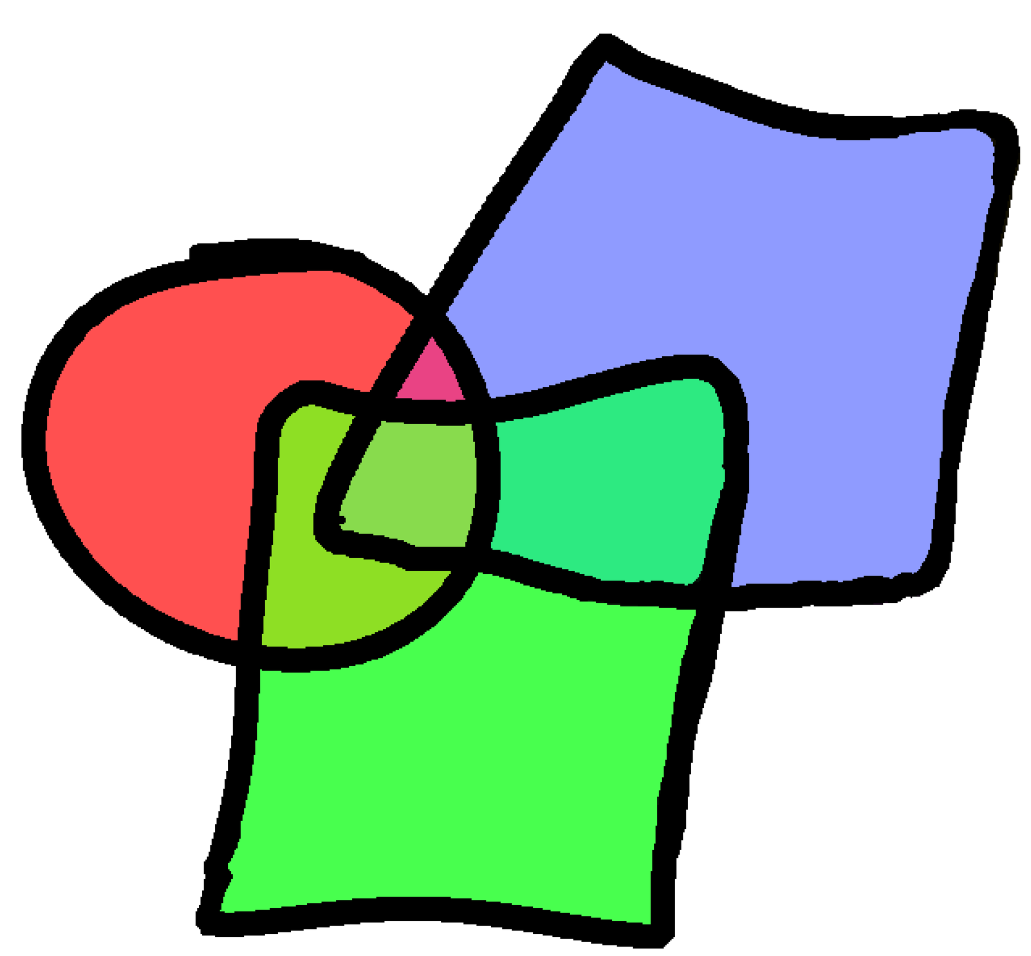
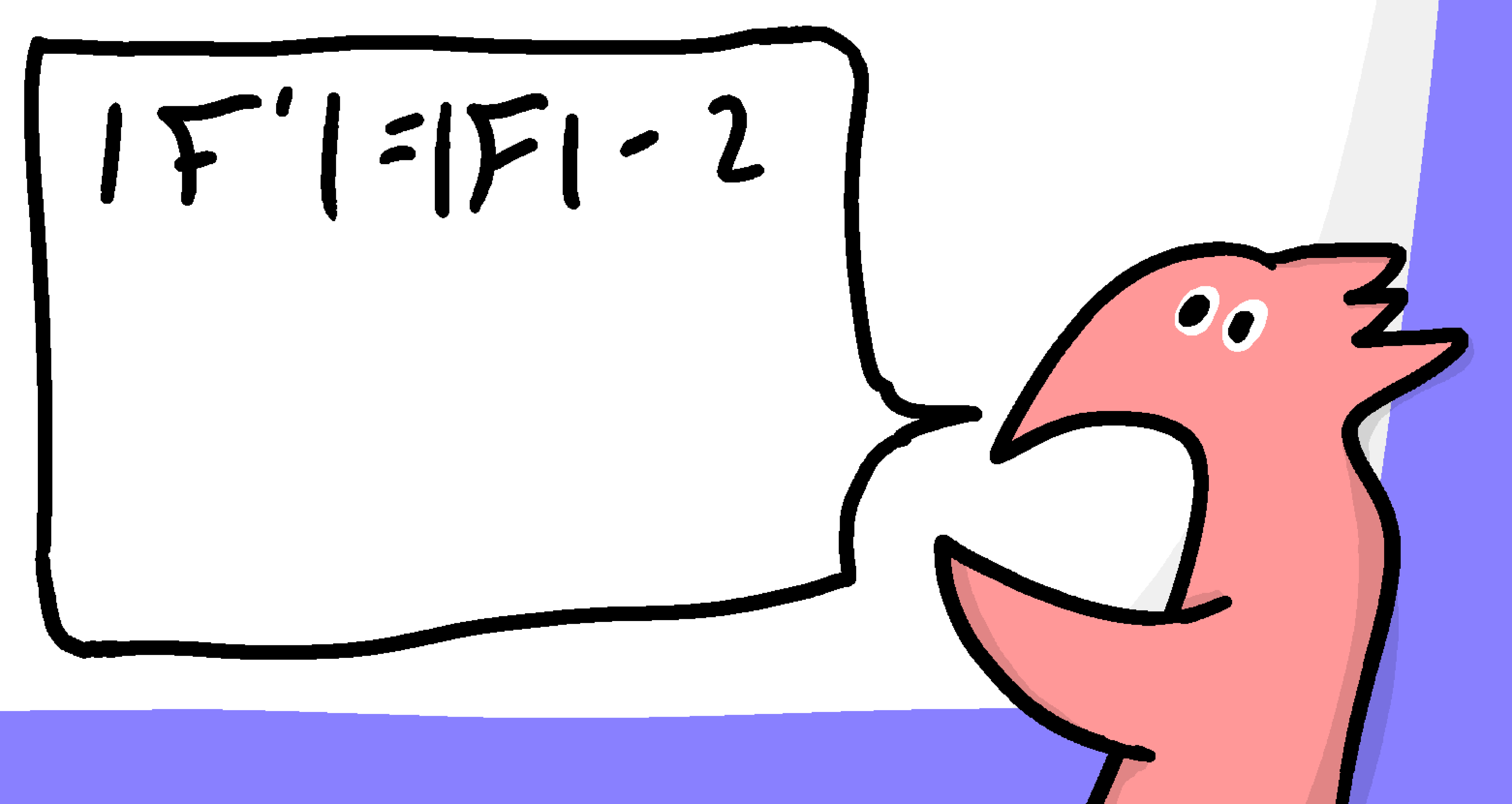
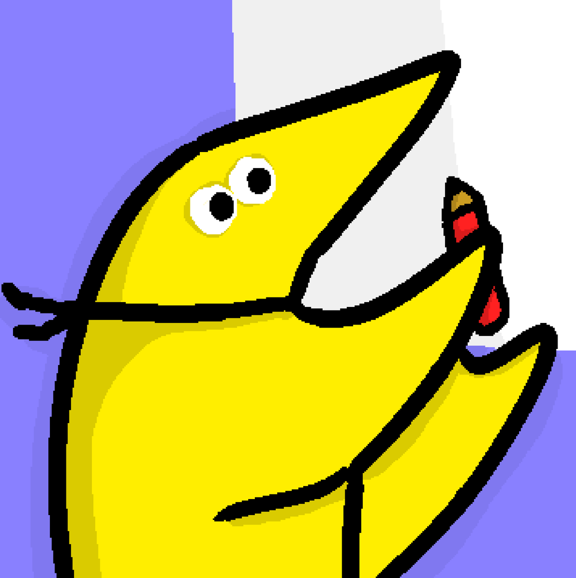
$\mathcal{F}' =$ 





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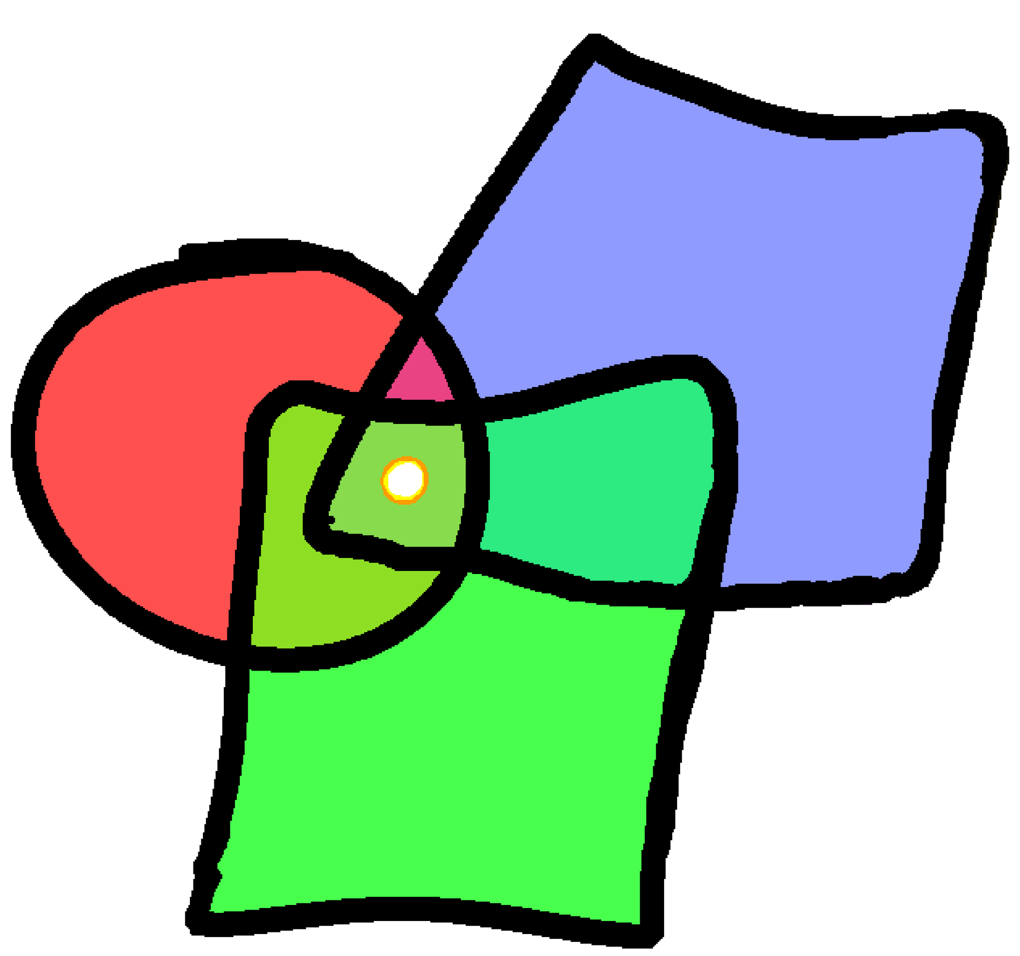
$F' =$ 

$|F'| = |F| - 2$

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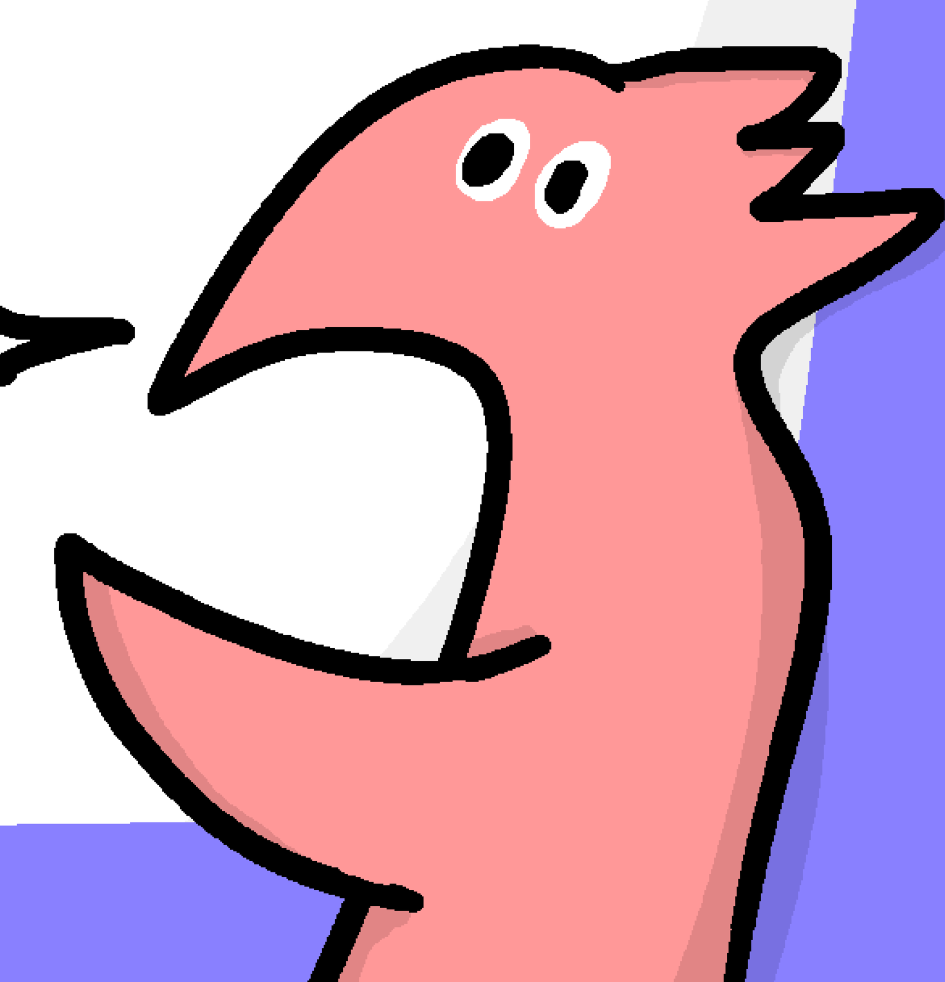

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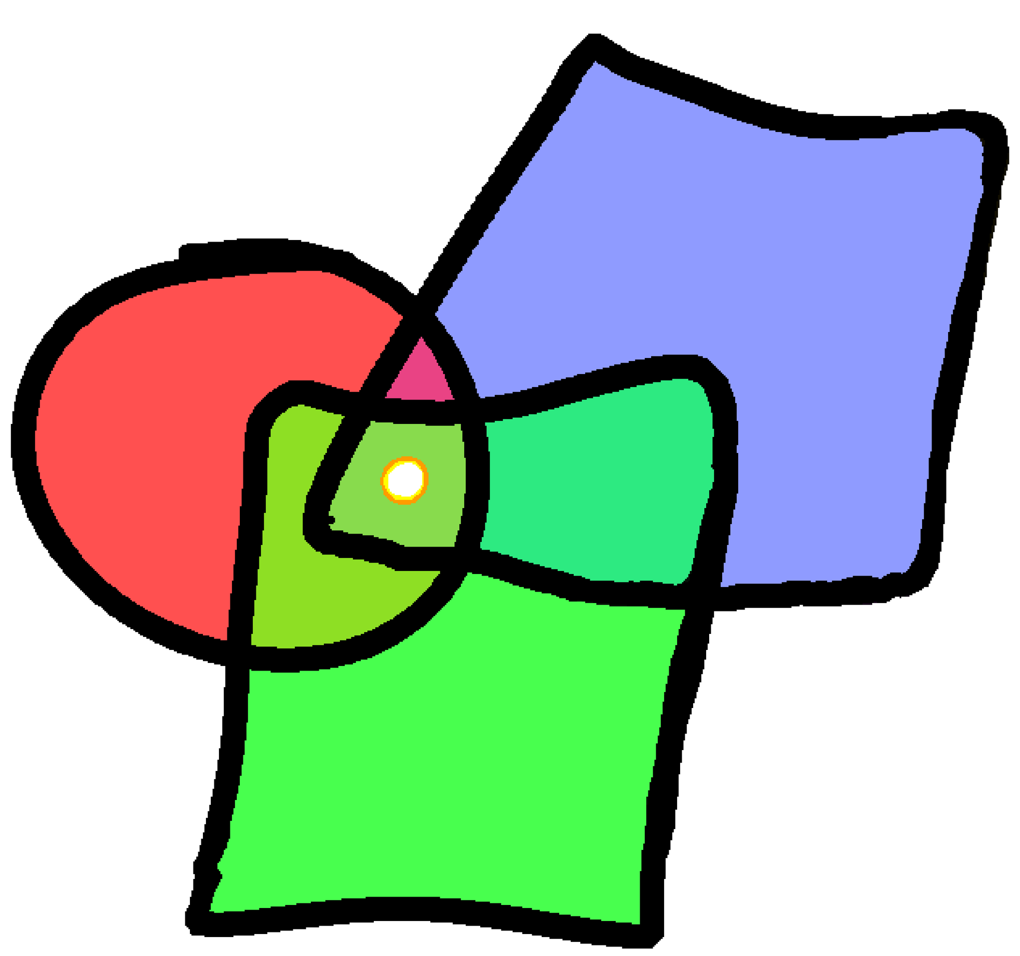
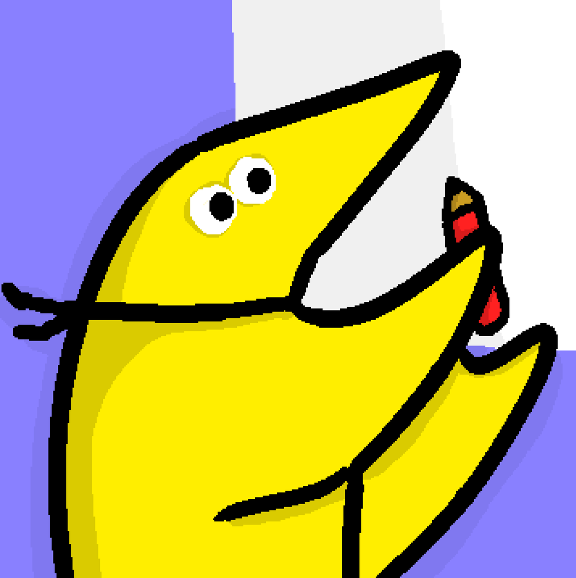
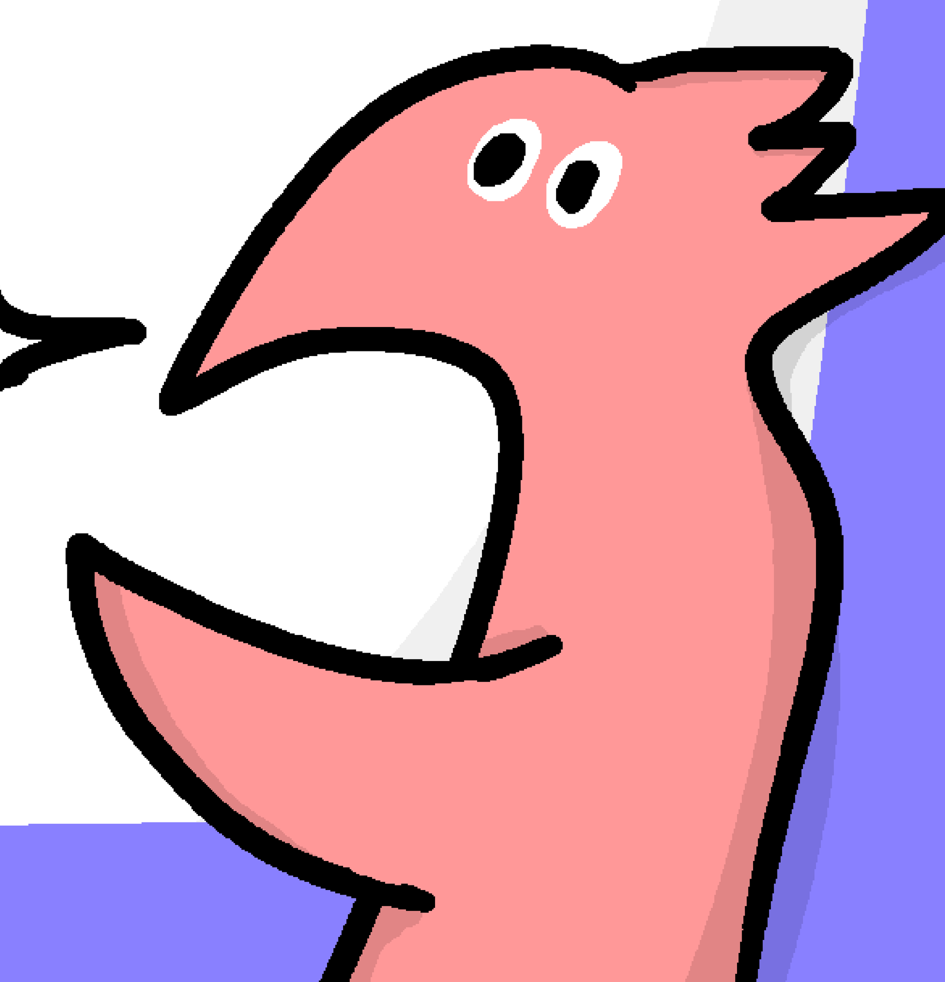
$F' =$ 

So, F has a point in common with tolerance 2





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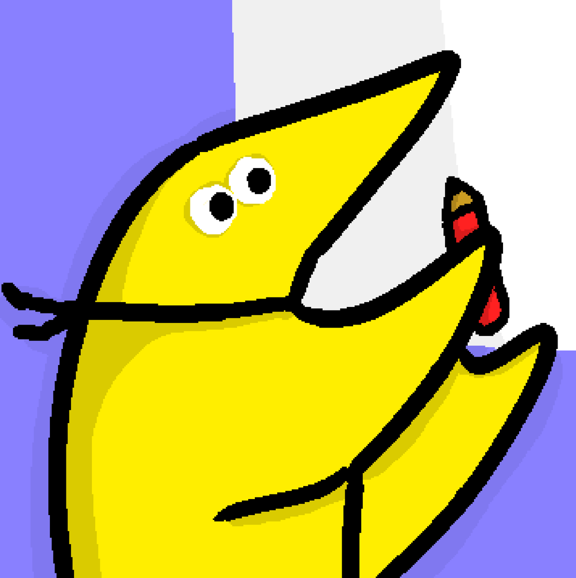

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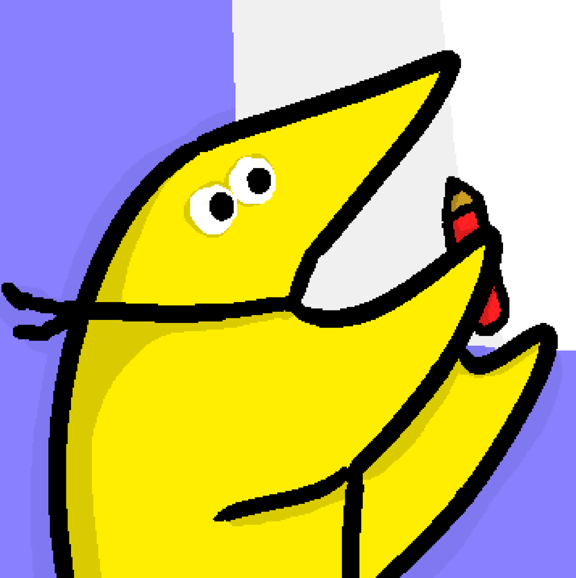

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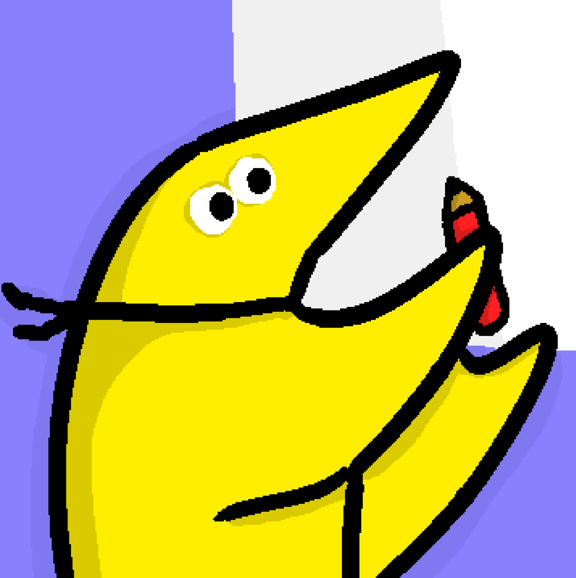

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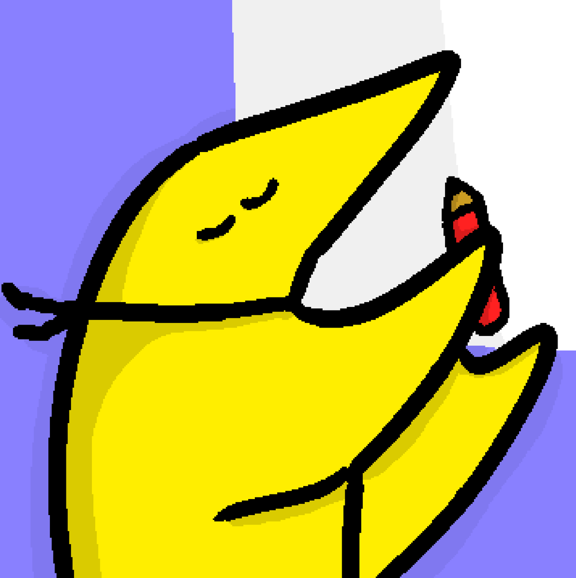

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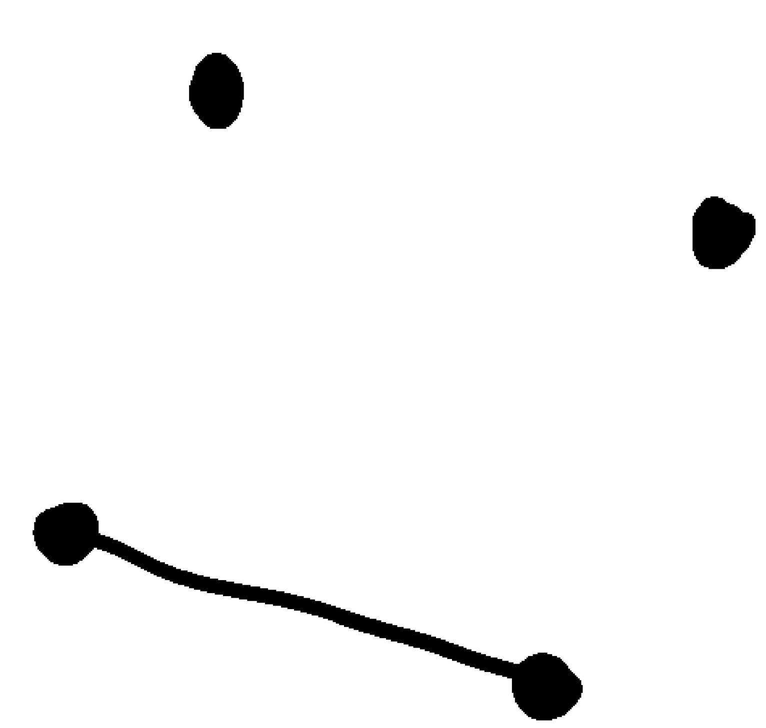
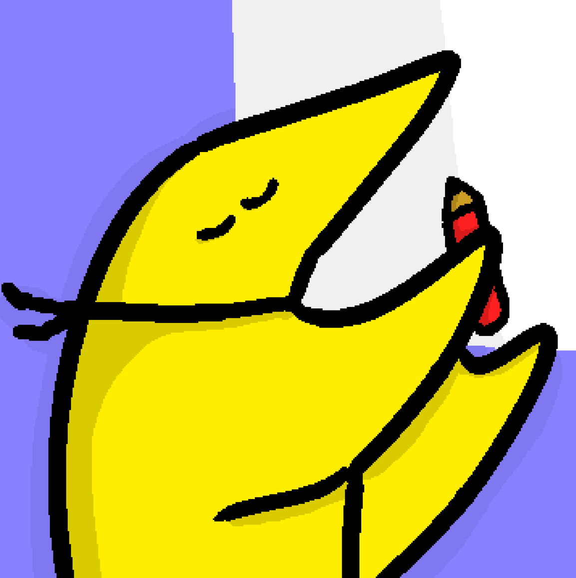
E.g.

$K =$ 



Tolerance Complexes

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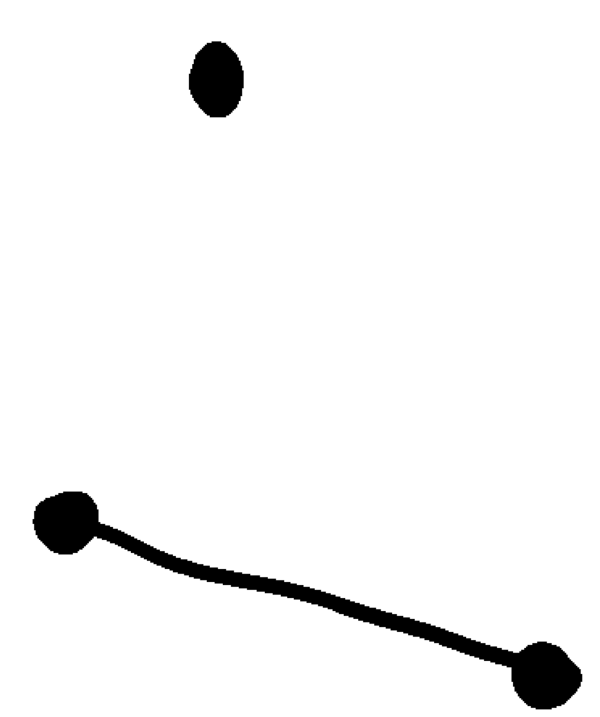
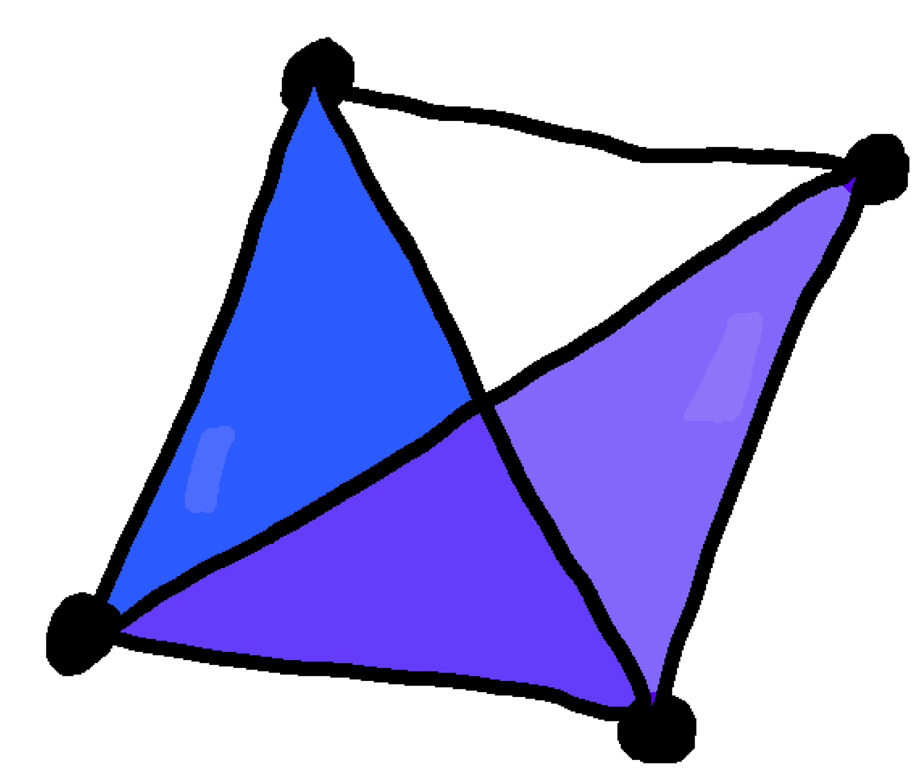
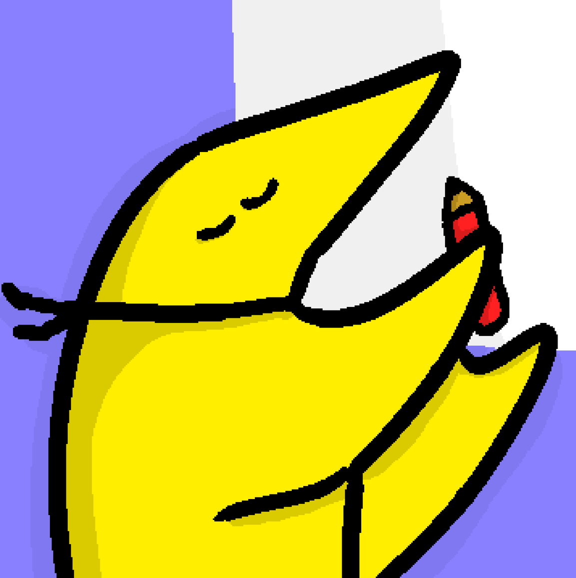
E.g. $K =$  $T_1(K) =$ 




Tolerance Complexes
F = family of sets


Tolerance Complexes
$\mathcal{F}$ = family of sets
$T_t(N(\mathcal{F})) = \{ \mathcal{F}' \subseteq \mathcal{F} \mid \mathcal{F}'$ has pt. in common with tolerance $t \}$

Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If K is d-representable, then

$$h(T_t(K)) \le h(d+1, t+1) - 1$$

Helly's property for tolerance complexes

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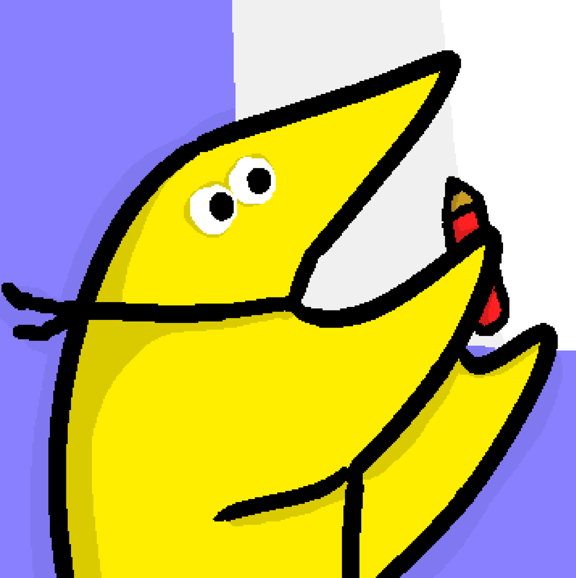
- If we assume K is d-collapsible / d-Leray, can we obtain a stronger conclusion?

Collapsibility and Leray numbers of $T_t(K)$

Conjecture: If K is d-Leray, then $T_t(K)$ is $(\eta(d+1, t+1) - 1)$-Leray.



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Conjecture: If K is d-collapsible, then

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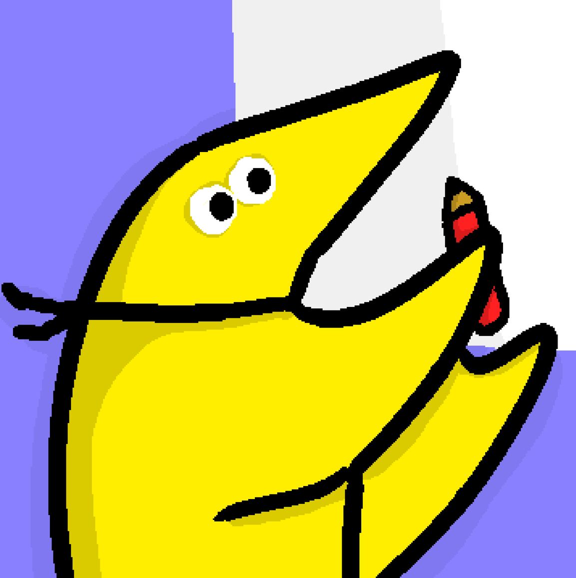

Extremal examples

$\mathcal{C}$ = facets of d-dim simplex,
$t+1$ copies of each.

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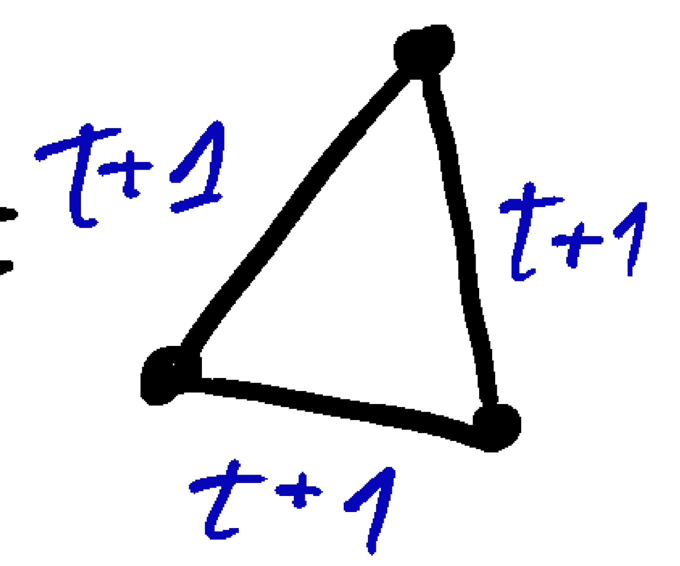

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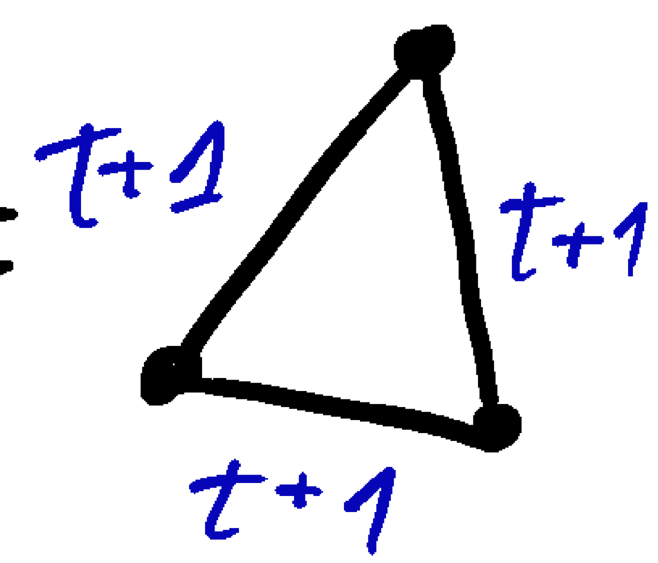
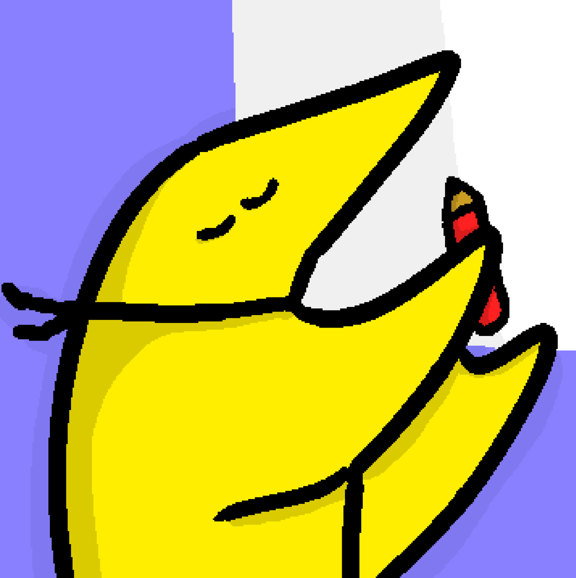
C = 

$N(C)$ = d-representable



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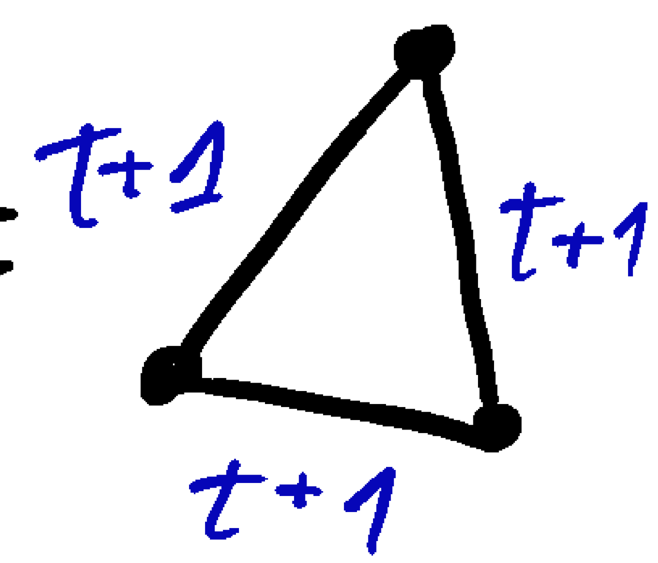
C = 

$N(C)$ = d-representable $\begin{pmatrix} d\text{-collapsible} \\ d\text{-Leray} \end{pmatrix}$

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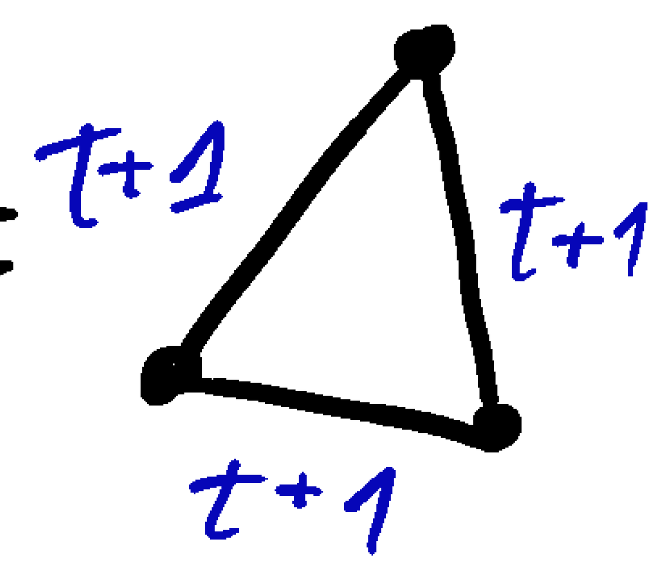
$T_t(N(\mathcal{C})) = ?$

$\mathcal{C} =$ 

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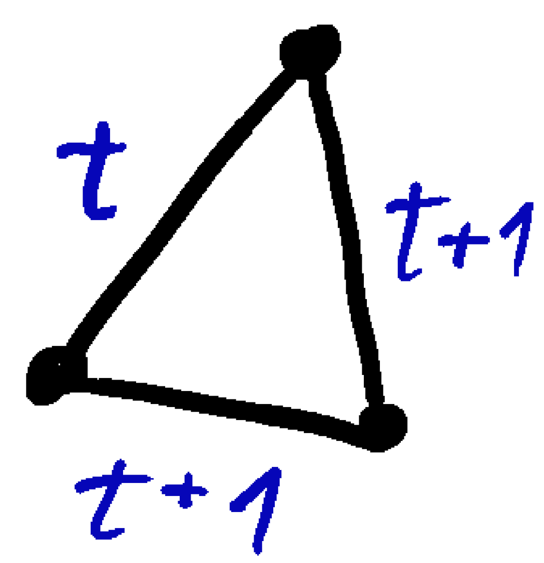
$C \setminus \{A\} =$ 

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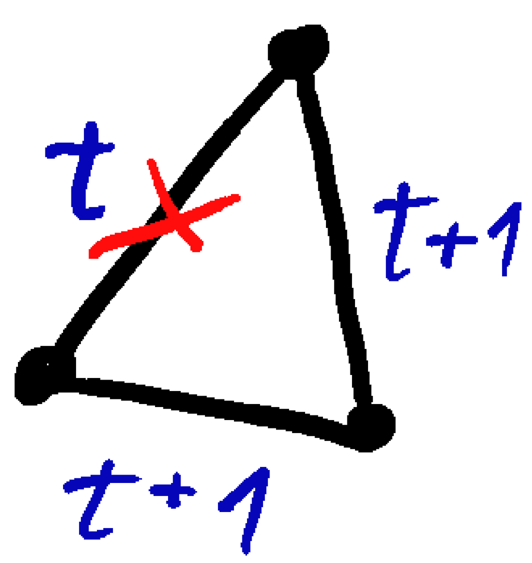
$\mathcal{C} \setminus \{A\} =$ 

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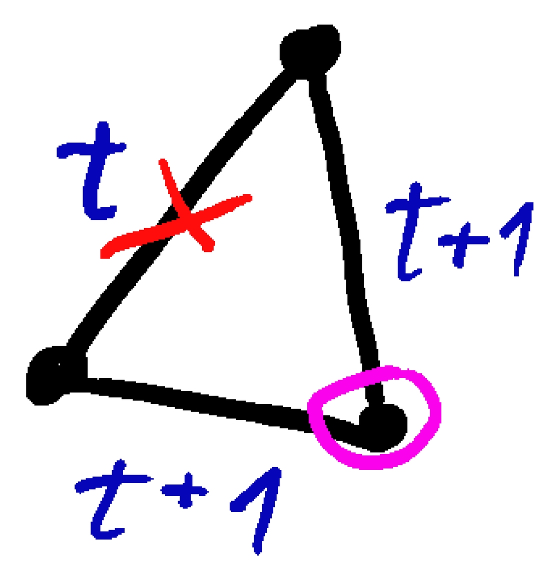
$\mathcal{C}\setminus\{A\} =$ 

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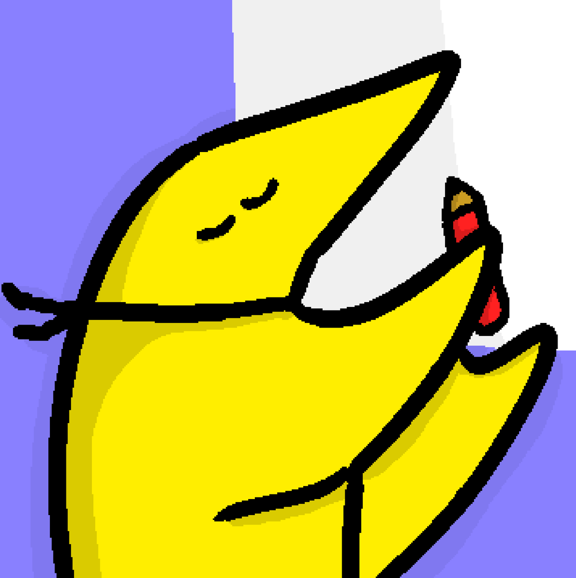

Extremal examples

$\mathcal{C}$ = facets of d-dim simplex, $t+1$ copies of each.

$T_t(N(\mathcal{C}))$ = boundary of $((d+1)(t+1)-1)$-dim. simplex

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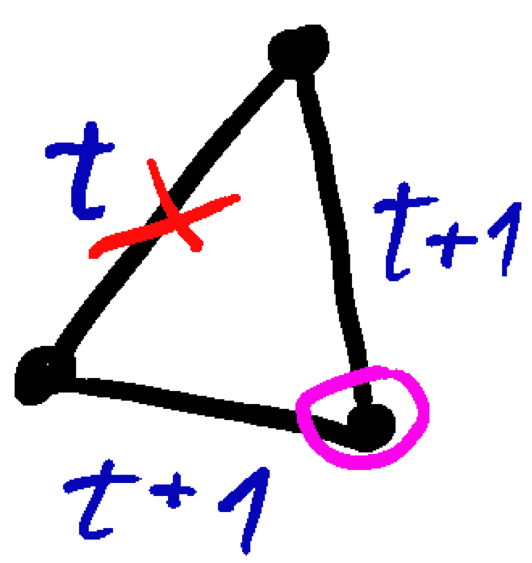

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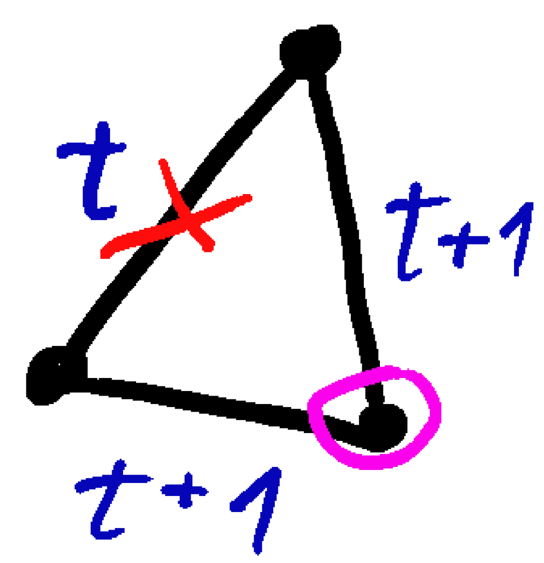
$\mathcal{C}\setminus\{A\}$ = 

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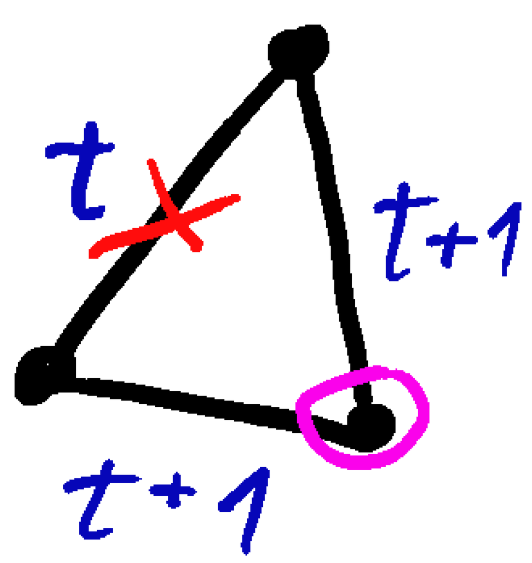

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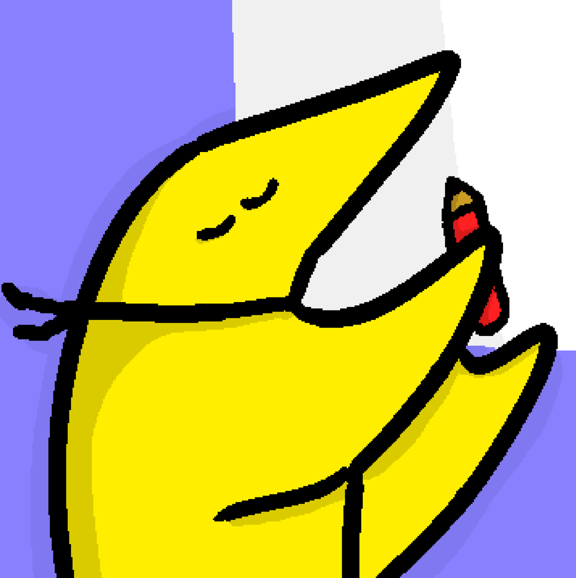
$T_t(N(\mathcal{C}))$ is **NOT** $((d+1)(t+1)-2)$-Leray



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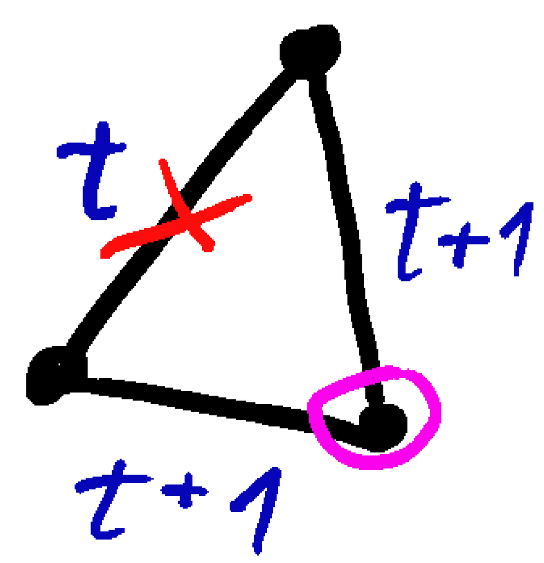
$\mathcal{C}\setminus\{A\} =$ 

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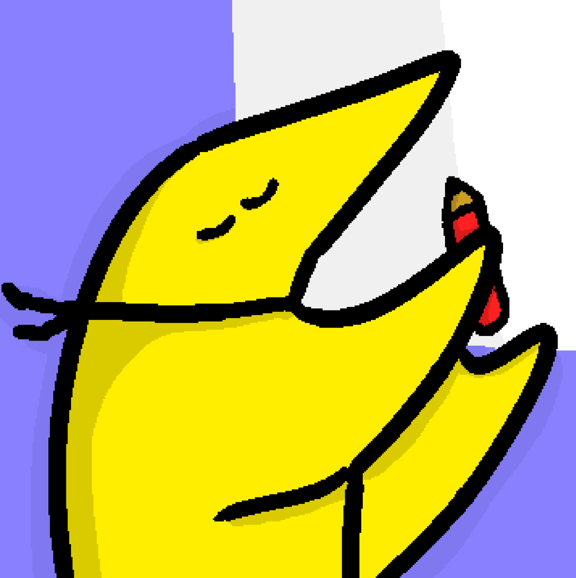
- For $d=1$, $(d+1)(t+1)-2 = 2t$



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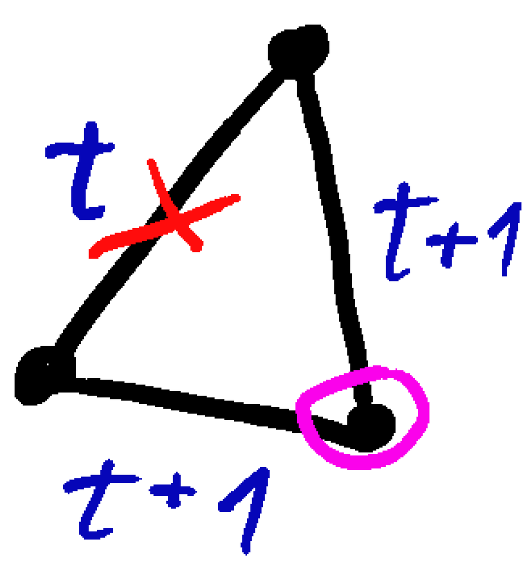

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Extremal examples

Montejano-Oliveros ('10):

$\exists\, \mathcal{C}$ family of convex sets in $\mathbb{R}^d$

s.t. $T_1(N(\mathcal{C}))$ is **not** $\left(\left\lfloor\left(\frac{d+3}{2}\right)^2\right\rfloor - 2\right)$-Leray

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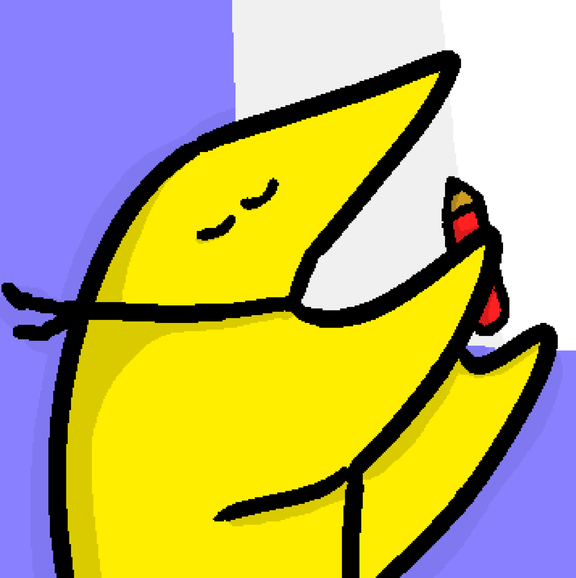

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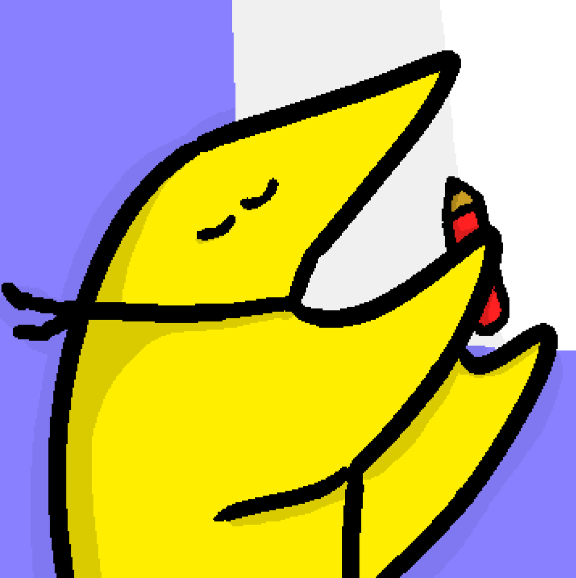
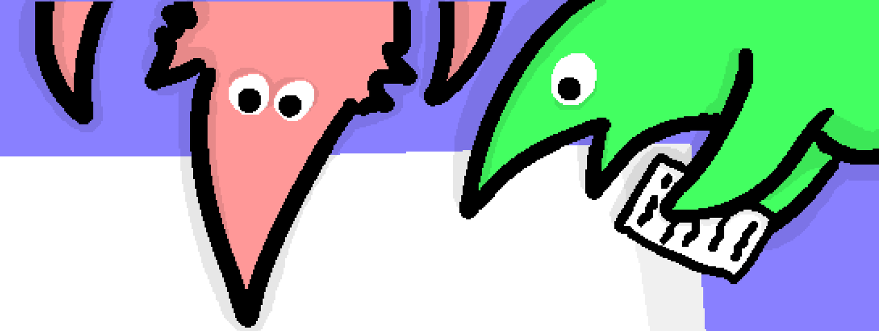
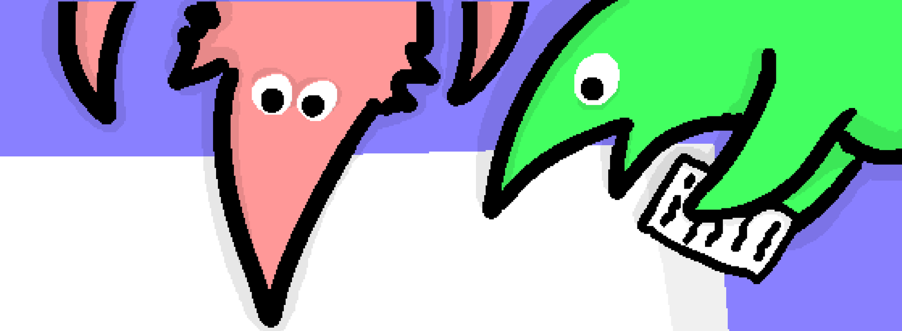




Main results

$$h(t,d) = \begin{cases} d & ; t=0 \\ \left[\sum_{s=1}^{\min\{t,d\}} \binom{d}{s} \left(h(t-s,d) + 1 \right) \right] + d & ; t>0 \end{cases}$$



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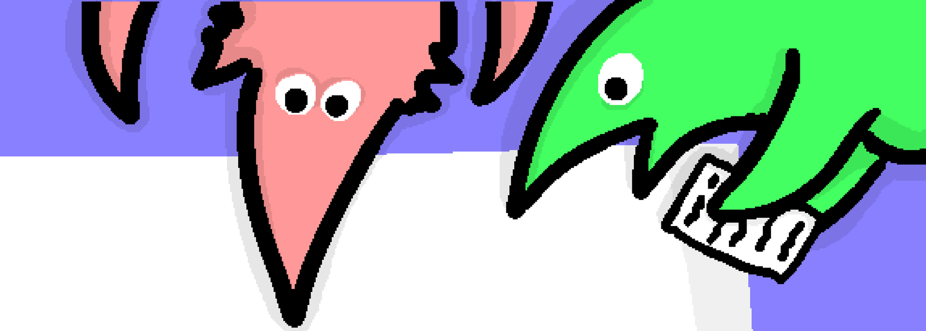
Thm (Kim-L. '21):

Let K be a d-collapsible complex. Then, $T_t(K)$ is $h(t,d)$-Leray.



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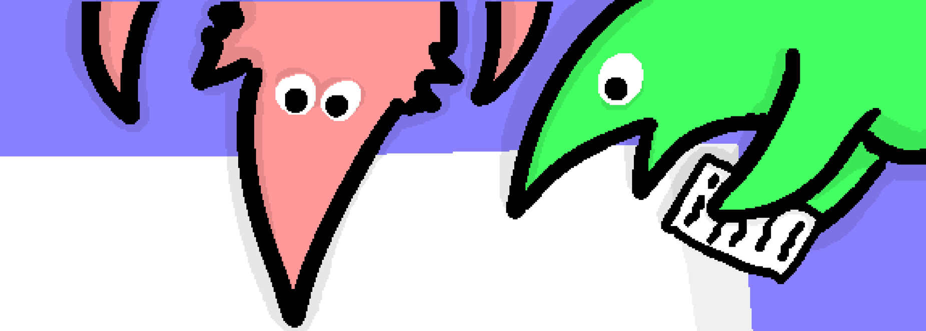

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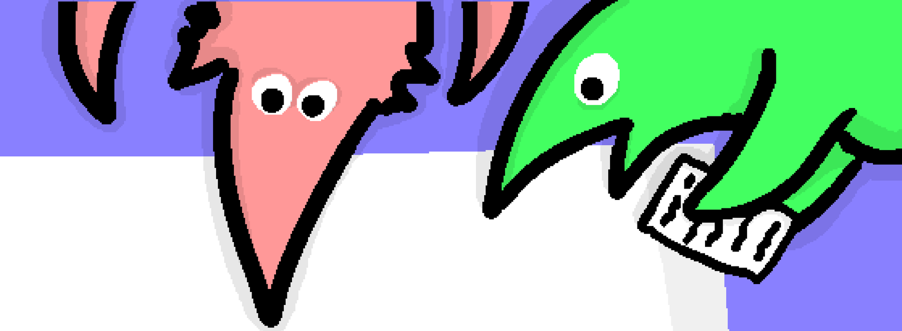
For fixed t,

$h(t,d) = O(d^{t+1})$



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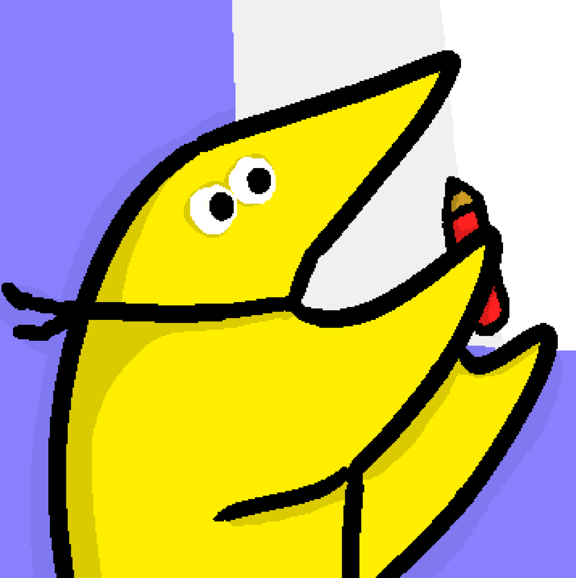

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$$? \Downarrow \quad \left\lfloor \left(\frac{d+3}{2}\right)^2 \right\rfloor - 1$$

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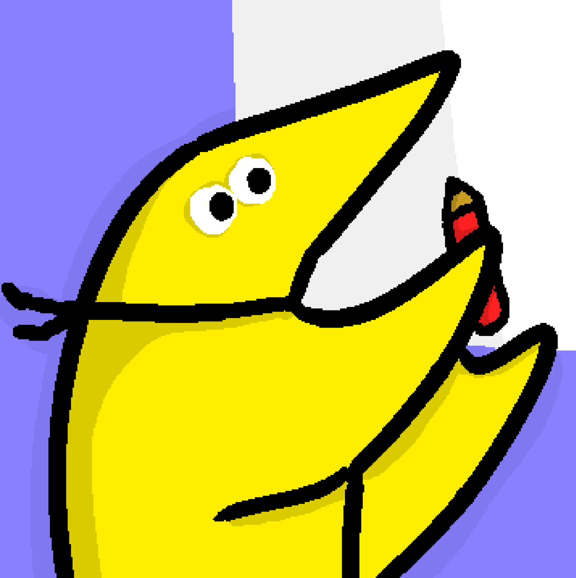

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- True for $d=2$:

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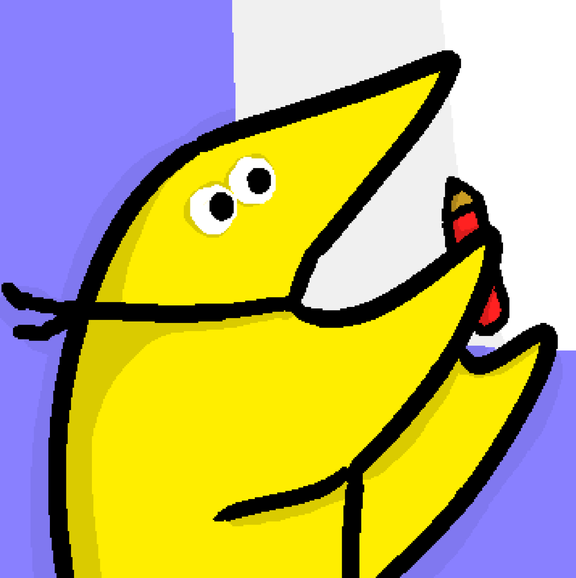
- True for $d=2$:

Thm (Kim-L. '21): Let K be 2-collapsible. Then, $T_1(K)$ is 5-Leray.



Some ideas from the proof

Some definitions:

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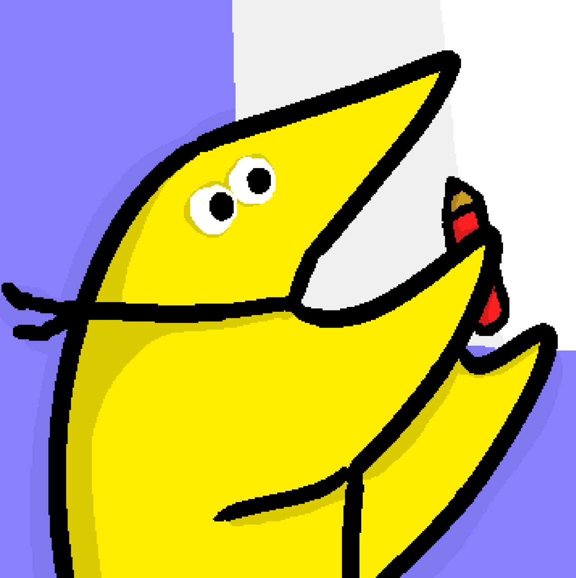
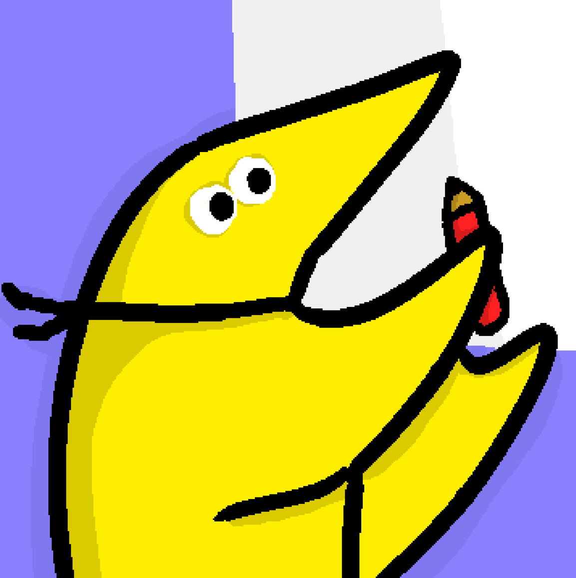
- $\mathrm{cost}(K,\sigma) = \{ \tau \in K : \sigma \not\subseteq \tau \}$



Some ideas from the proof

Thm (Tancer '10): K is d-collapsible iff either $\dim(K) < d$ or



Some ideas from the proof

Thm (Tancer '10): K is d-collapsible iff either $\dim(K) < d$ or $\exists\, \sigma \in K$, $|\sigma| = d$, contained in unique max face $\tau \neq \sigma$ and

$$\operatorname{cost}(K, \sigma) \text{ is } d\text{-collapsible}$$

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Let K be d-col.

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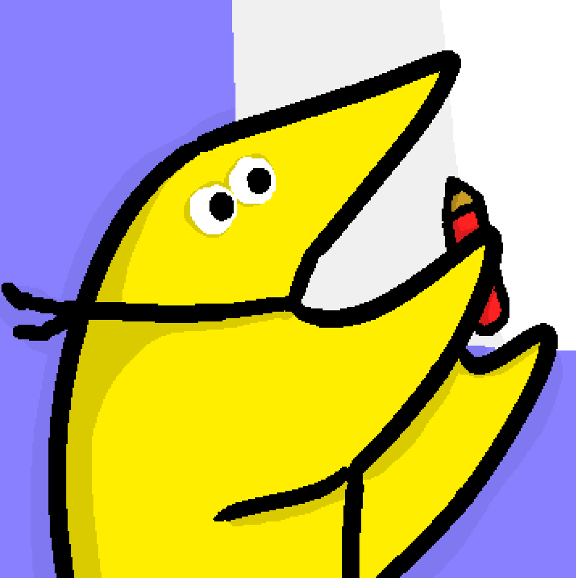
If $\dim(K) < d$ then

$$\dim(T_t(K)) < d + t < h(t,d)$$



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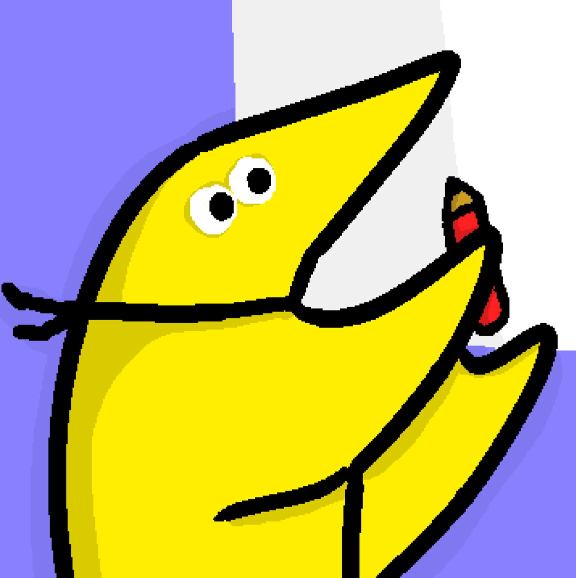

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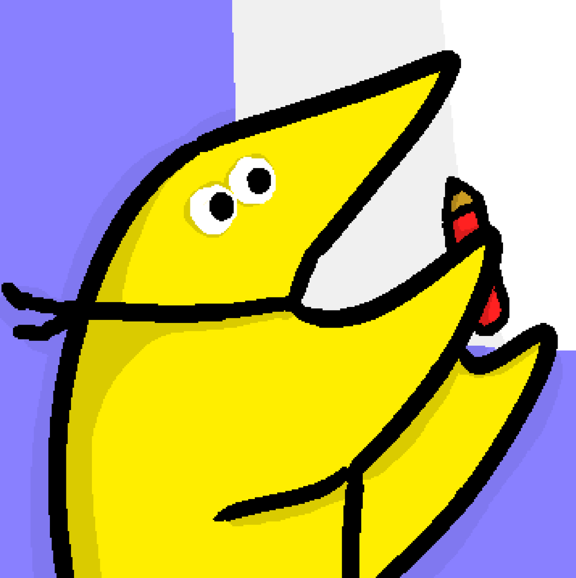
Otherwise: $\exists\ \sigma \in K$, $|\sigma| = d$, s.t. σ is contained in unique max. face $\tau = \sigma \cup U$ $(U \neq \emptyset)$, $\mathrm{cost}(K,\sigma)$ d-col.



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We want to show: $\widetilde{H}_k(T_t(k)) = 0$

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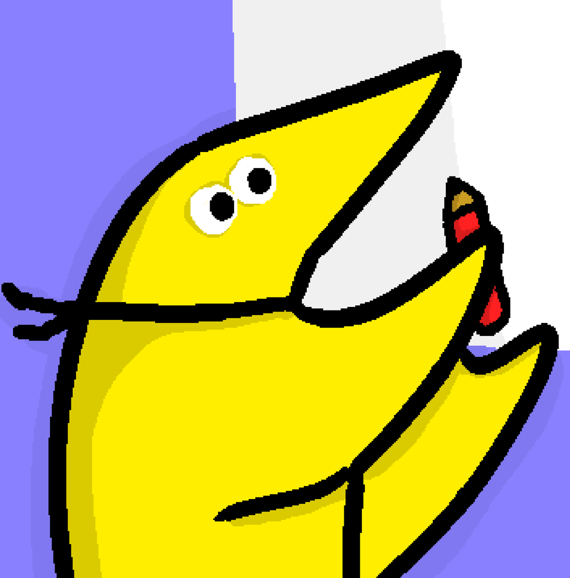

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Look at long exact seq. of the pair $T_t(\mathrm{cost}(k,\sigma)) \subseteq T_t(k)$:

Some ideas from the proof

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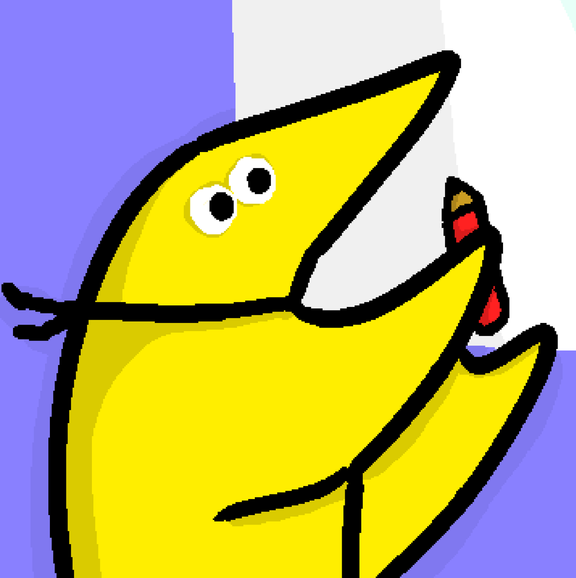

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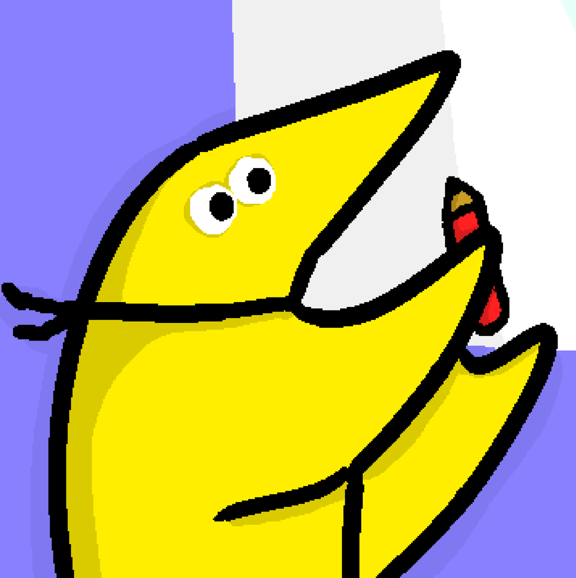

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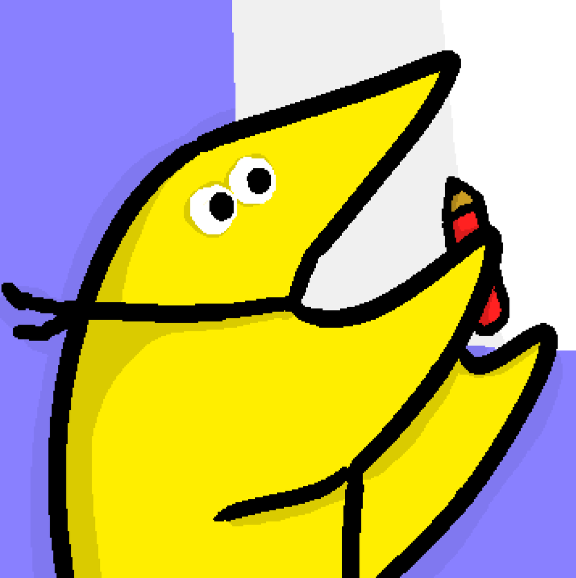
$$H_k\big(T_t(K), T_t(\mathrm{cost}(K,\sigma))\big) \cong \bigoplus_{\substack{W\subseteq V\setminus(\sigma\cup\upsilon) \\ |W|=t}} \tilde{H}_{k-d-1}\Big(\bigcup_{\substack{\sigma'\subseteq\sigma, \\ 1\le|\sigma'|\le t}} T_{t-|\sigma'|}\big(\mathrm{lk}(K,\sigma\setminus\sigma')[\upsilon\cup W]\big)\Big)$$



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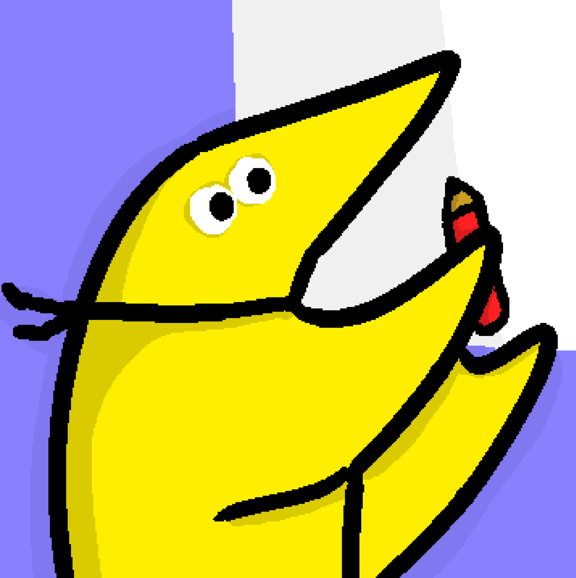
d-collapsible

(Khmelnitsky '18)



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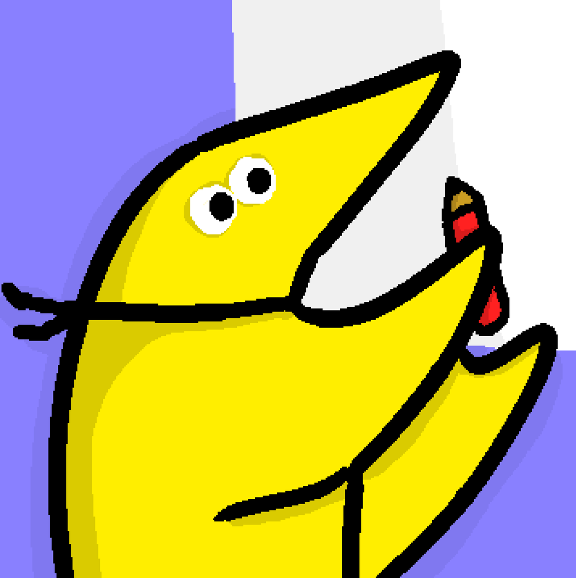
$h(t-|\sigma'|, d)$-Leray

(by induction on t)



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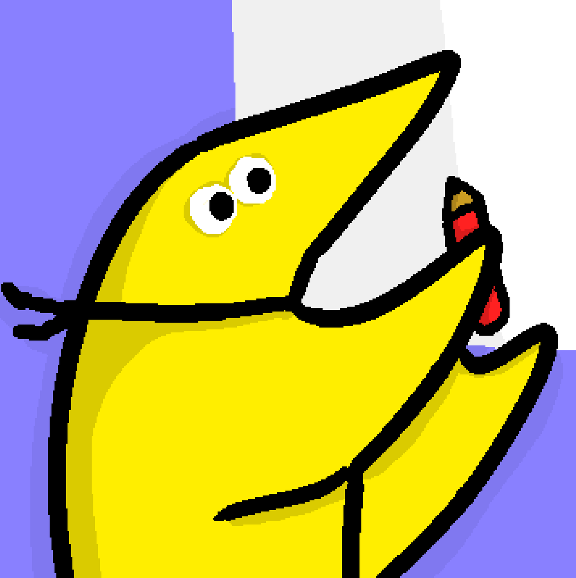
$$\Big[\Big(\sum_{\substack{\sigma'\subseteq\sigma\\ 1\le|\sigma'|\le t}} h(t-|\sigma'|,d)+1\Big)-1\Big]\text{-Leray}$$

(Kalai-Meshulam '06)



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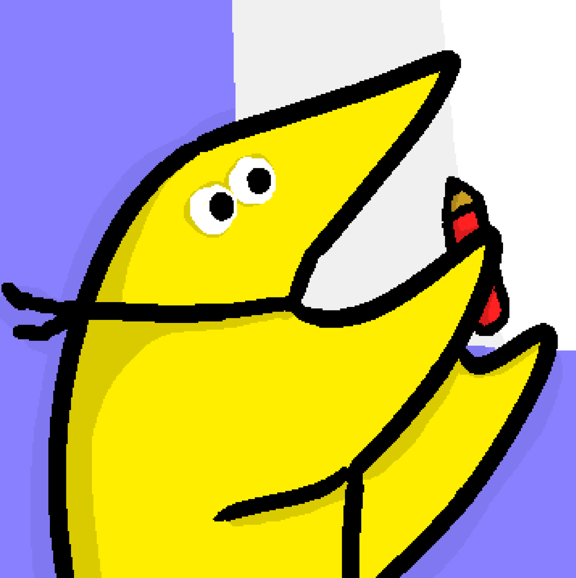

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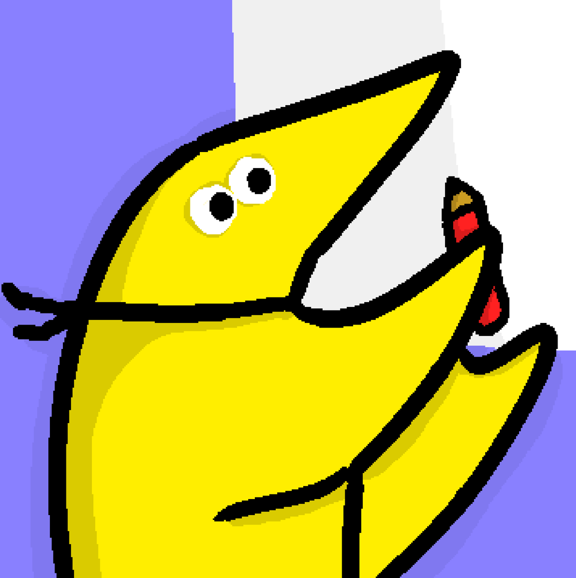
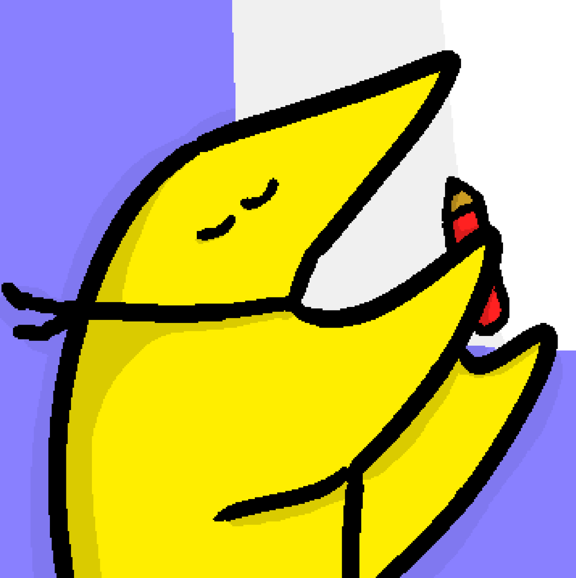


A geometric application

Thm (Kim, L. 21'): Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$ be families of convex sets in the plane.



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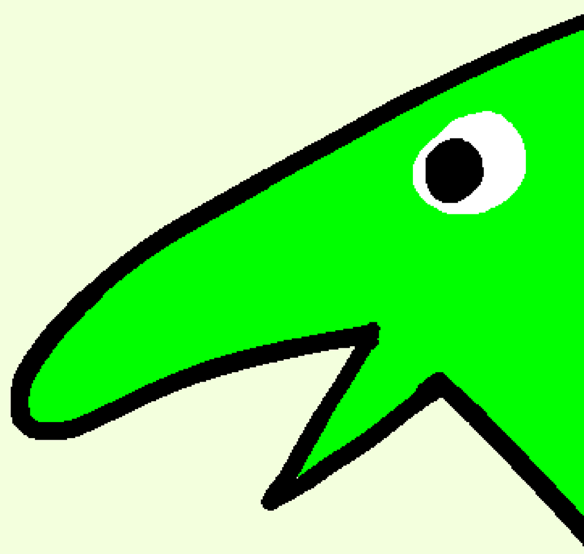
Follows from application of Kalai and Meshulam's topological colorful Helly theorem ('05)



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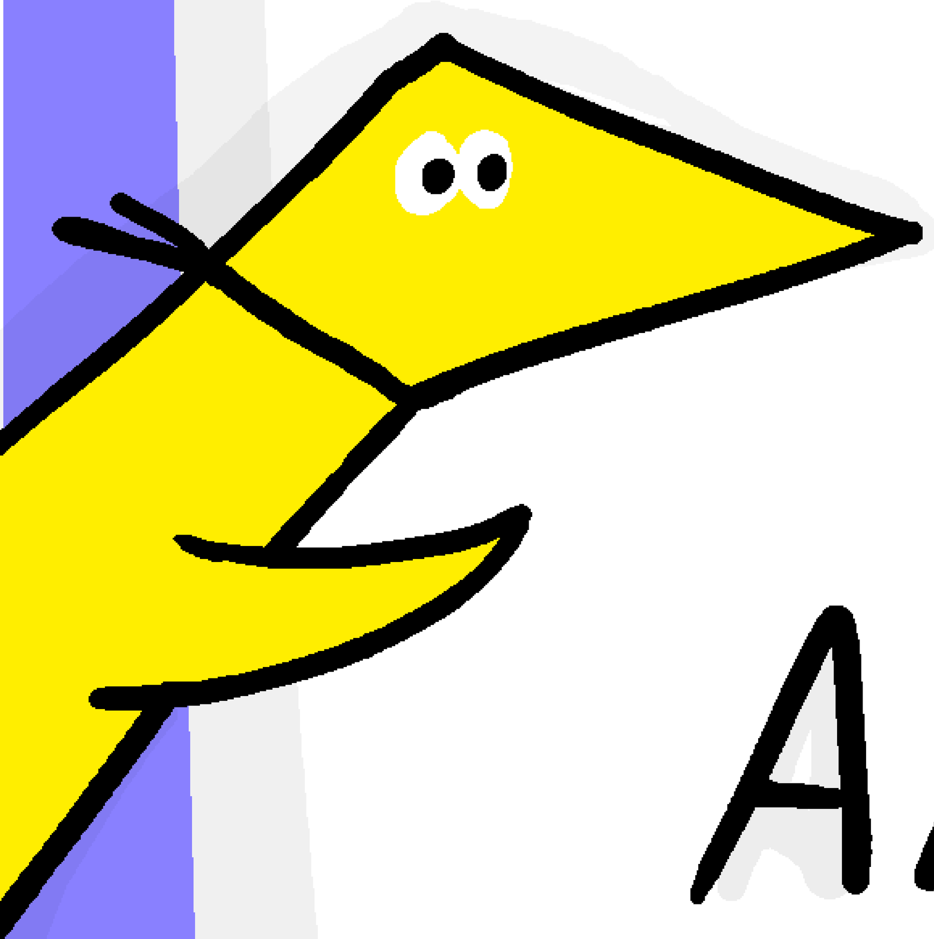
THANKS FOR LISTENING!

Leray numbers of tolerance complexes



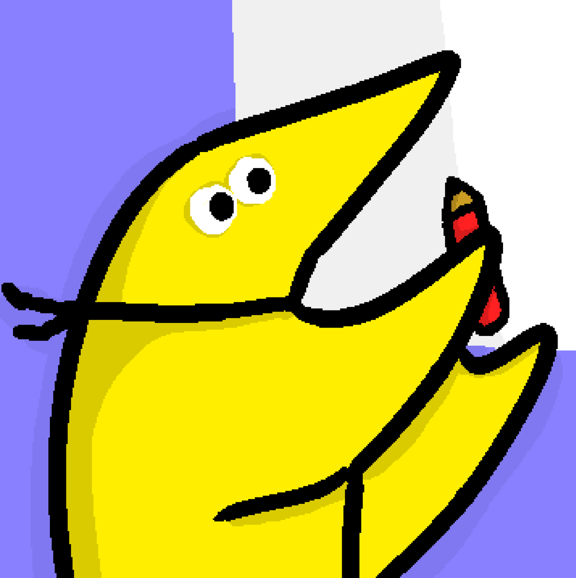
Alan Lew
Hebrew University of Jerusalem

(Joint w/ Minki Kim)

Copenhagen-Jerusalem Combinatorics Seminar

Simplicial complexes

- V = finite set



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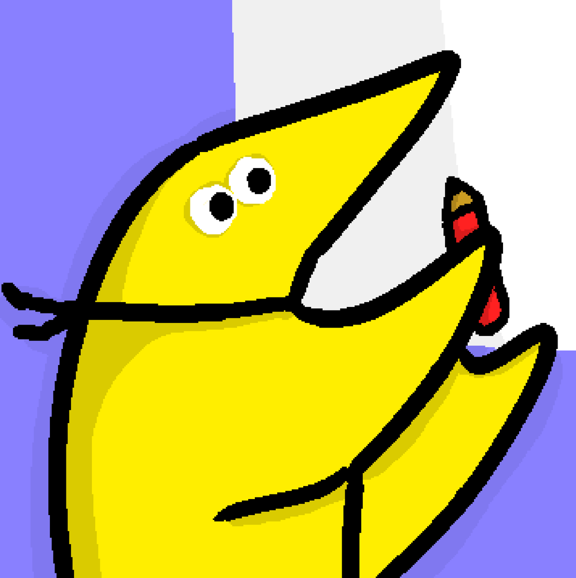

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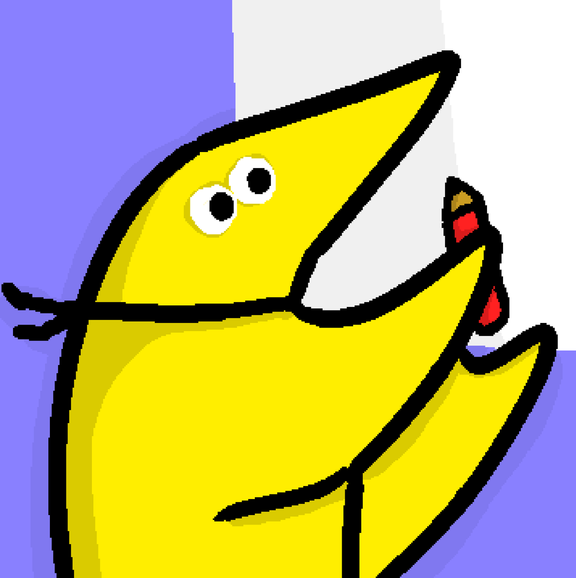

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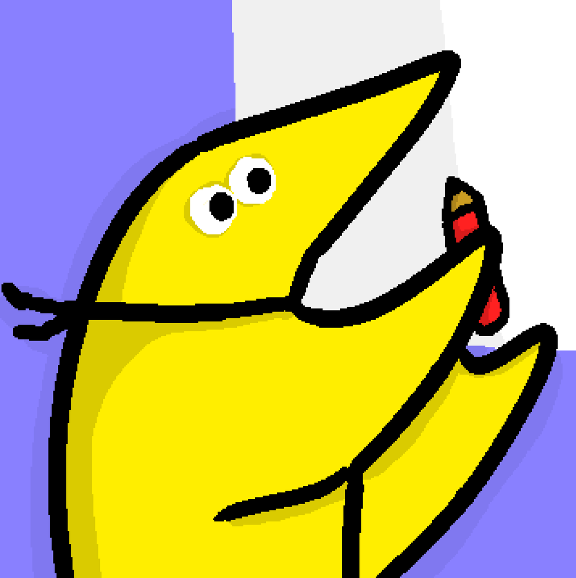

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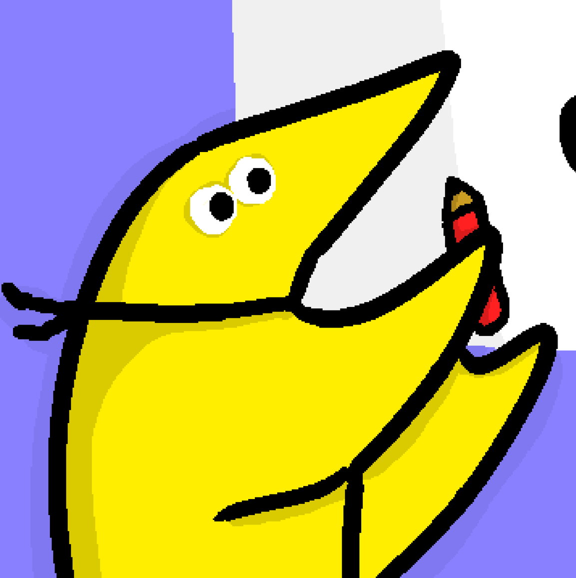
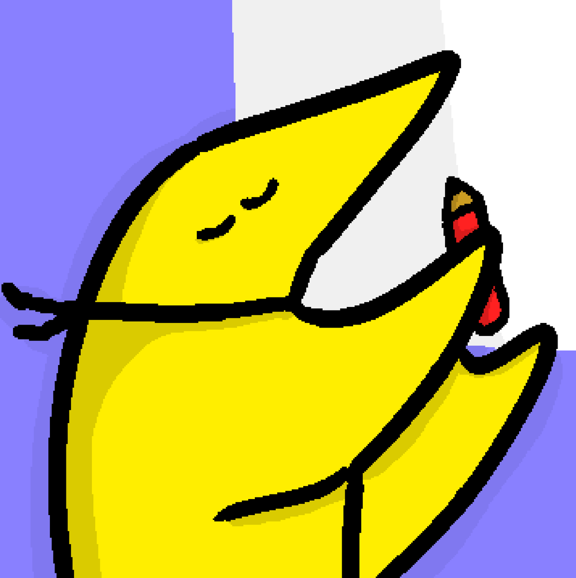


Simplicial complexes

- We can view simp. complexes as geometric objects:



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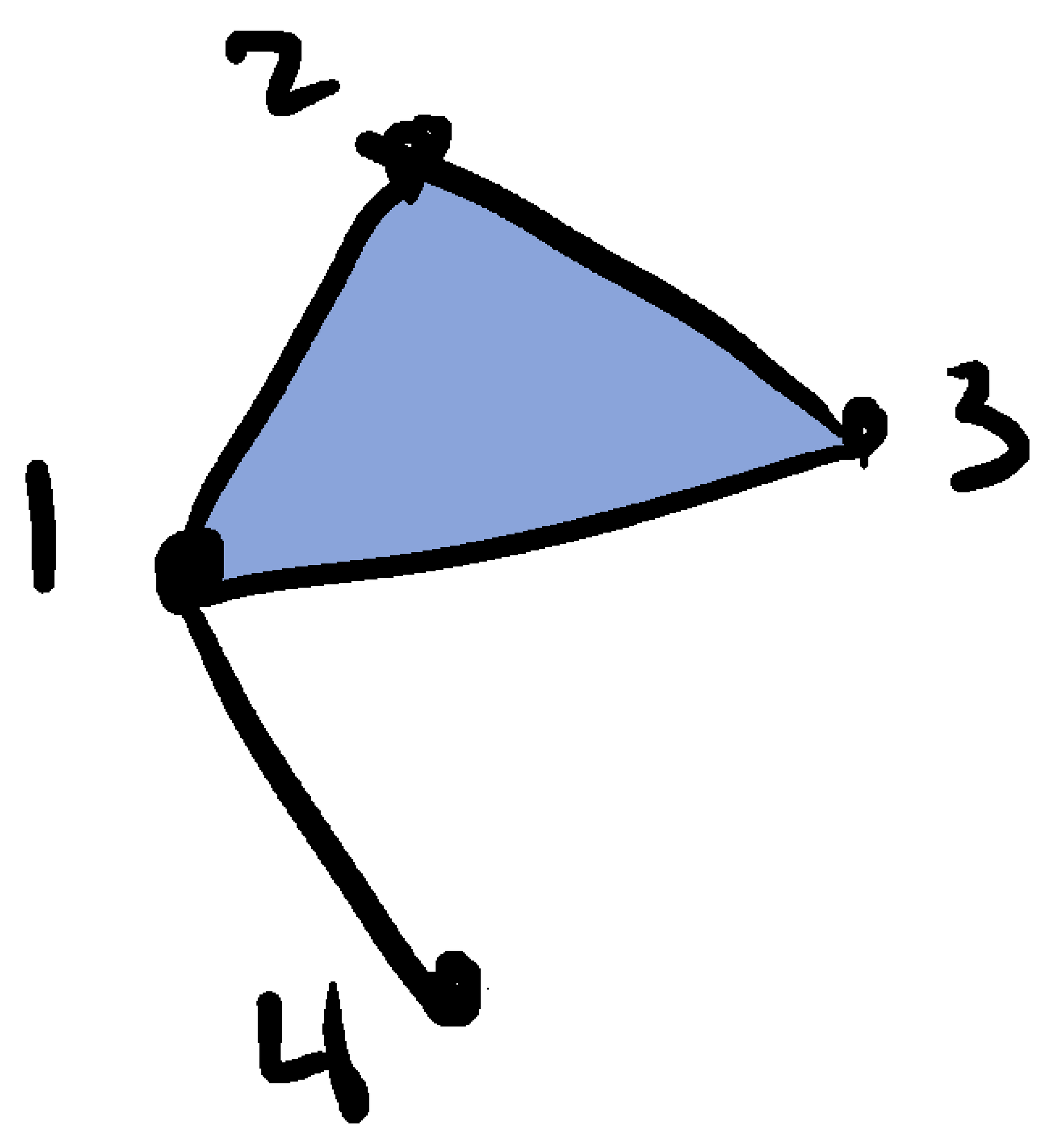

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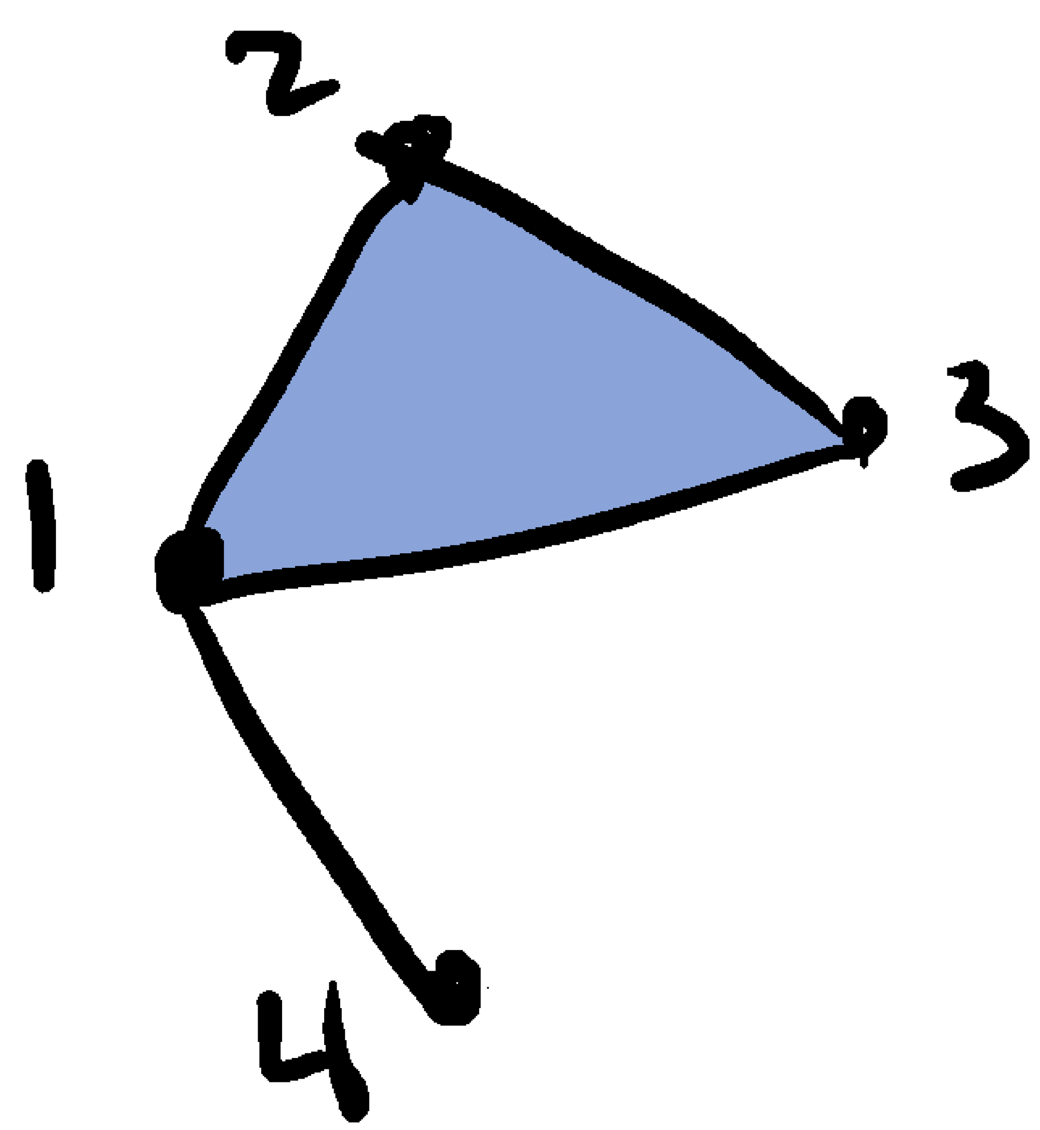
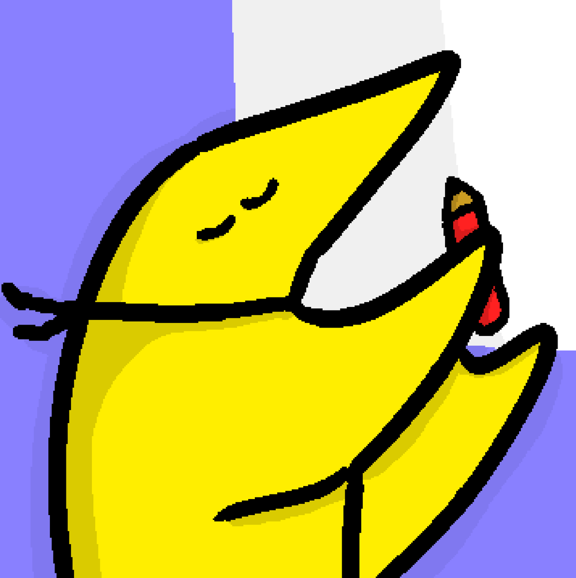
$K =$ 

$$\iff K = \left\{ \{1,2,3\}, \{1,4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset \right\}$$



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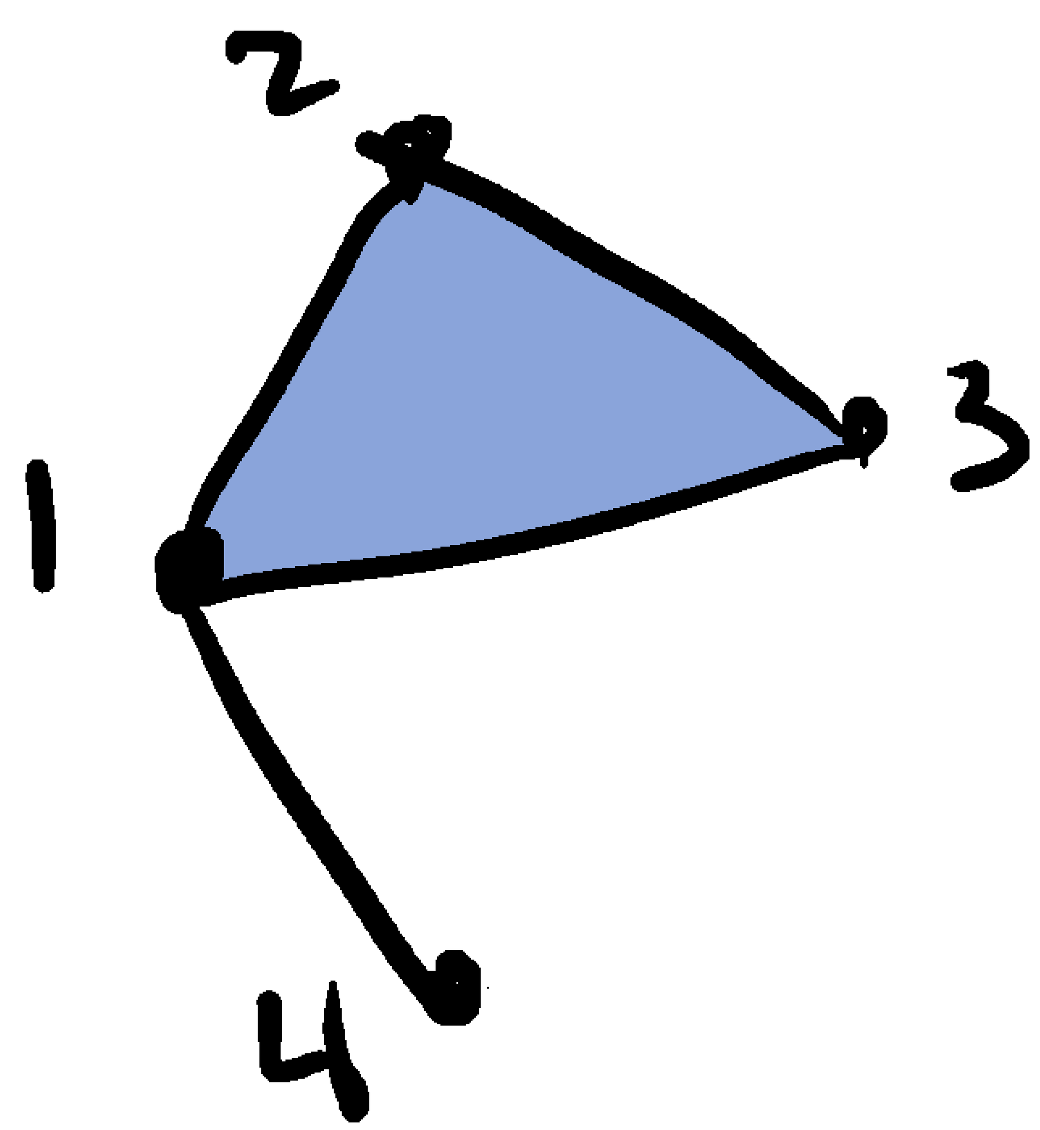
$K =$ 

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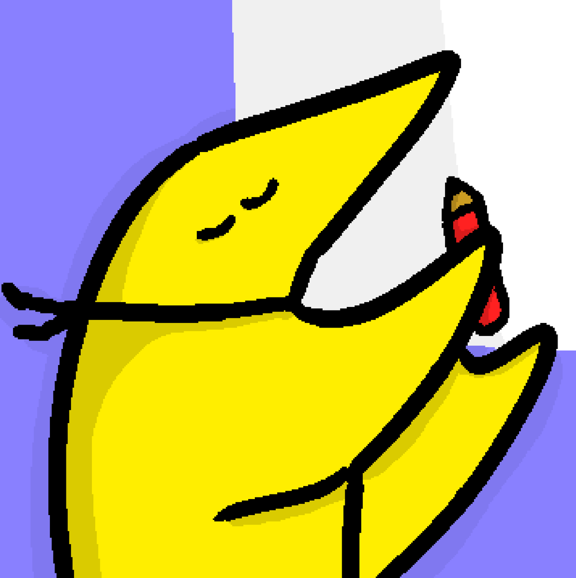
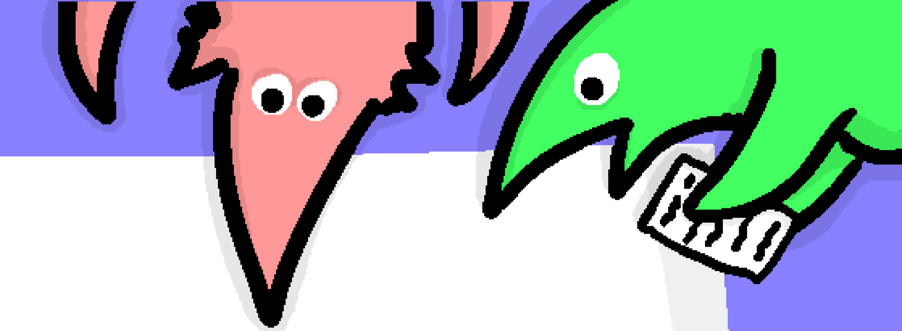
- we can study the topology of a complex.



Homology

- $\tilde{H}_i(K)$ = i-dimensional reduced homology group of K (with real coefficients)



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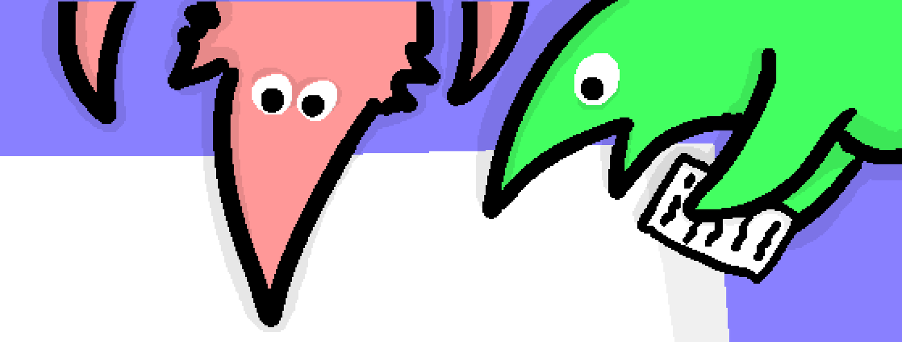

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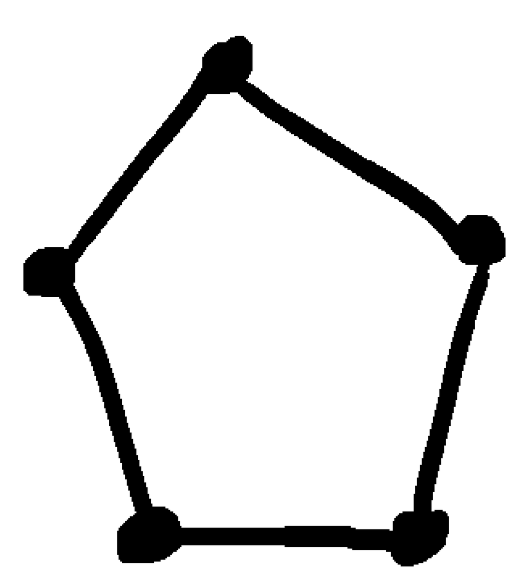
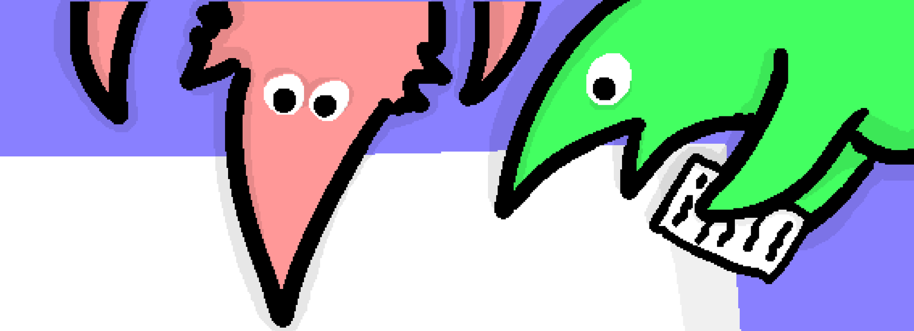
$K =$ 



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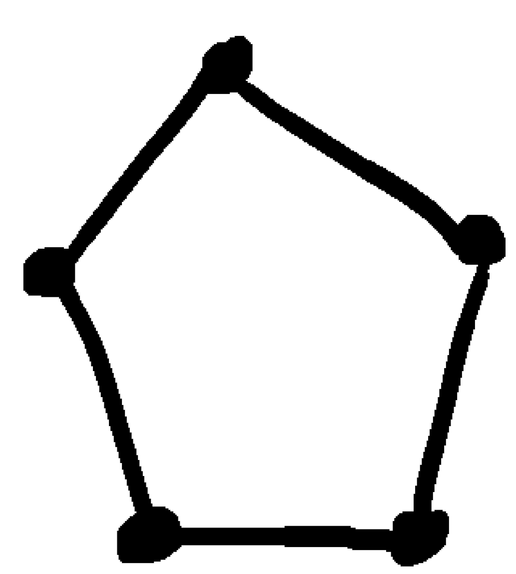
$K =$ 

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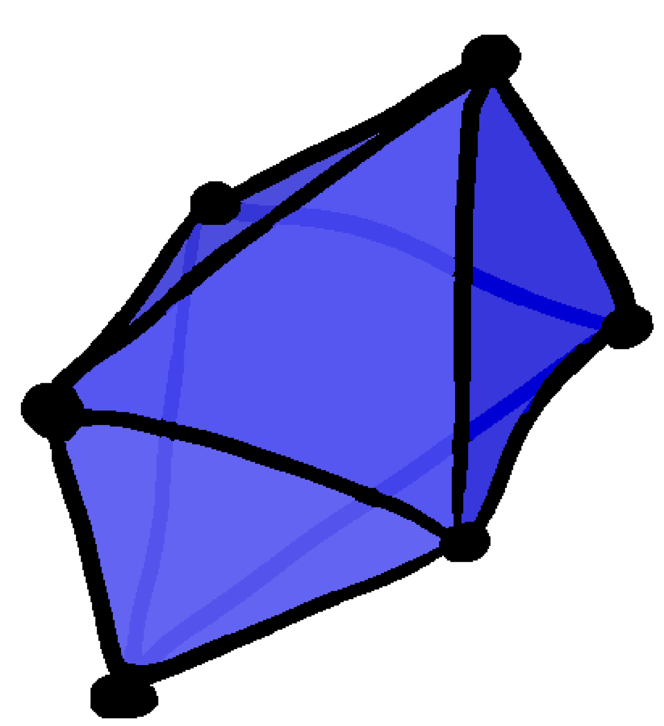
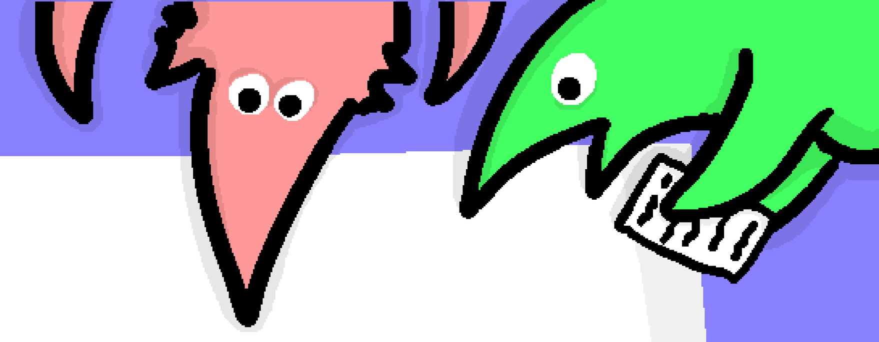
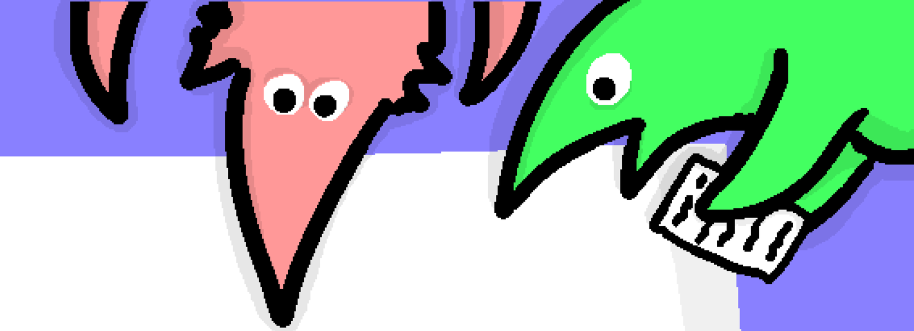
K = 





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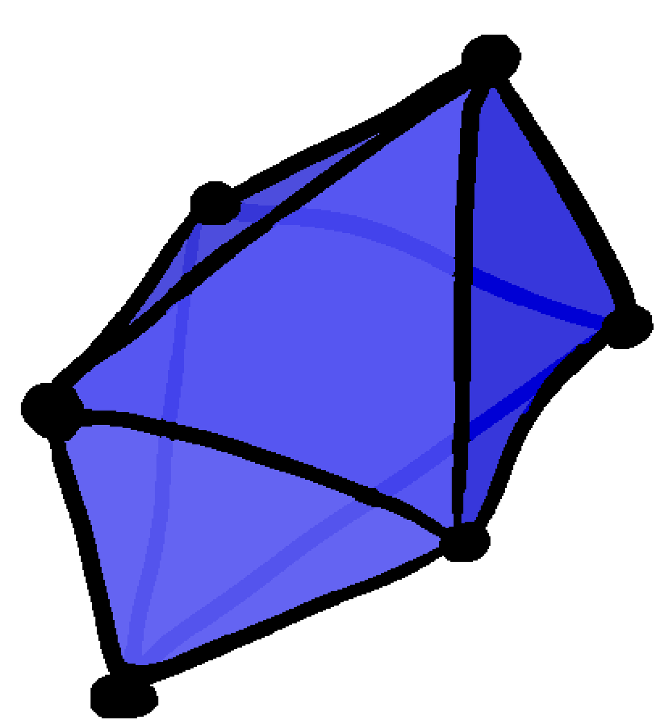
$K =$ 

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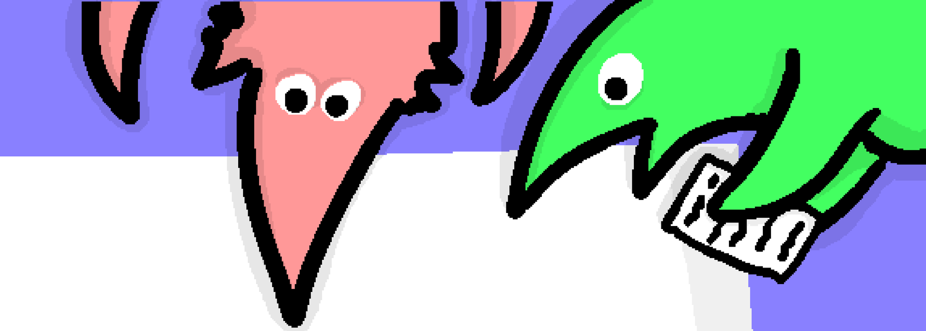
K = d-dimensional sphere



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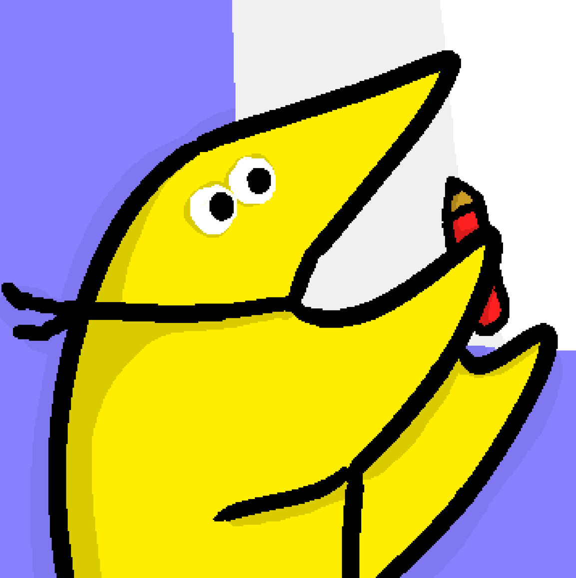

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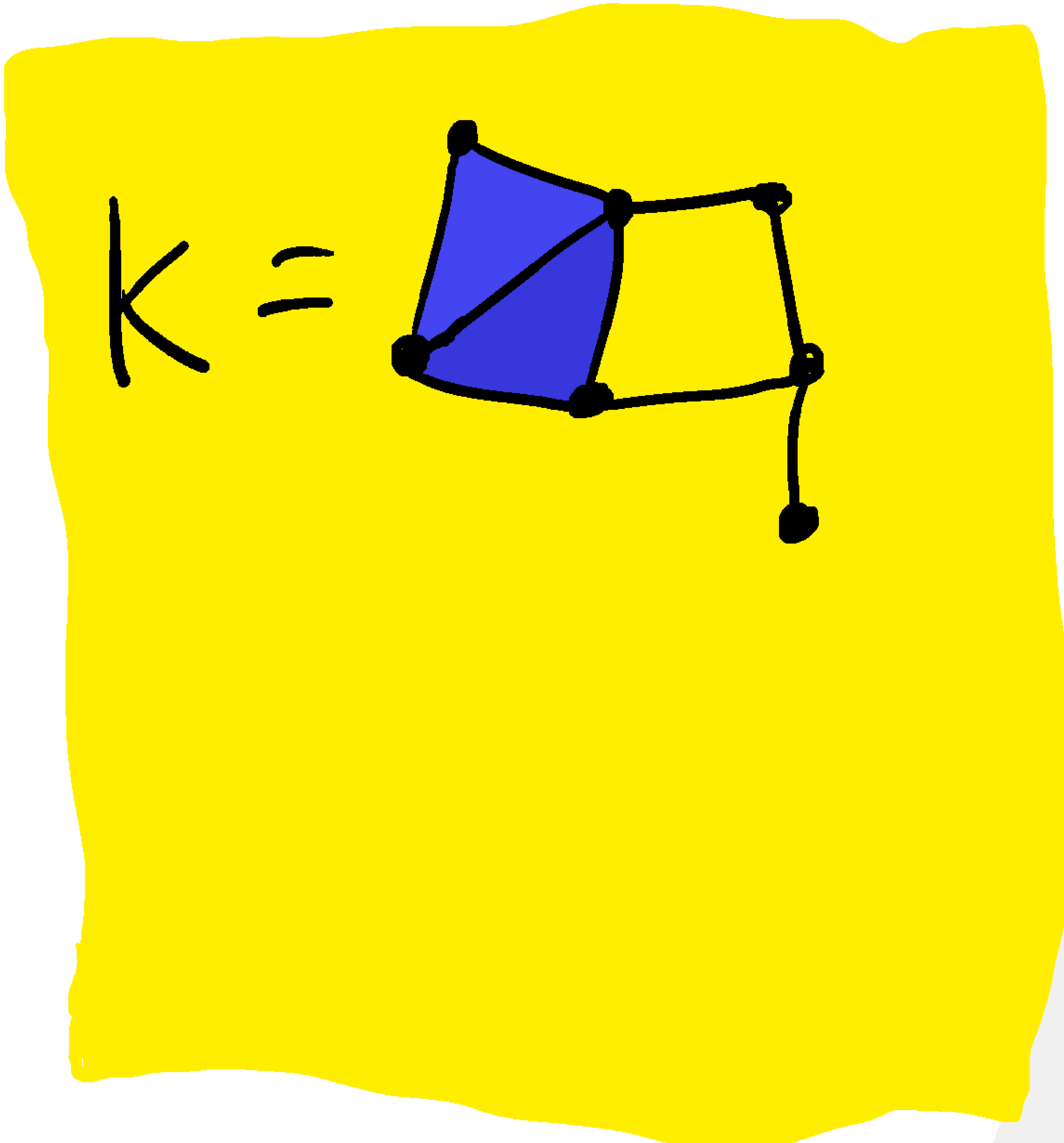

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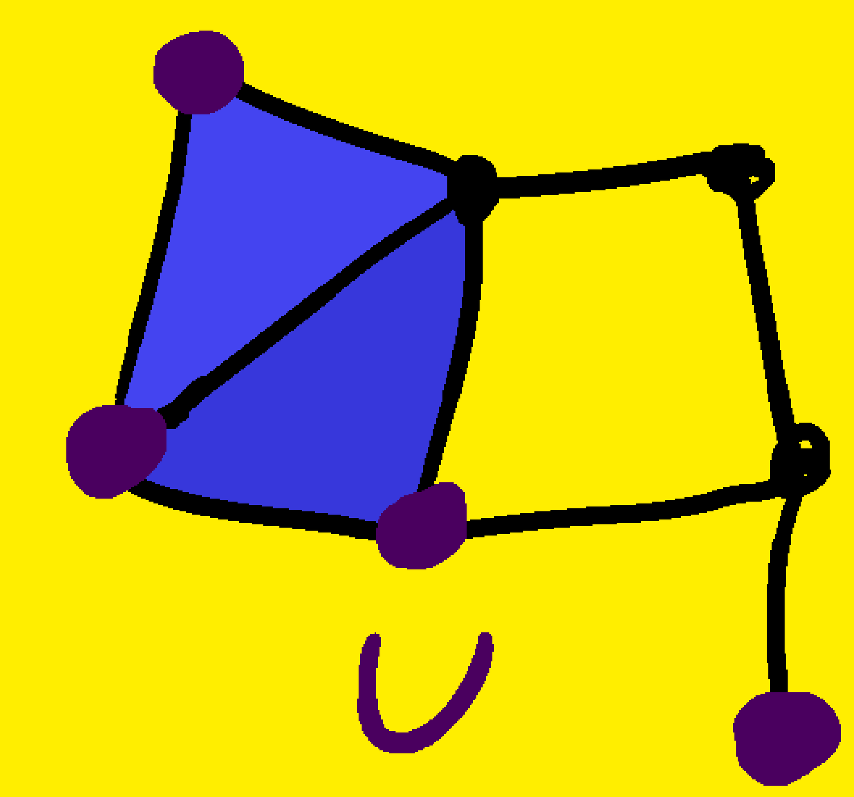
$K =$ 

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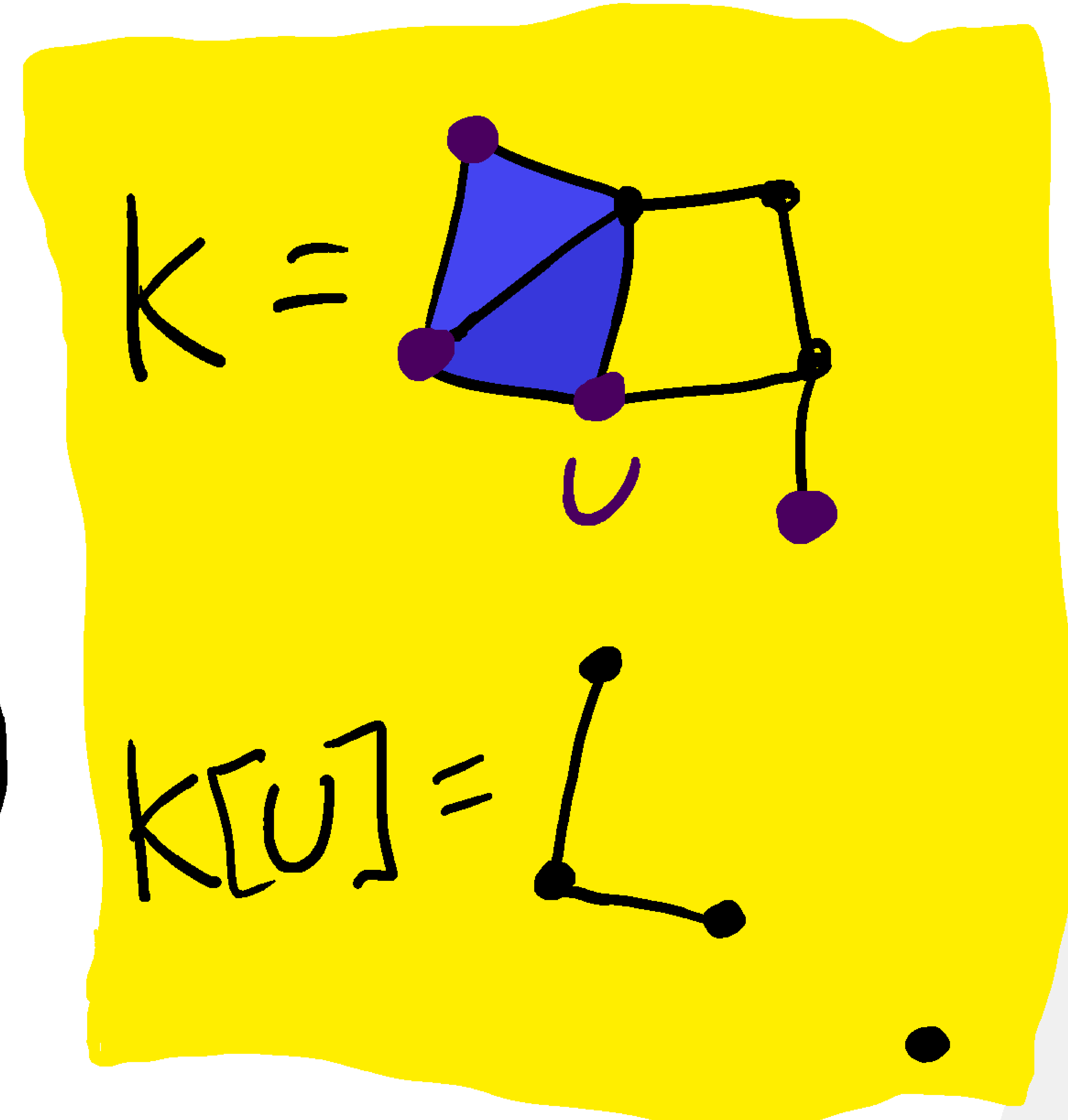

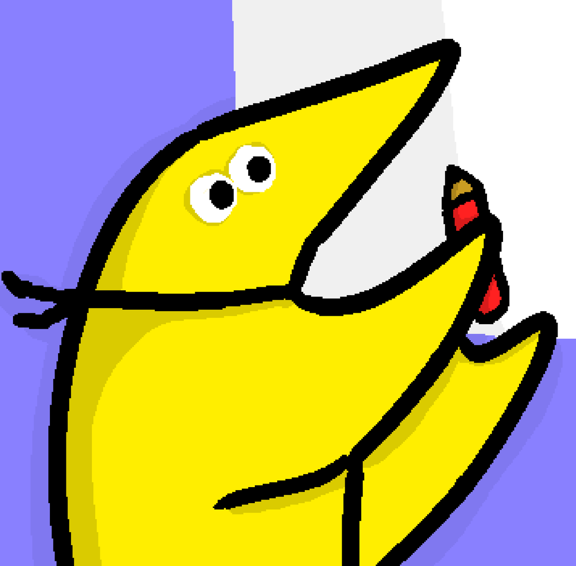

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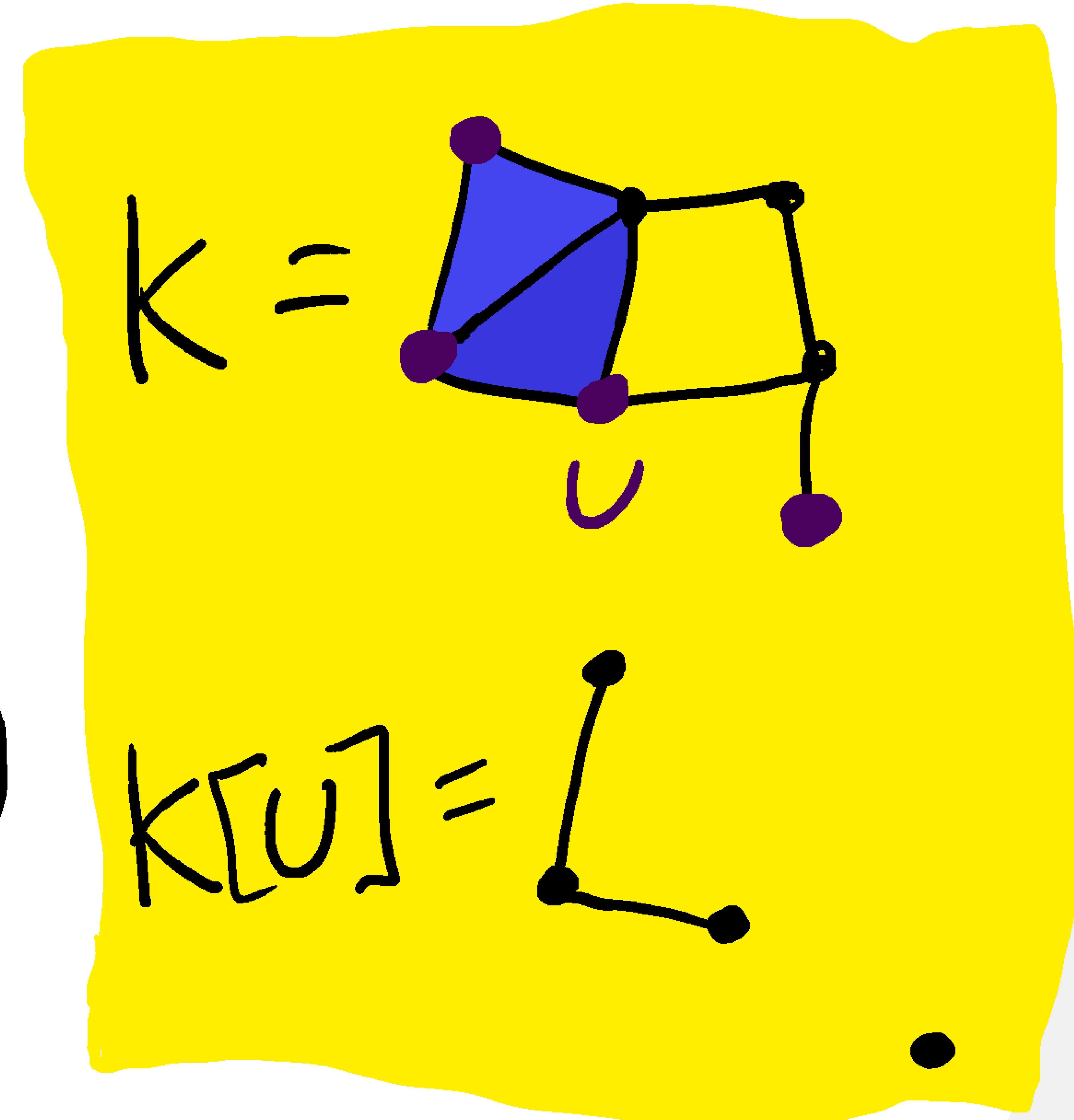

- K is d-Leray if

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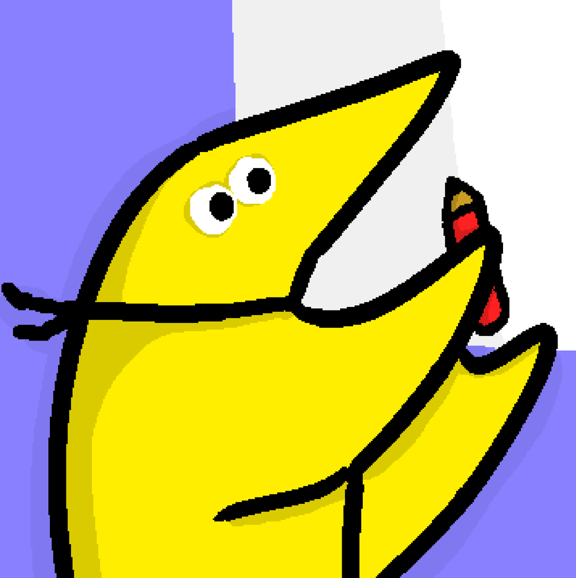

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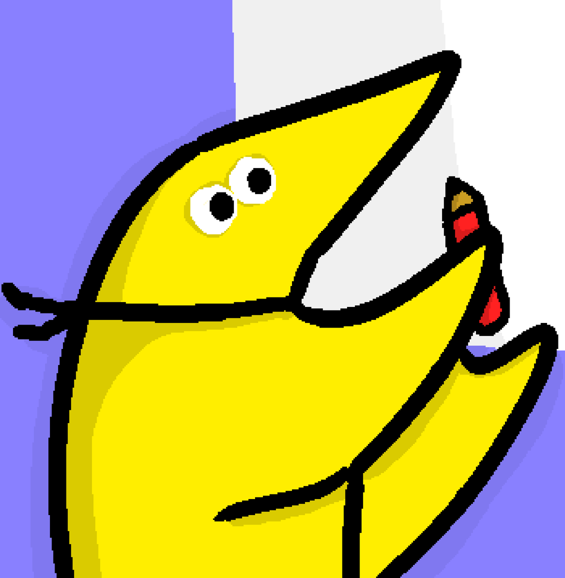
Leray number of K = minimal d s.t. K is d-Leray.



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"homological dimension of K"

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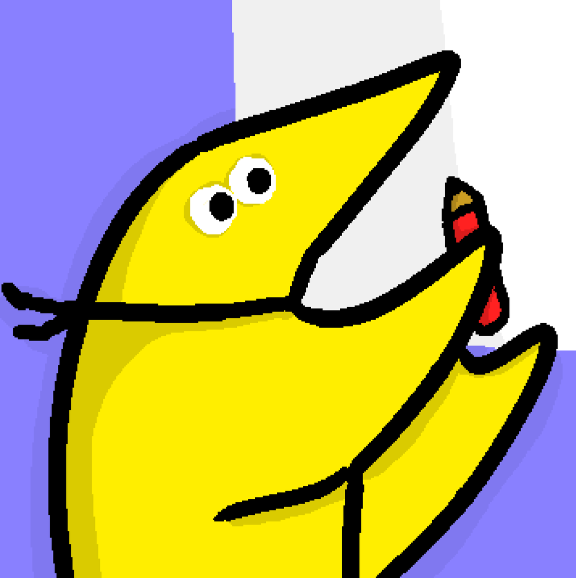

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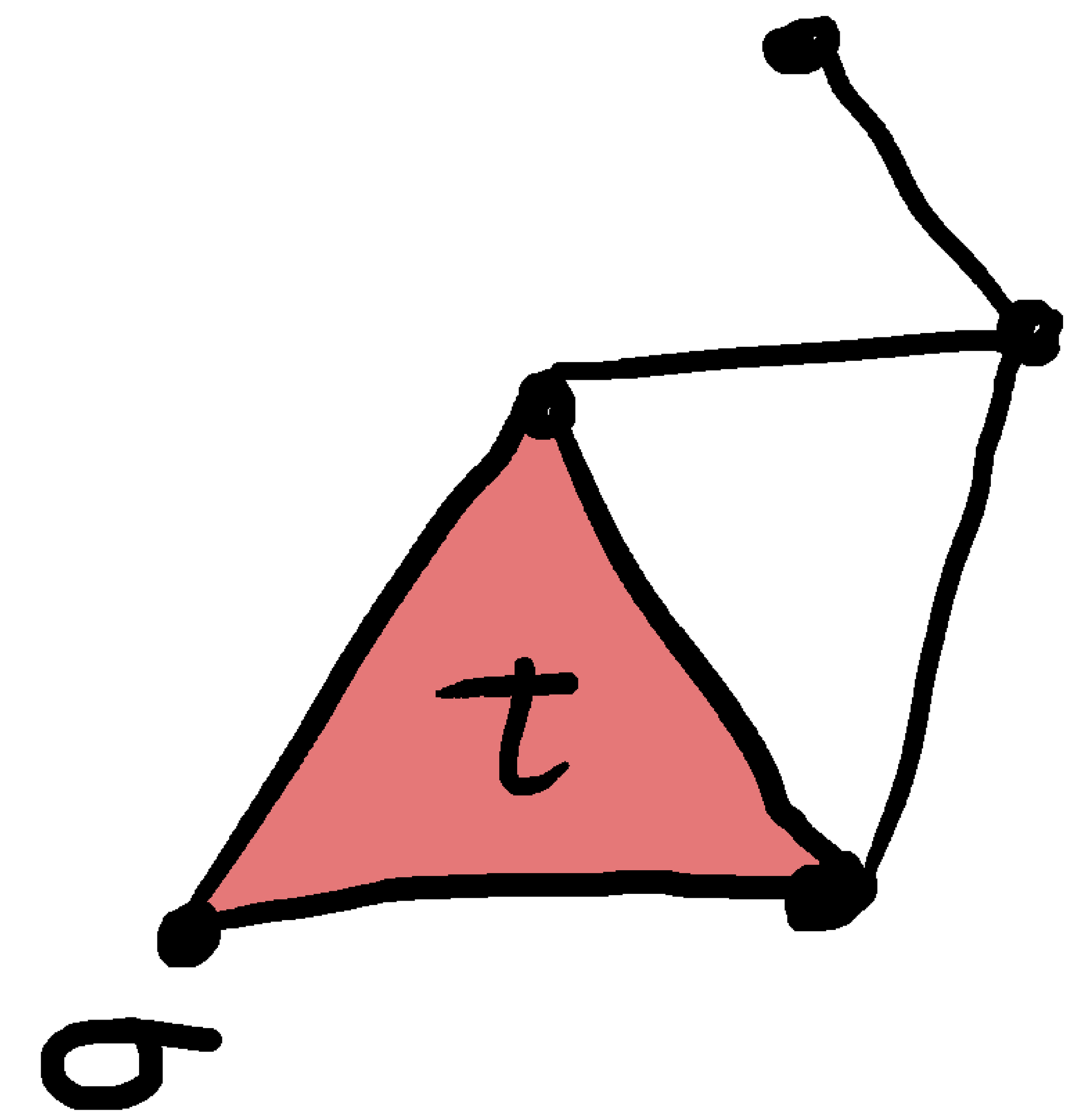
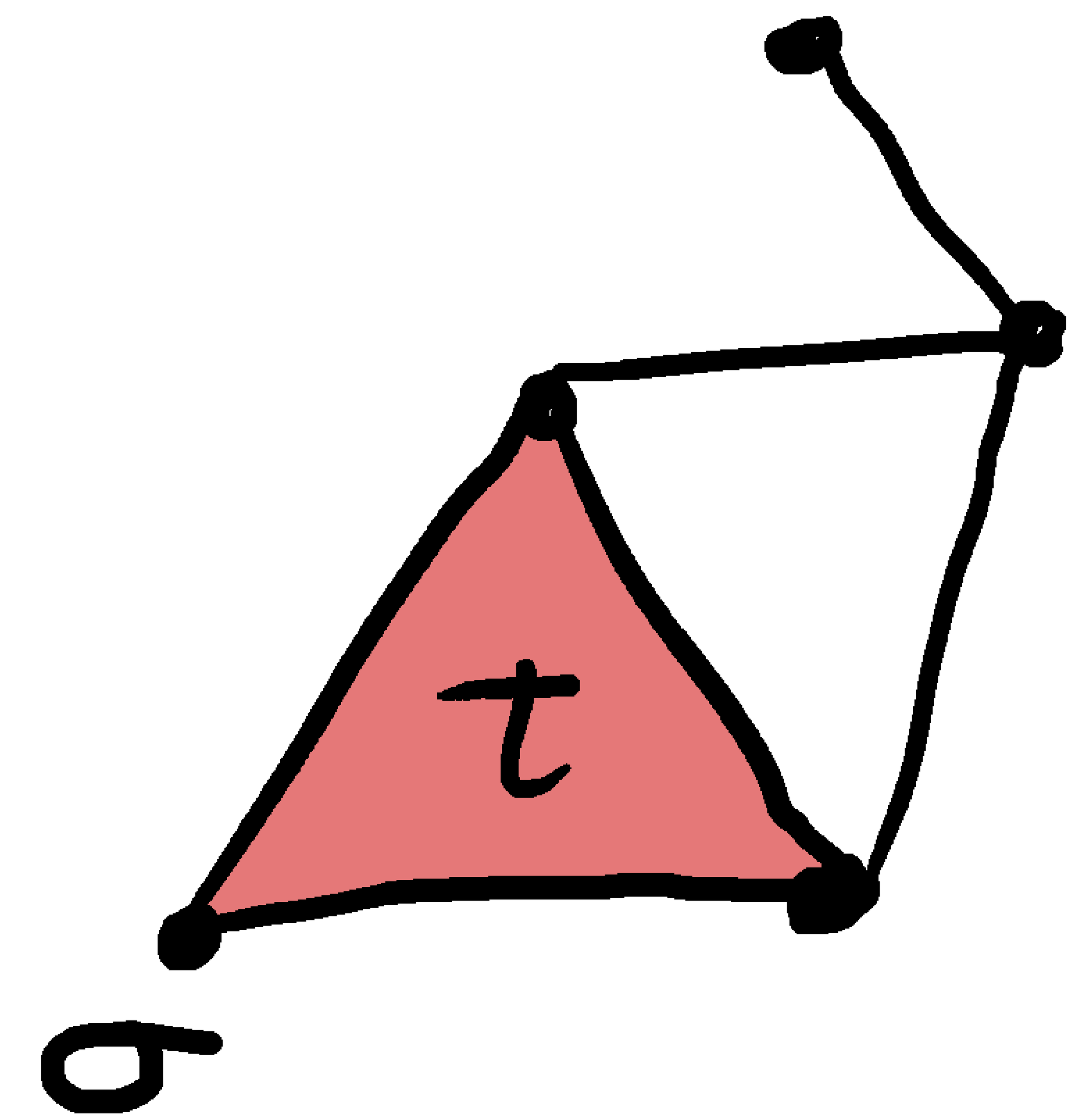


Collapsibility

Let $\sigma \in K$ s.t. $|\sigma| \leq d$

and σ is contained in unique maximal face $\tau \in K$

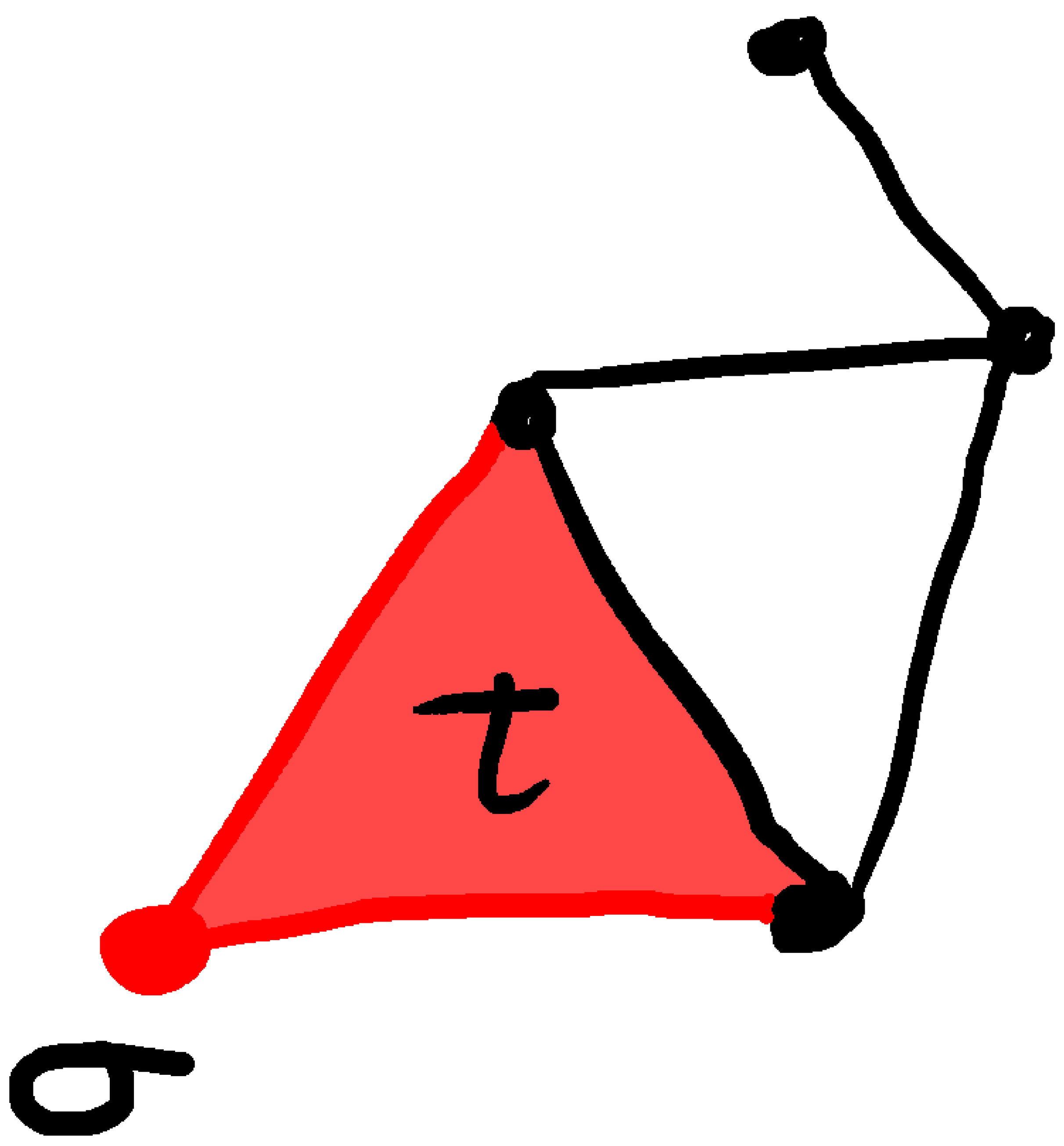


Elementary d-collapse:

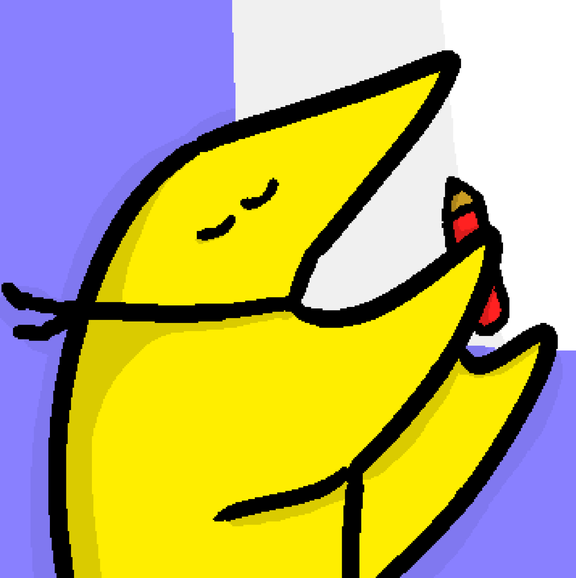
Collapsibility

Let $\sigma \in K$ s.t. $|\sigma| \leq d$

and σ is contained in unique maximal face $t \in K$



Elementary d-collapse:



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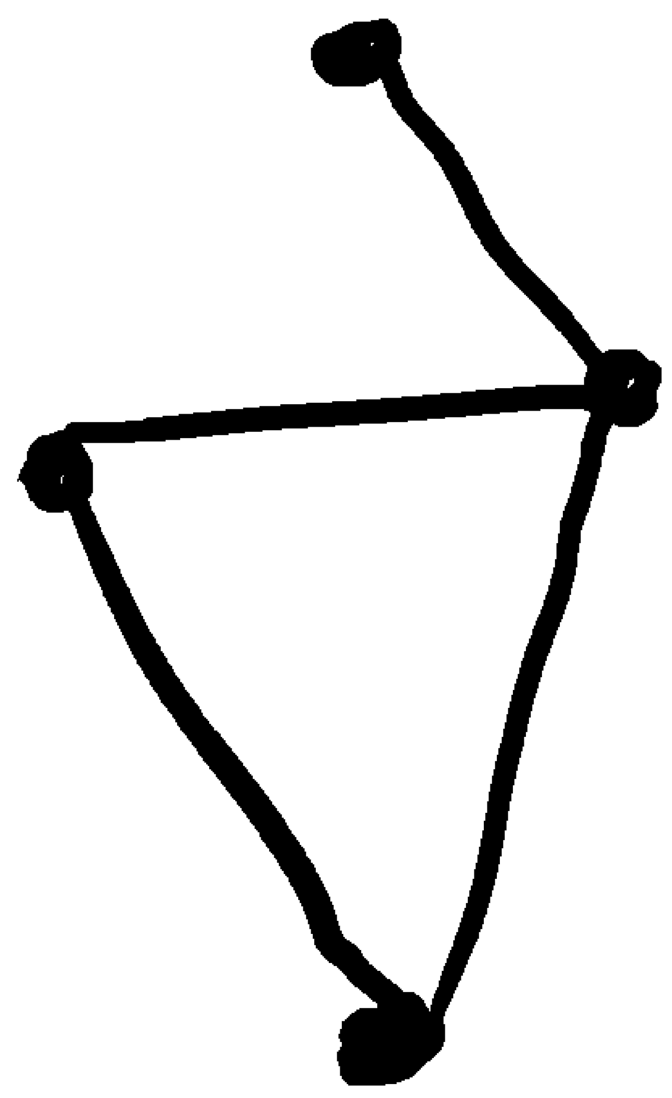

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Collapsibility

Let $\sigma \in K$ s.t. $|\sigma| \le d$

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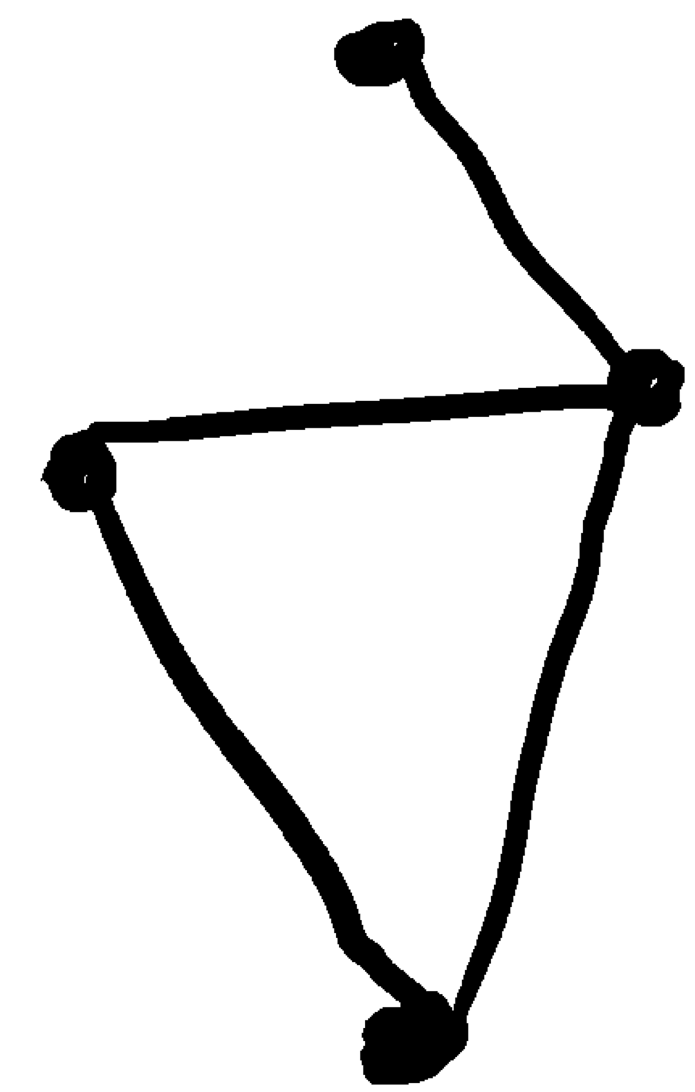
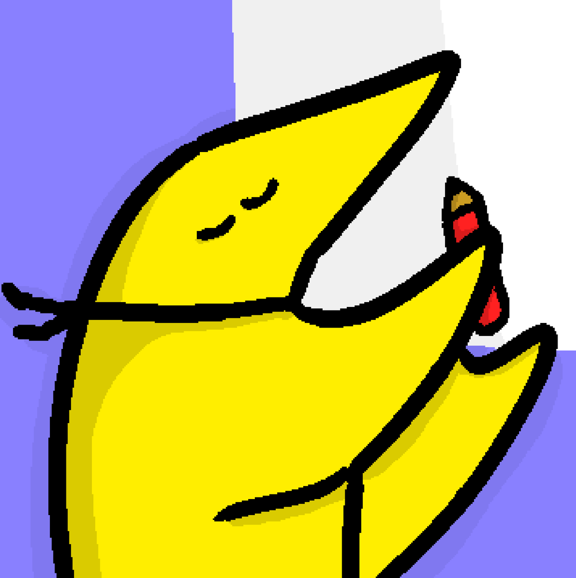


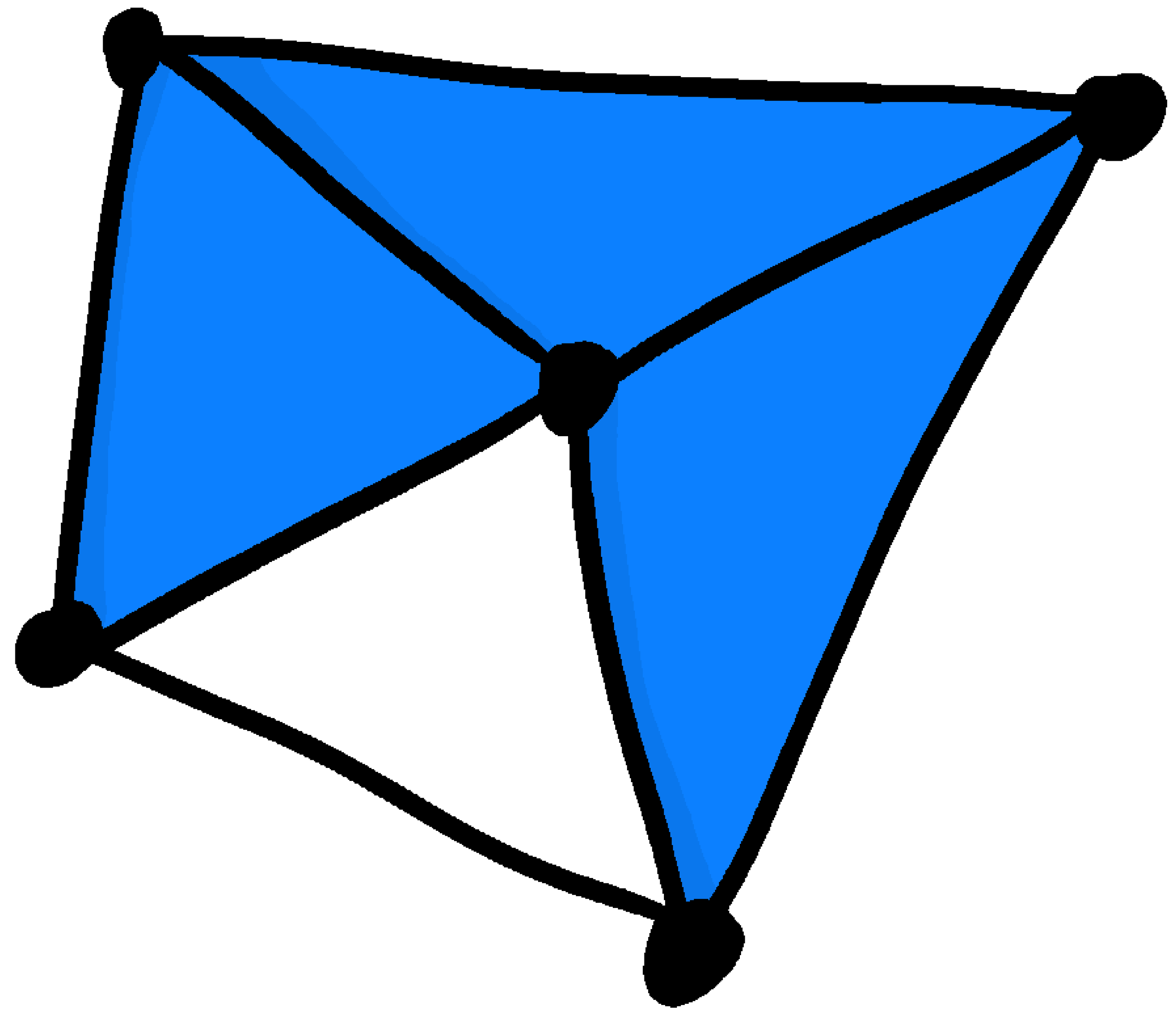
Elementary d-collapse:

- If $\exists$ sequence of elem. d-coll. from K to $\emptyset$: K is **d-collapsible**



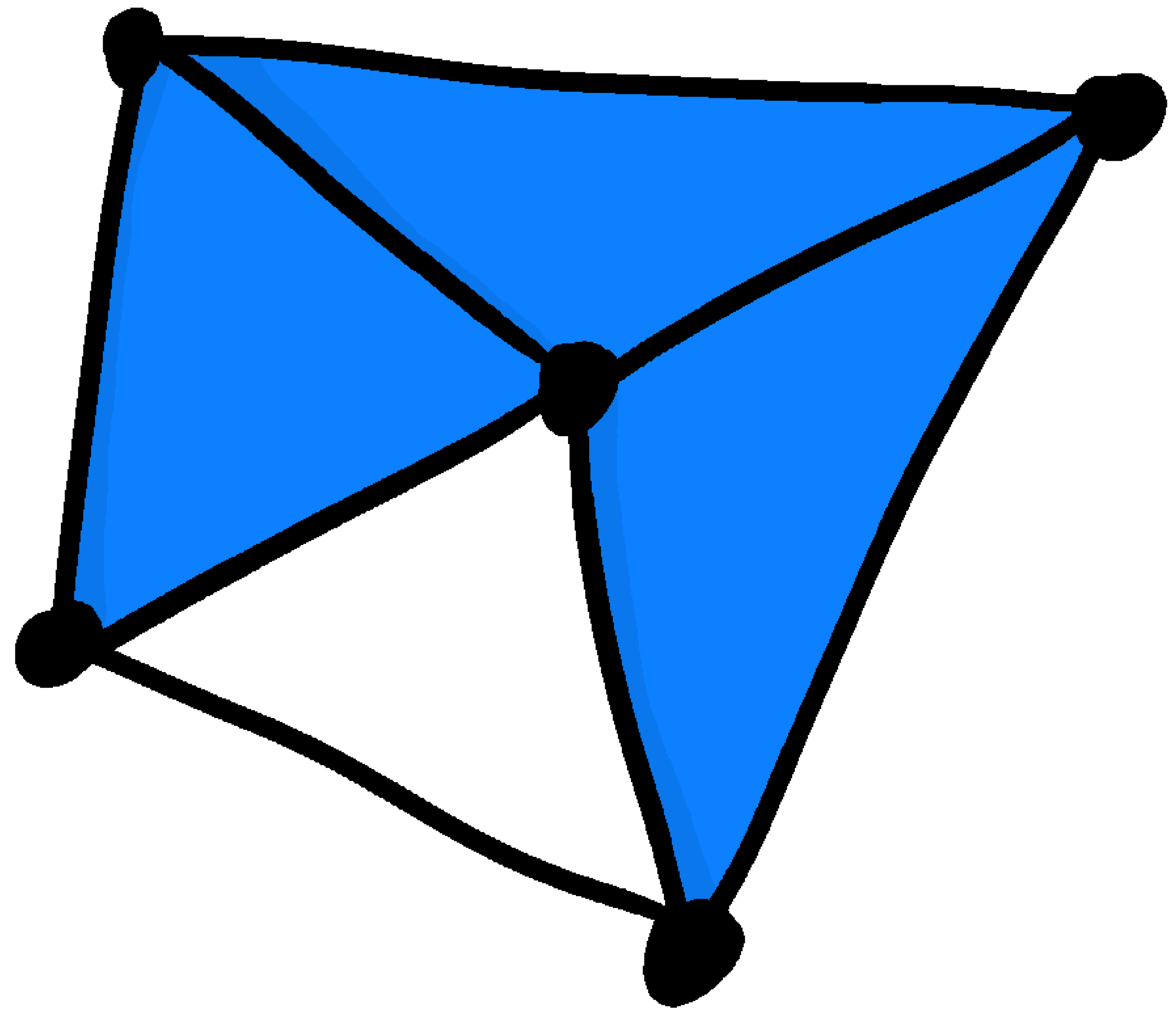
Collapsibility

E.g.

$K =$ 

Collapsibility

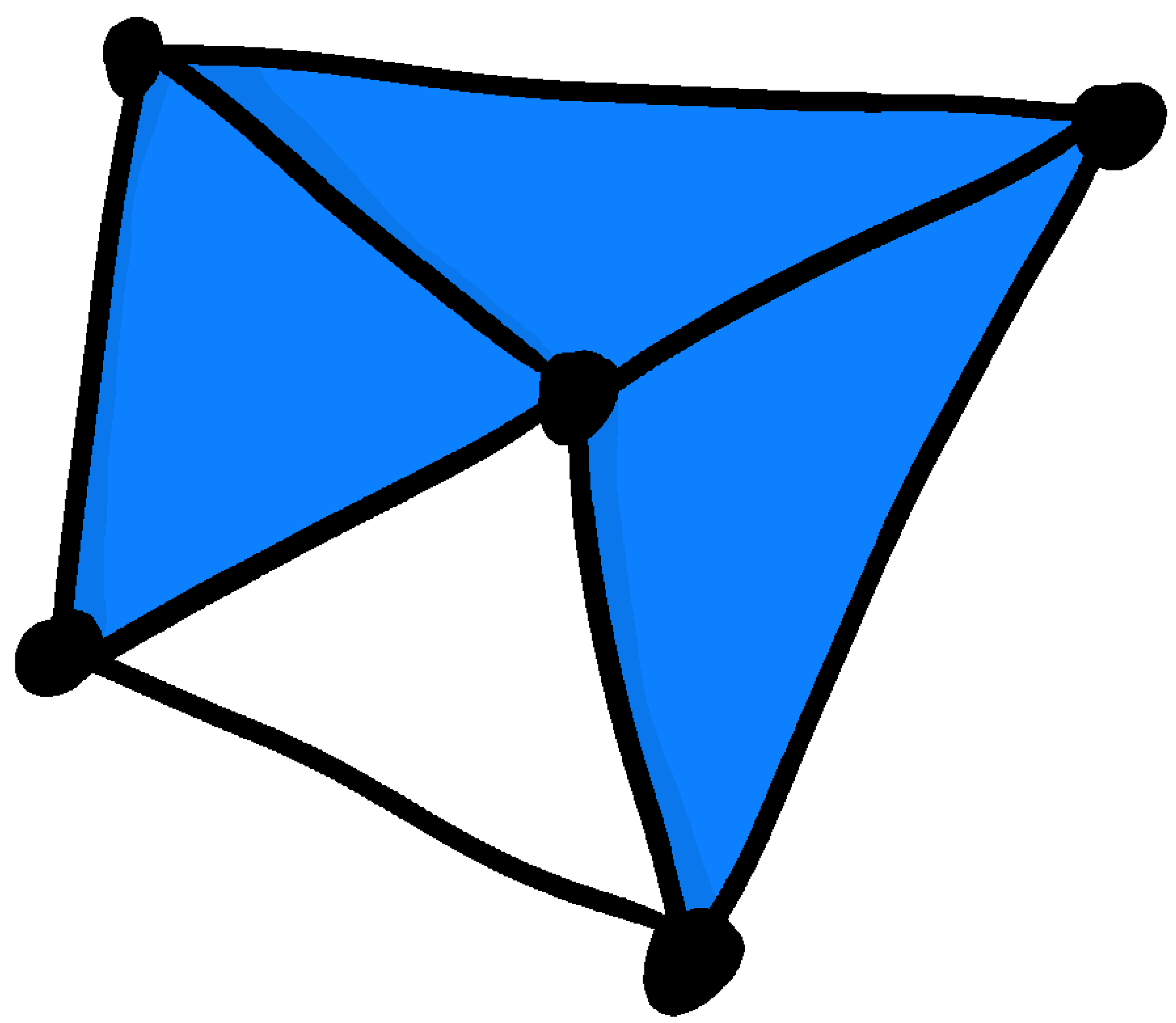
E.g.

$K =$ 

K is not 1-collapsible

Collapsibility

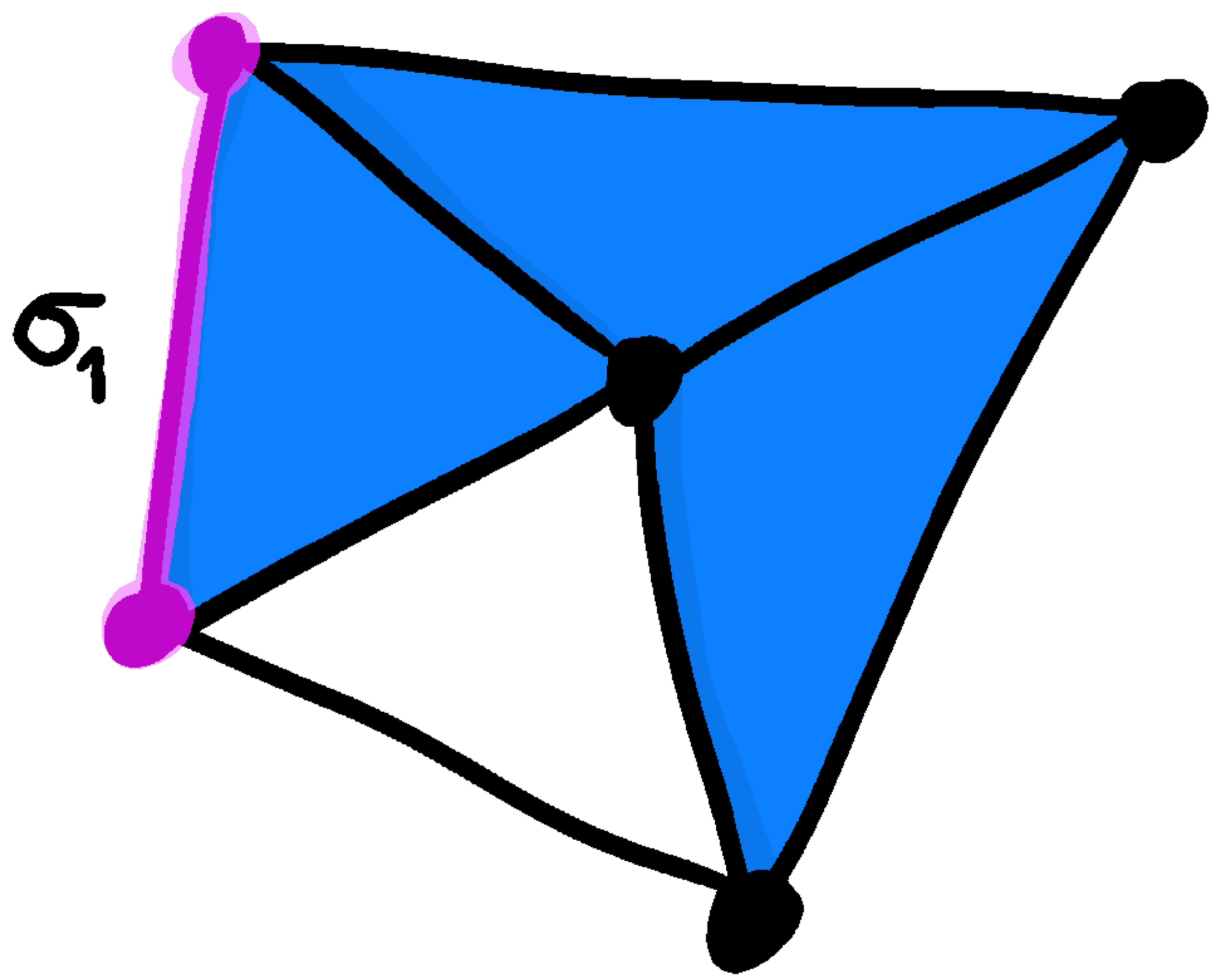
E.g.

$K =$ 

we will show that K is 2-collapsible.

Collapsibility

E.g.

$K =$ 

We will show that K is 2-collapsible.

Collapsibility

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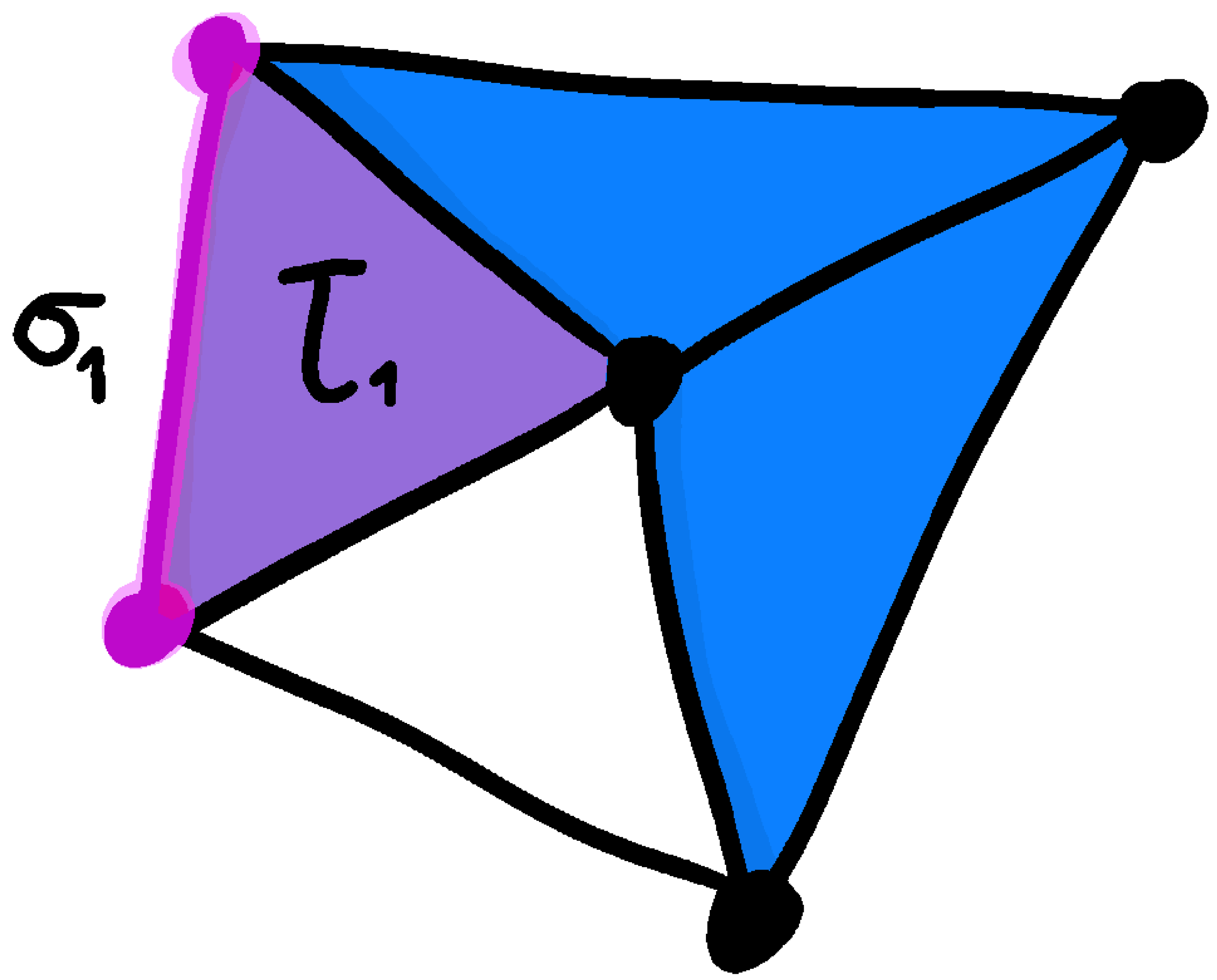
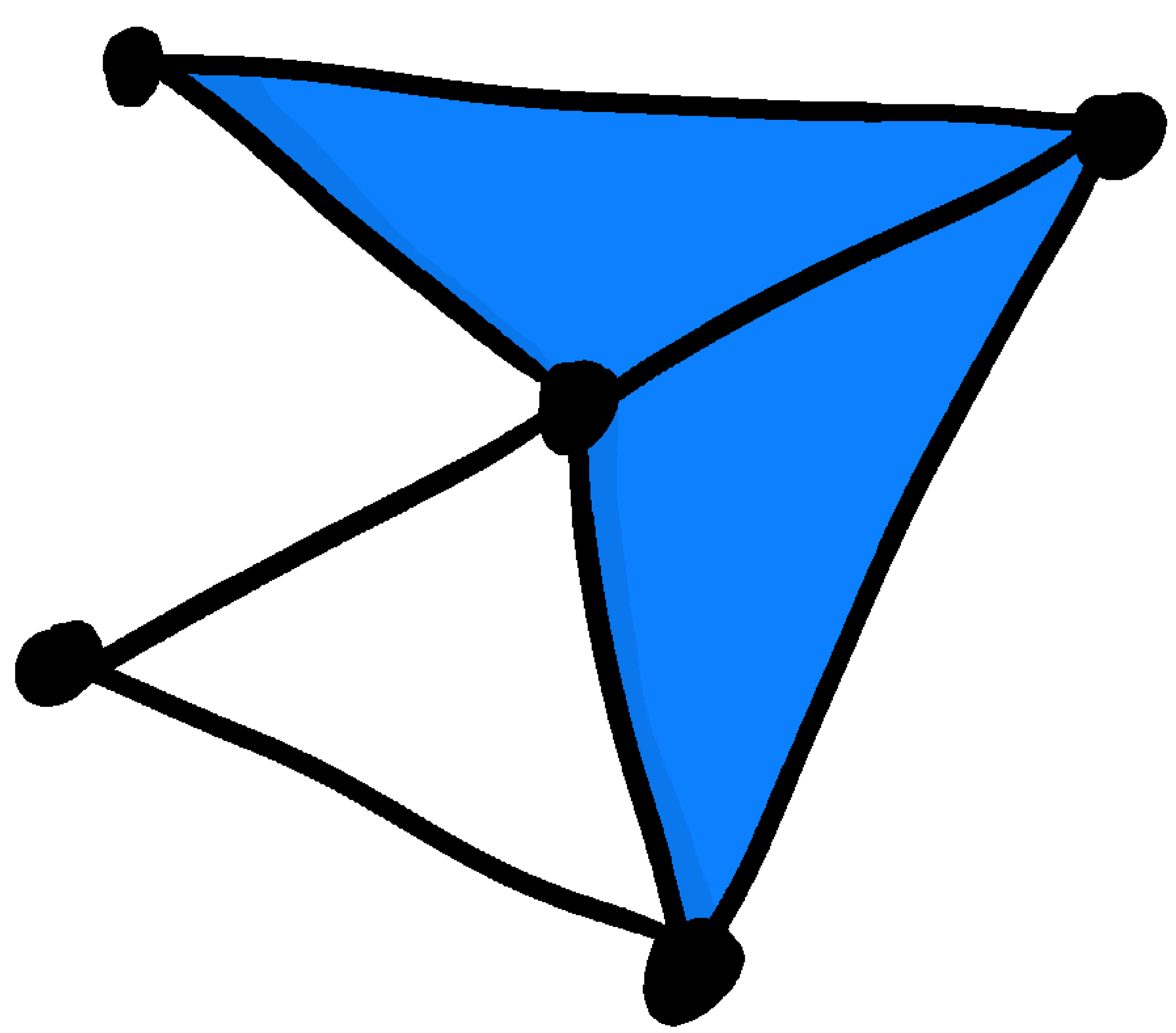
$K =$ 

We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_1 =$ 

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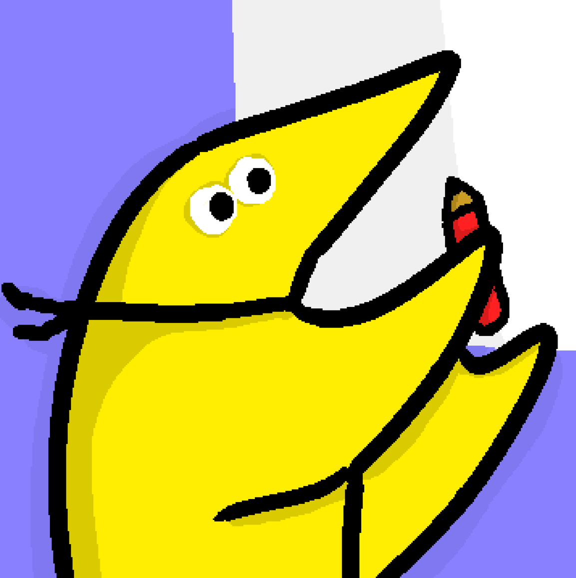

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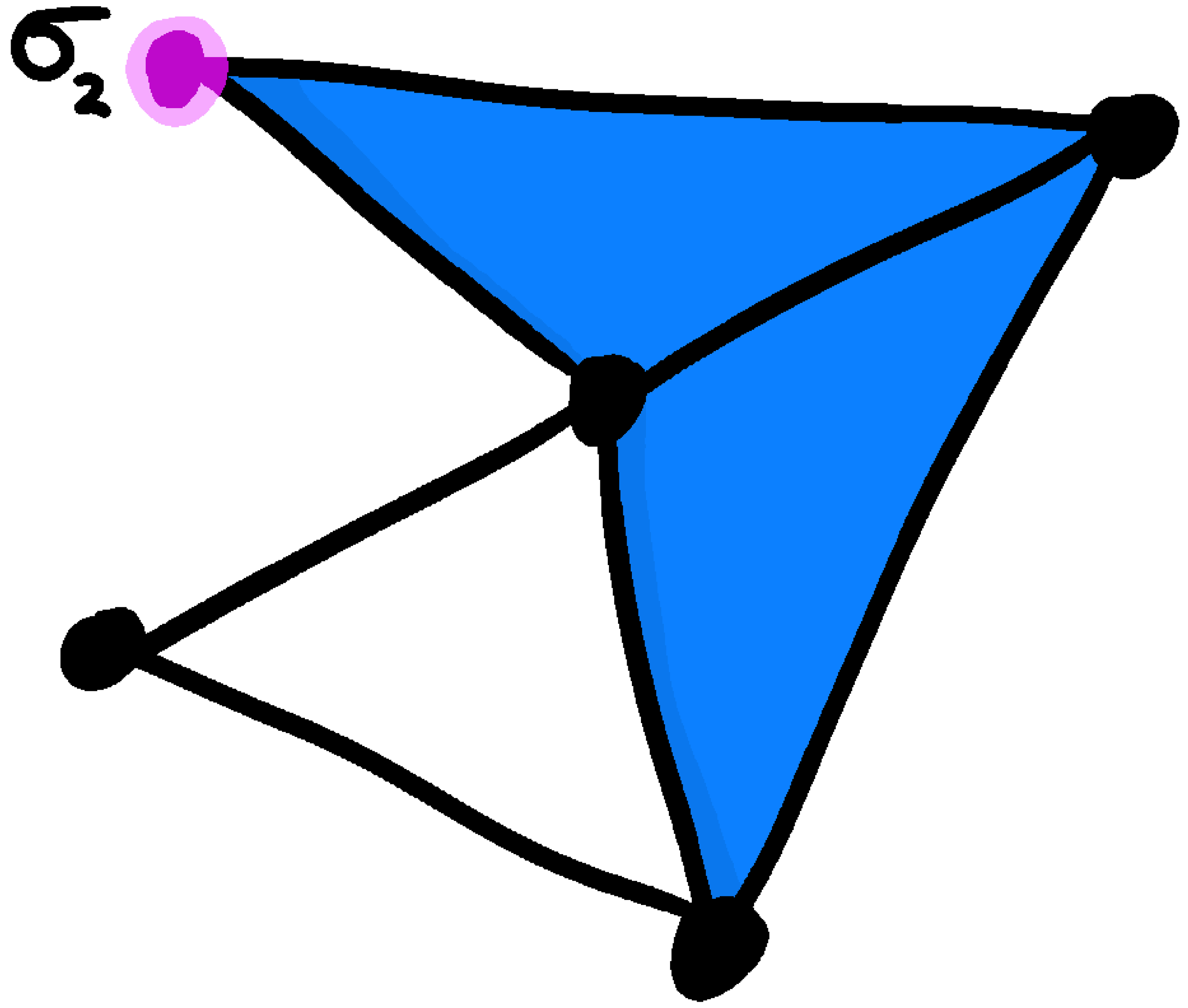
$K_1 =$ 

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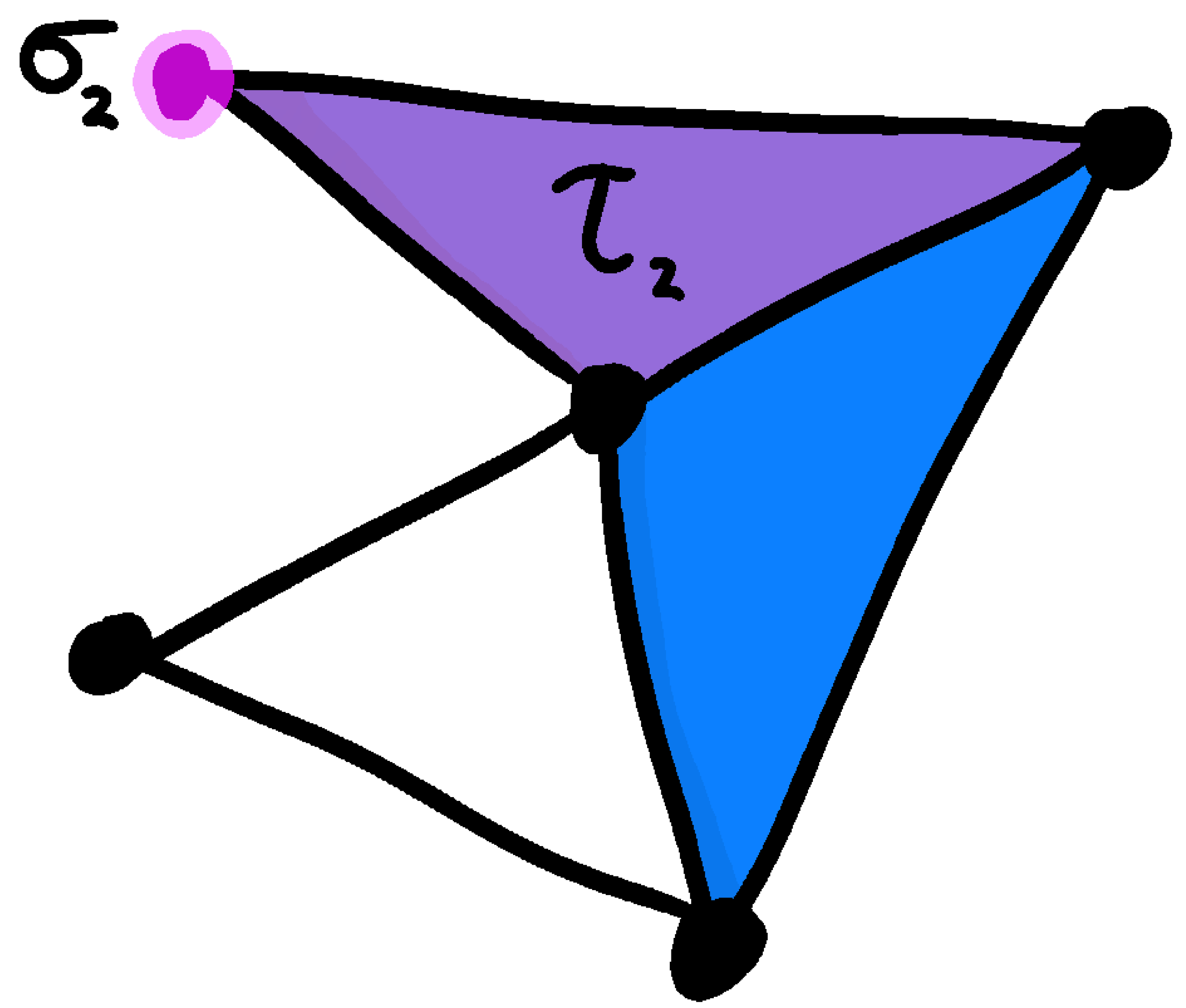
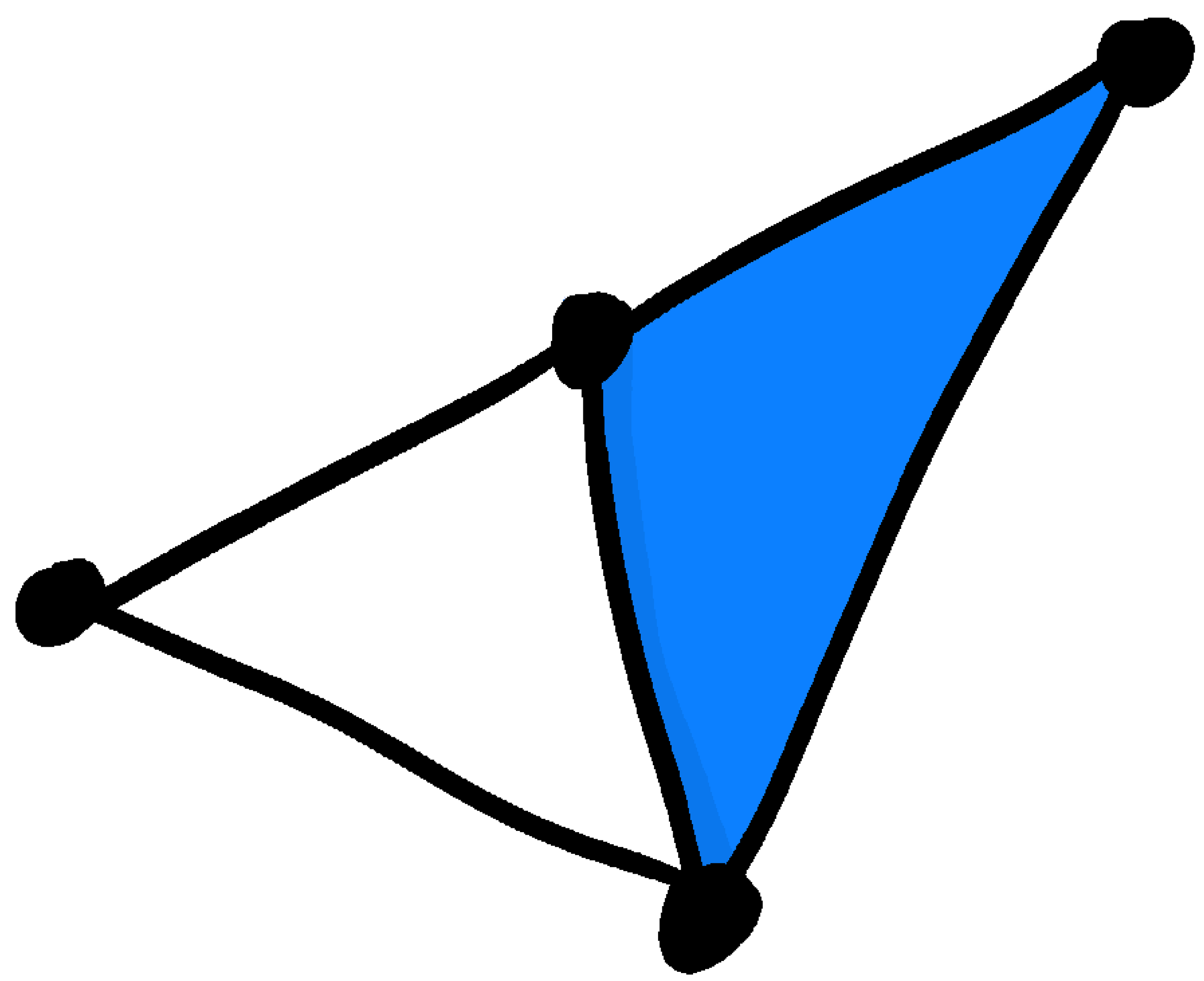
$K_1 =$ 

We will show that K is 2-collapsible.

Collapsibility

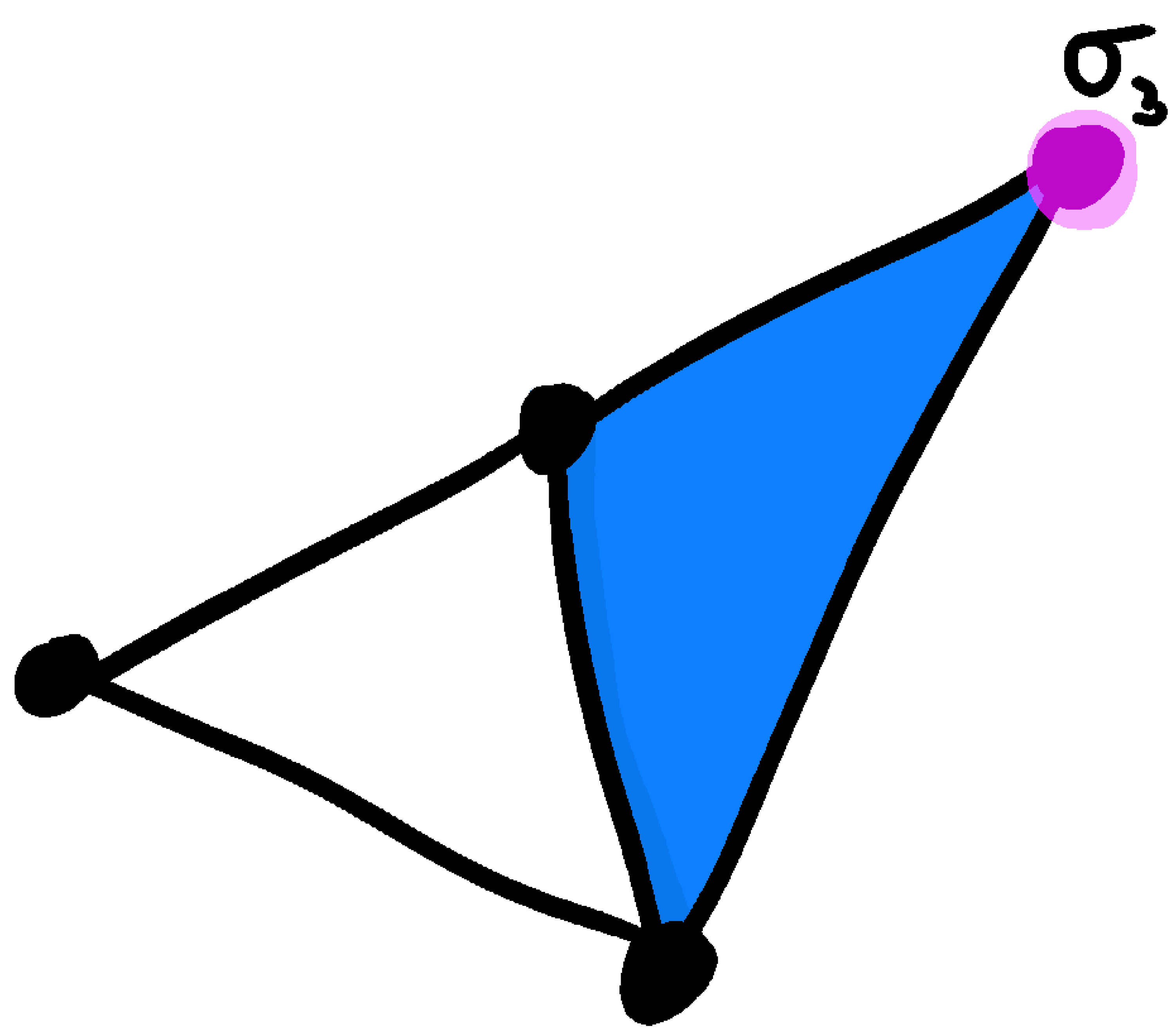
E.g.

$K_2 =$ 

We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_2 =$ 

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Collapsibility

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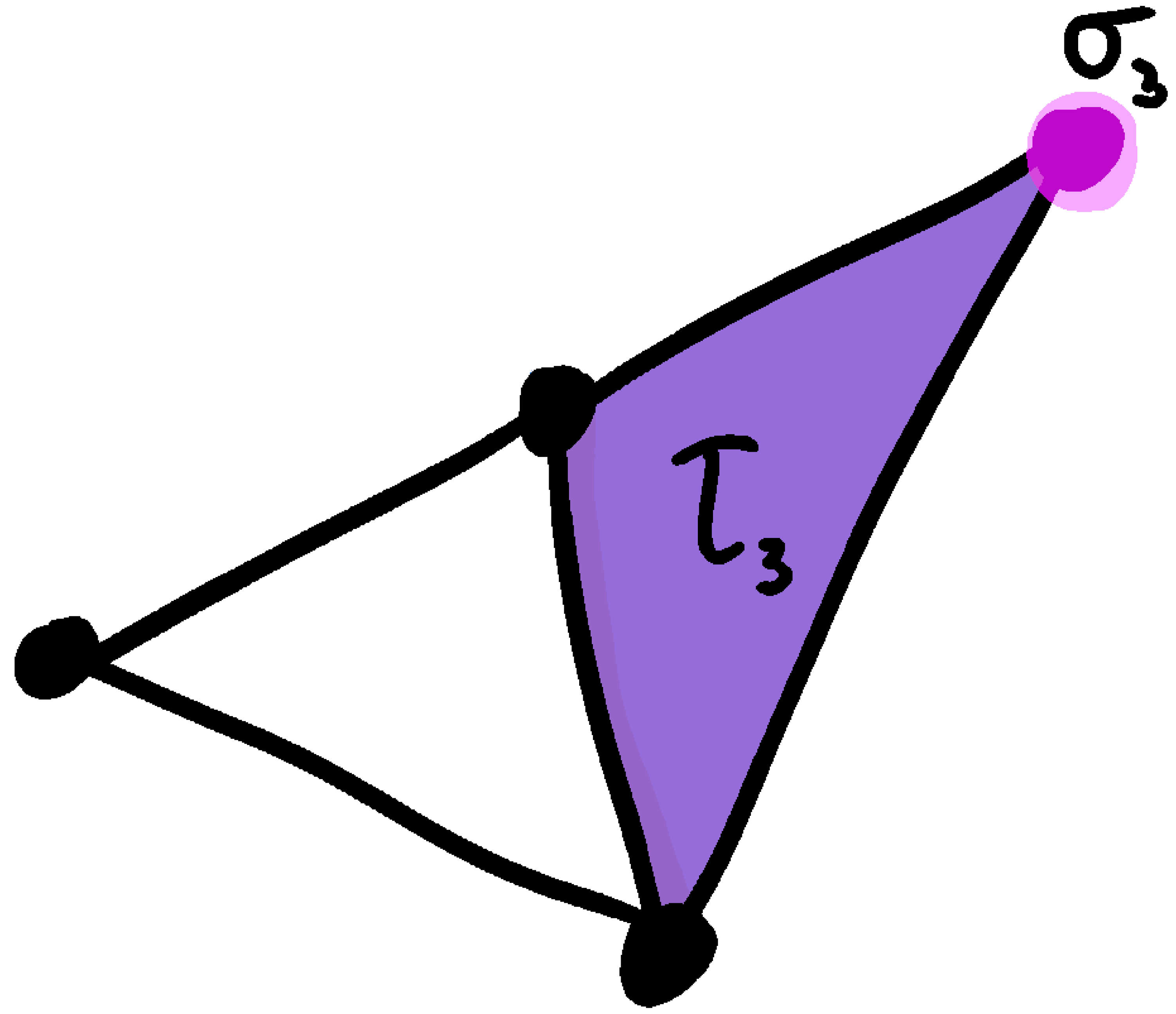
$K_2 =$

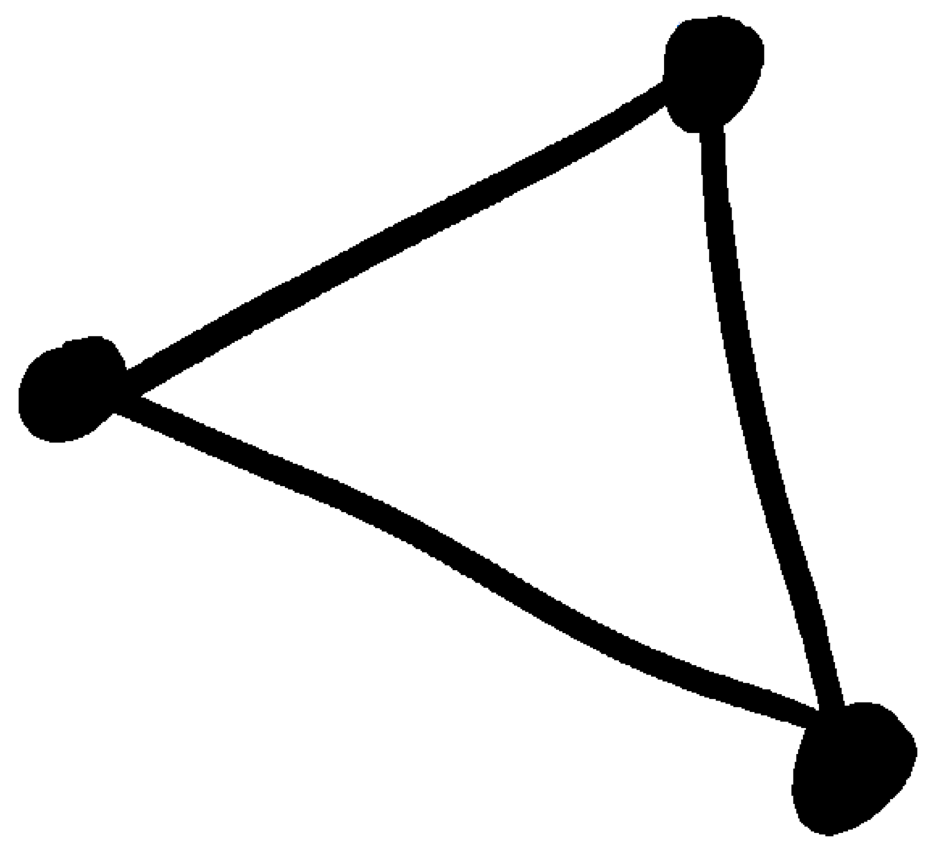


We will show that K is 2-collapsible.

Collapsibility

E.g.

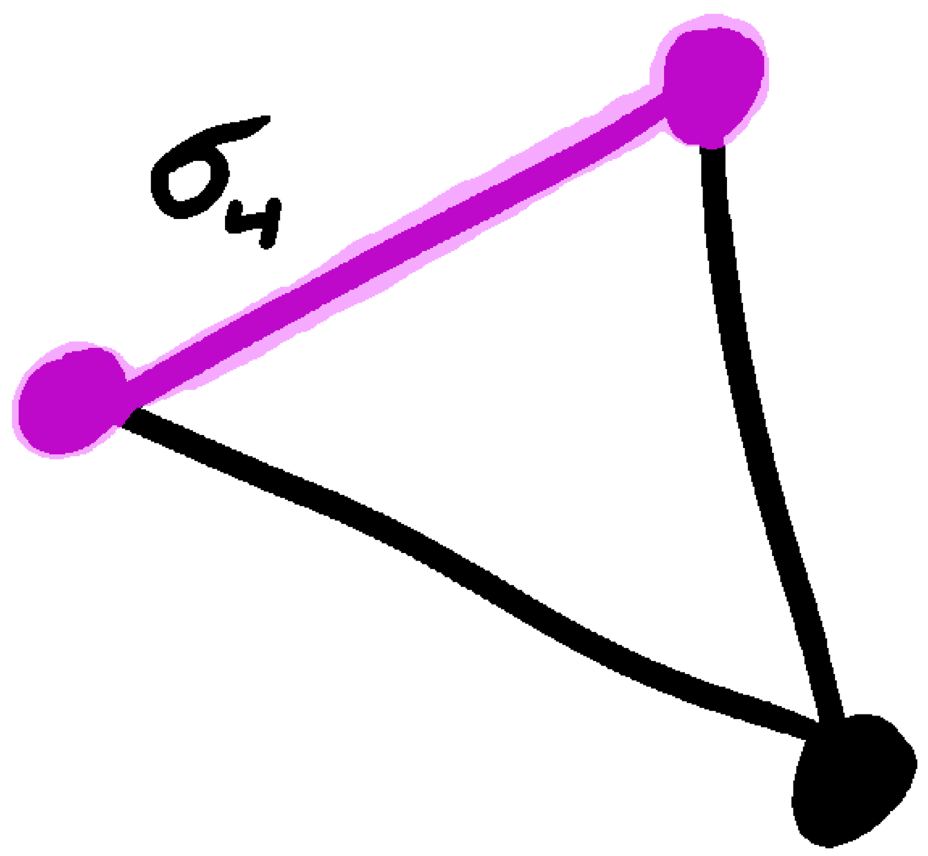
$K_3 =$



We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_3 =$ 

We will show that K is 2-collapsible.

Collapsibility

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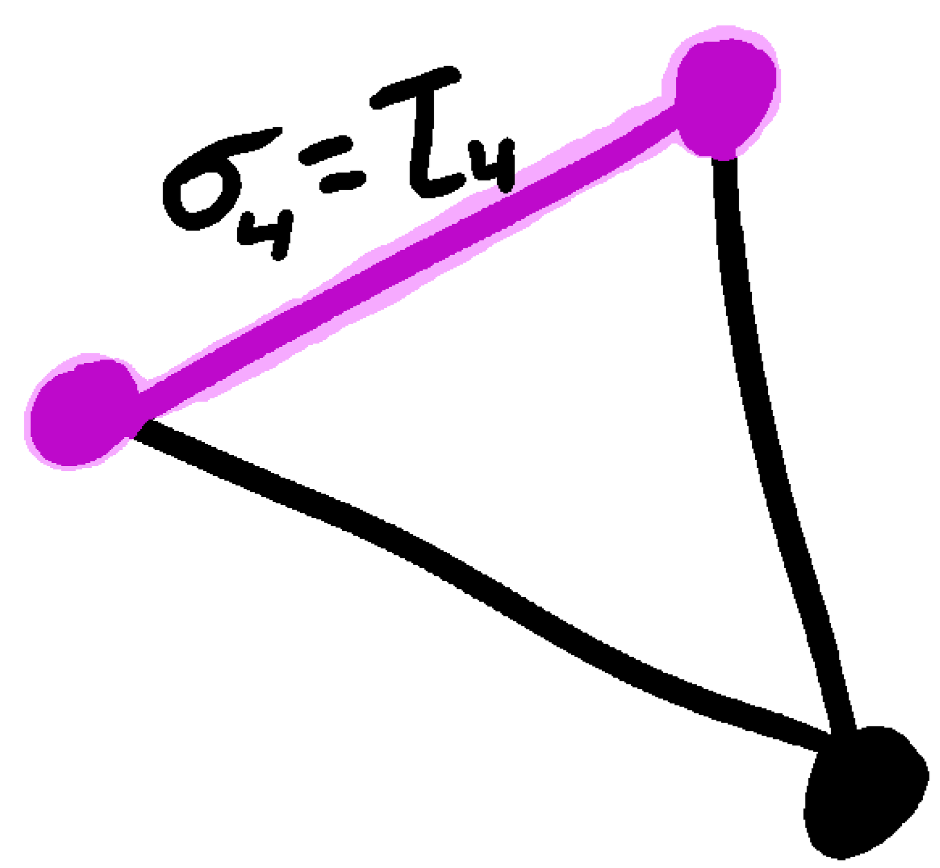
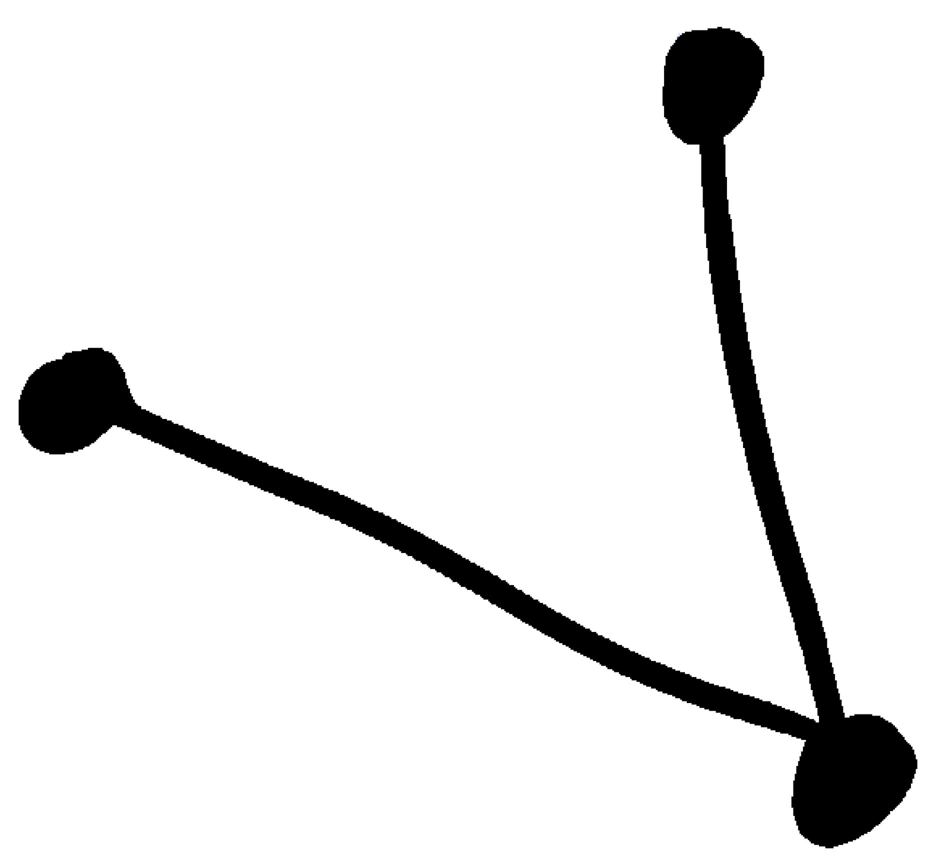
$K_3 =$



We will show that K is 2-collapsible.

Collapsibility

E.g.

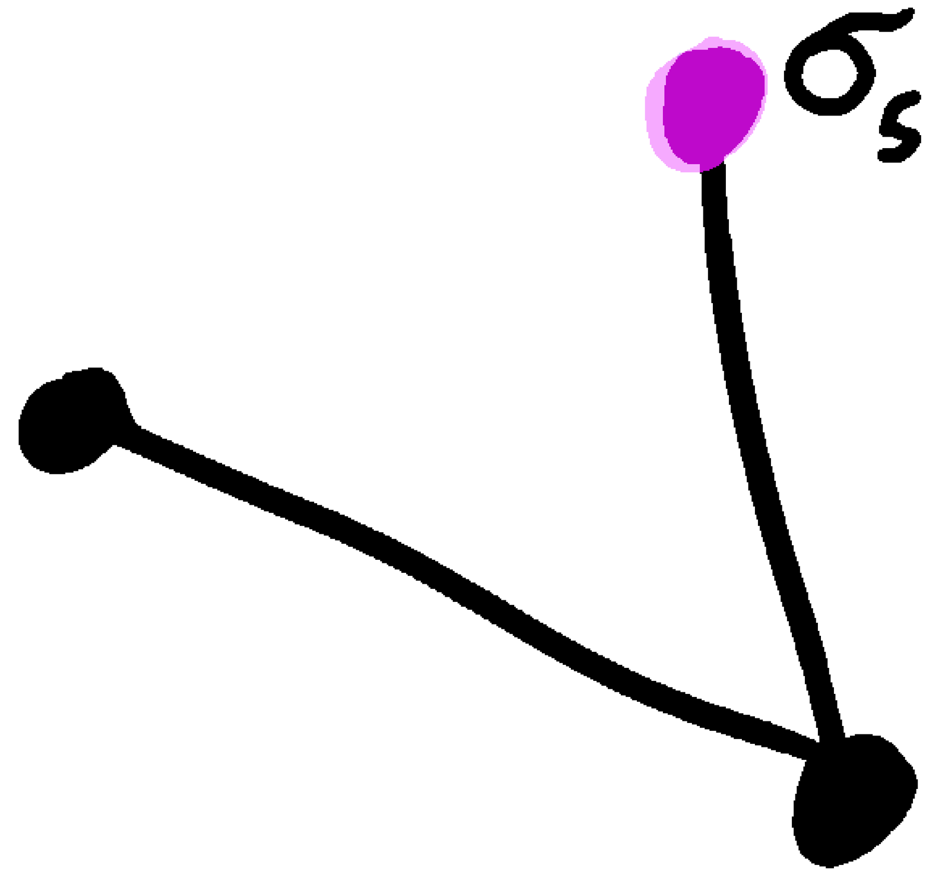
$K_4 =$ 

We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_4 =$



We will show that K is 2-collapsible.

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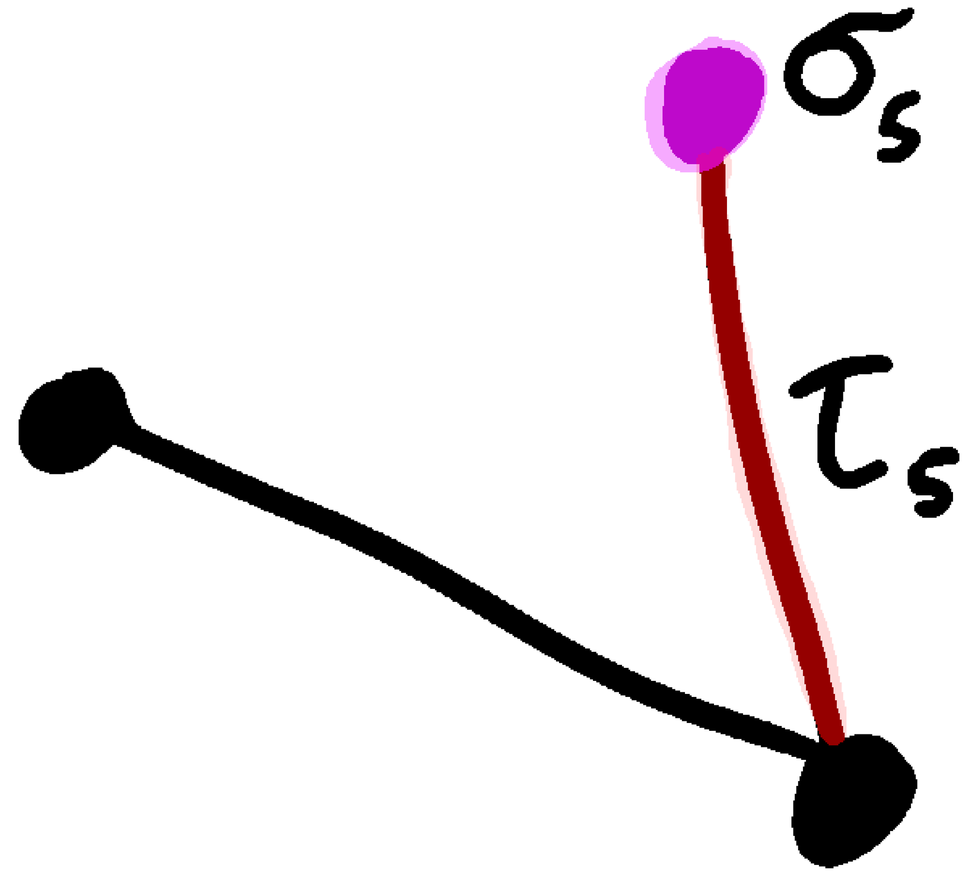
$K_4 =$

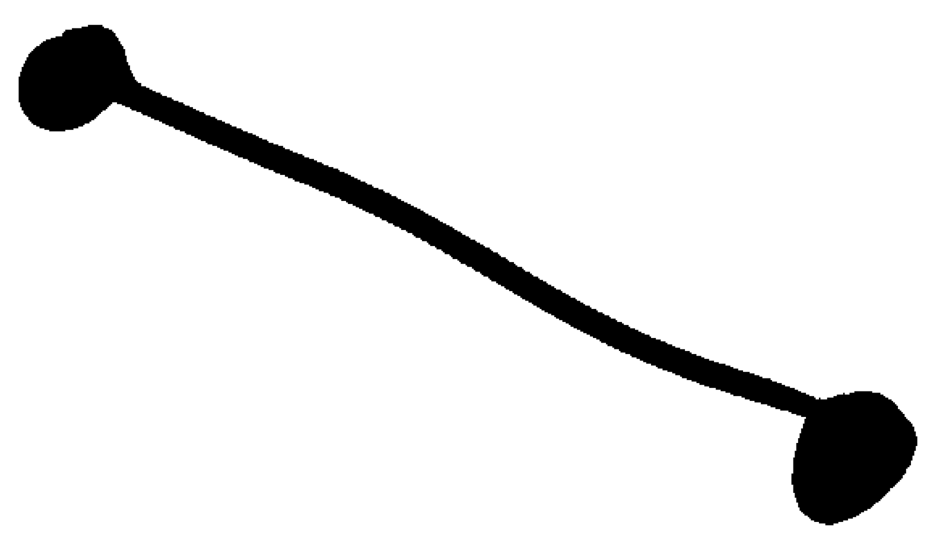


We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_5 =$



We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_5 =$

We will show that K is 2-collapsible.

Collapsibility

E.g.

$K_5 =$

$\sigma_6 = \emptyset$

We will show that

K is 2-collapsible.

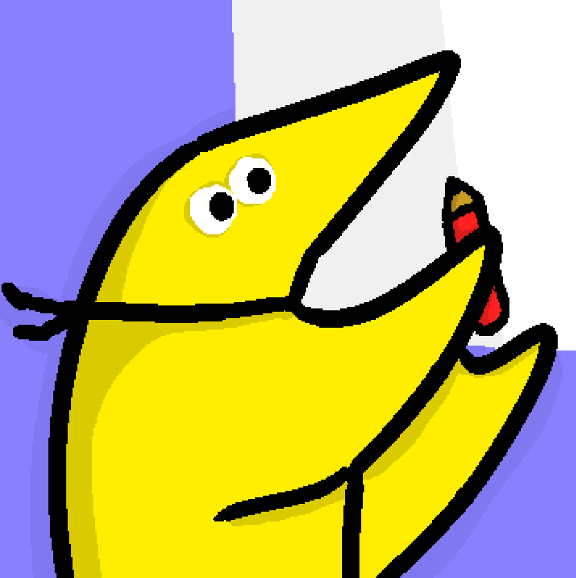
Collapsibility

E.g.

$K_6 =$

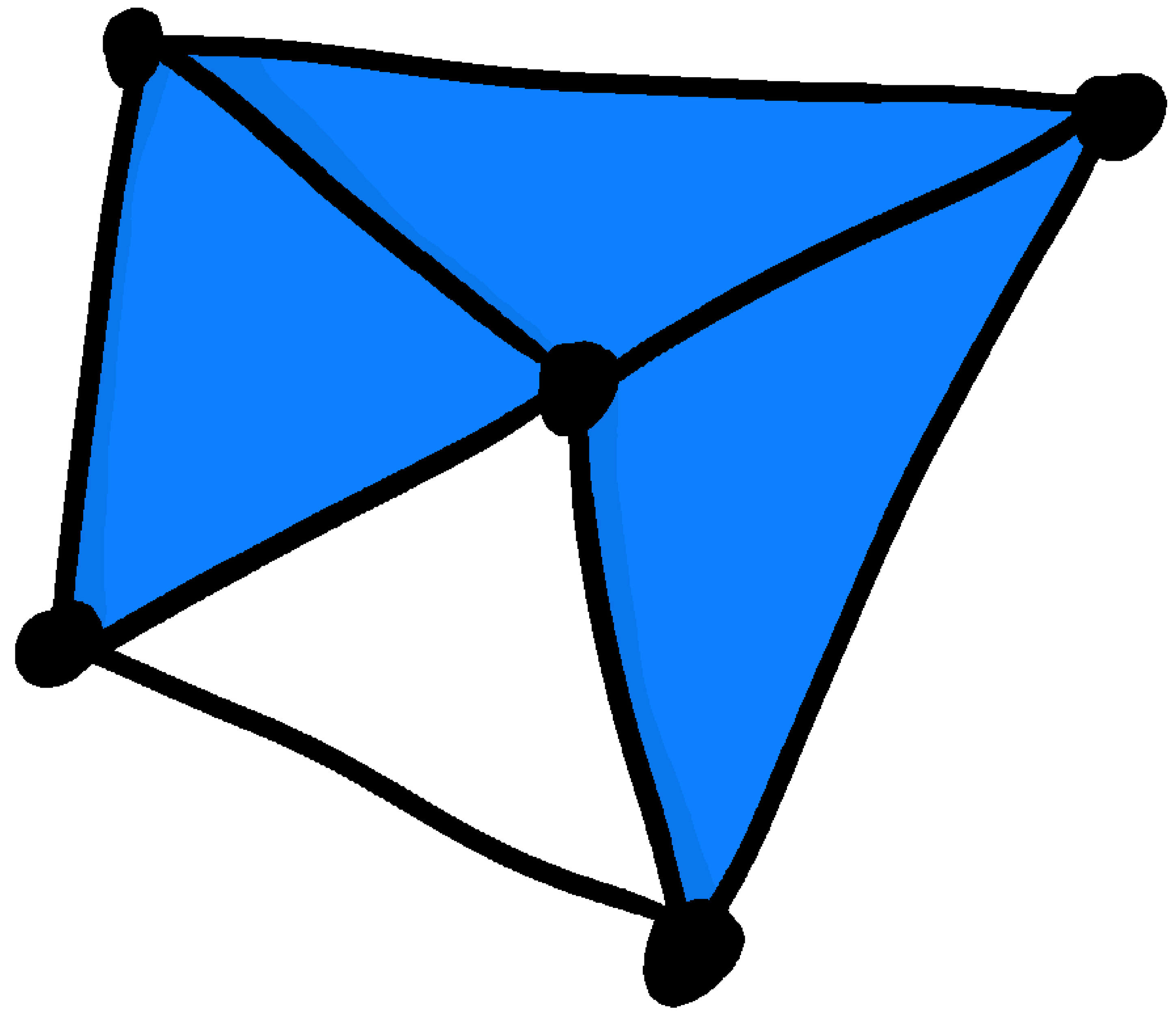
We will show that

K is 2-collapsible.



Collapsibility

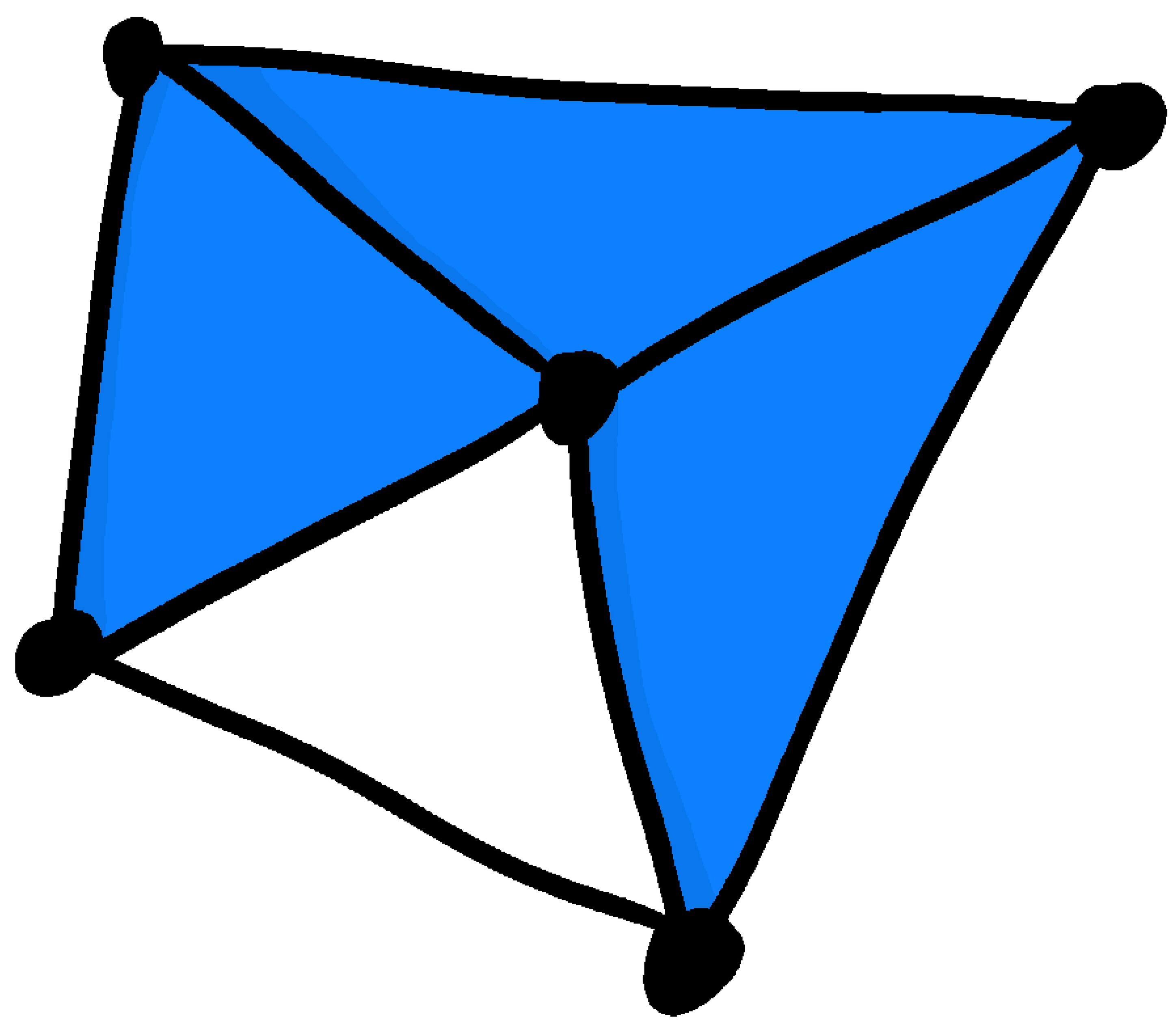
E.g.

$K =$ 

K is 2-collapsible.

Collapsibility

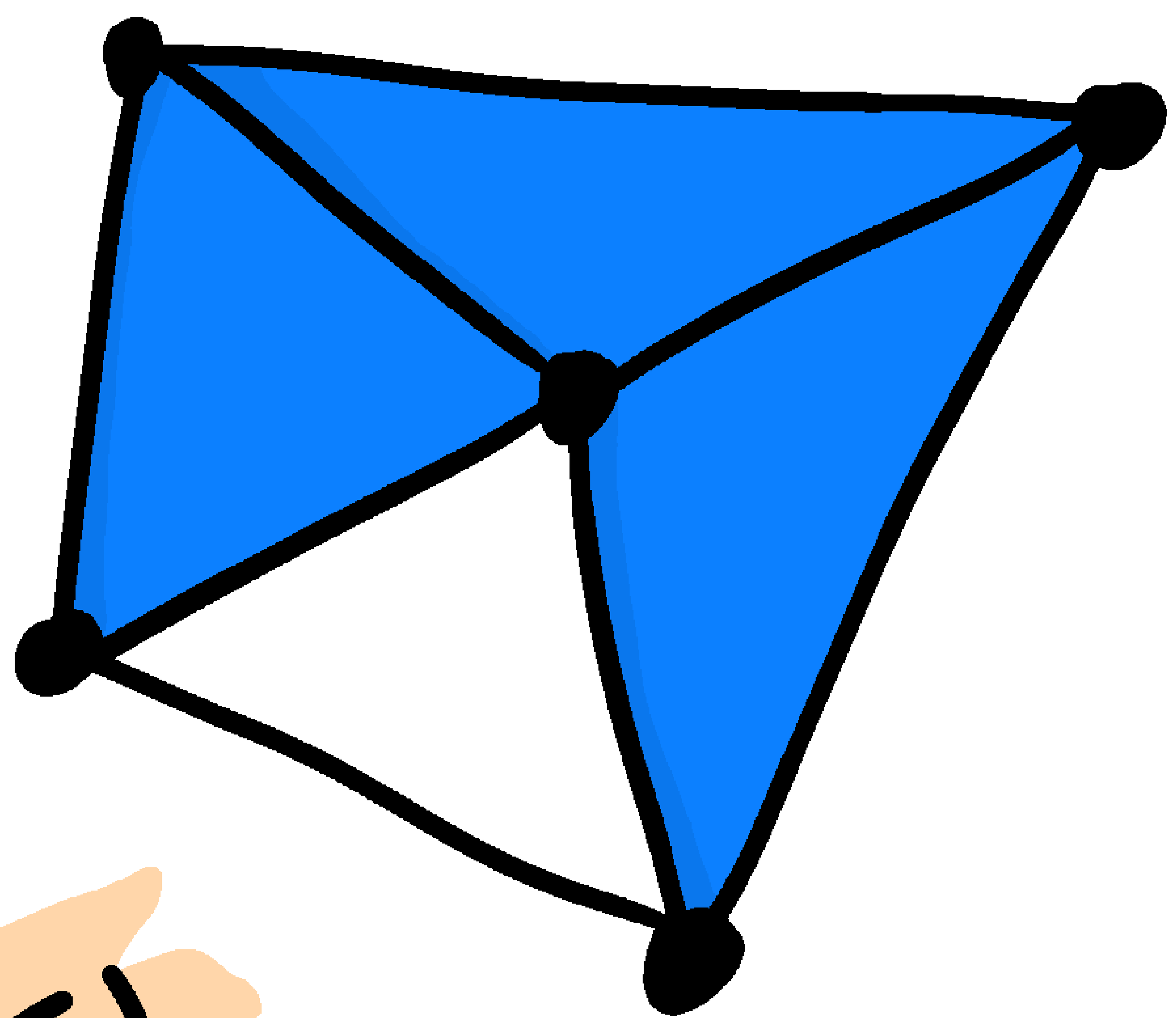
E.g.

$K =$  K is 2-collapsible.

- Collapsibility of K = minimal d s.t. K is d-collapsible.

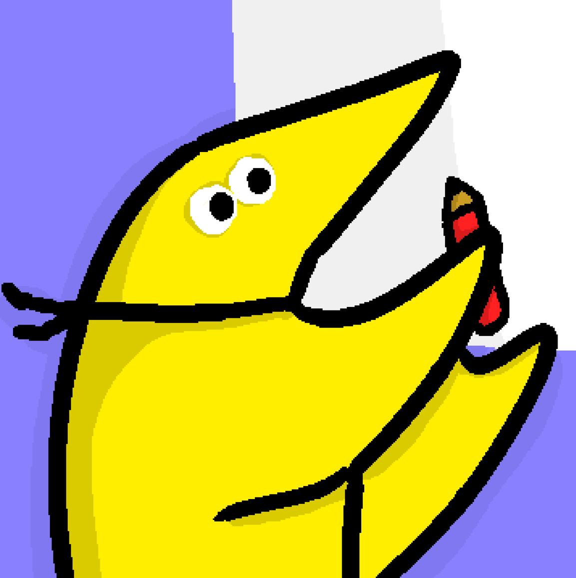
Collapsibility

E.g.

$K =$  K is 2-collapsible.

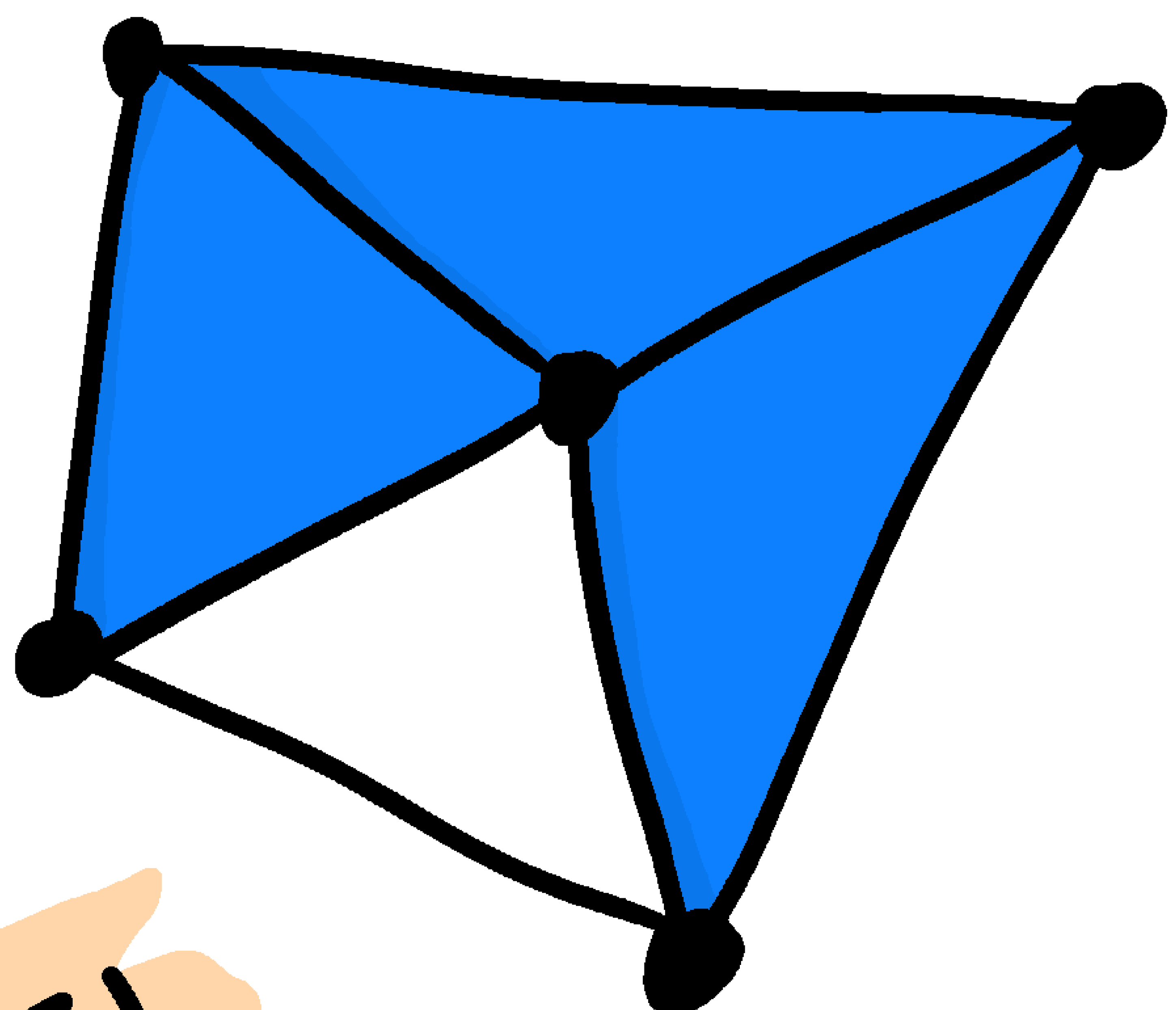
$C(K)$

- Collapsibility of K = minimal d s.t. K is d-collapsible.



Collapsibility

E.g.

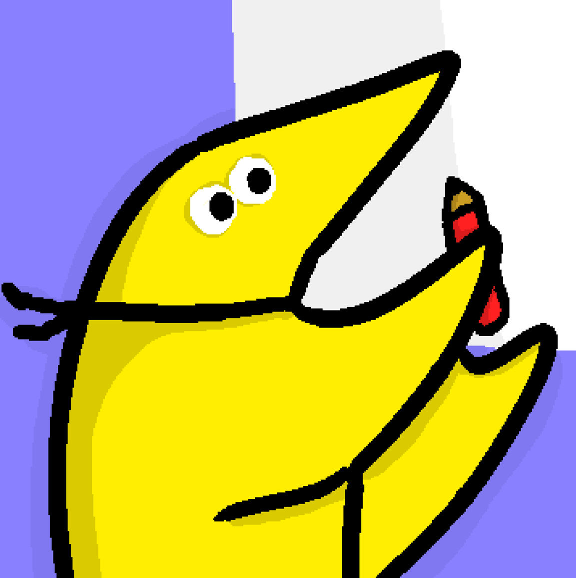
$K =$ 

K is 2-collapsible.

$C(K) = 2$

$C(K)$

- Collapsibility of K = minimal d s.t. K is d-collapsible.



Representability

- $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ family of sets.

Representability

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Representability

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  - A vertex is assigned to each set in the family.

Representability

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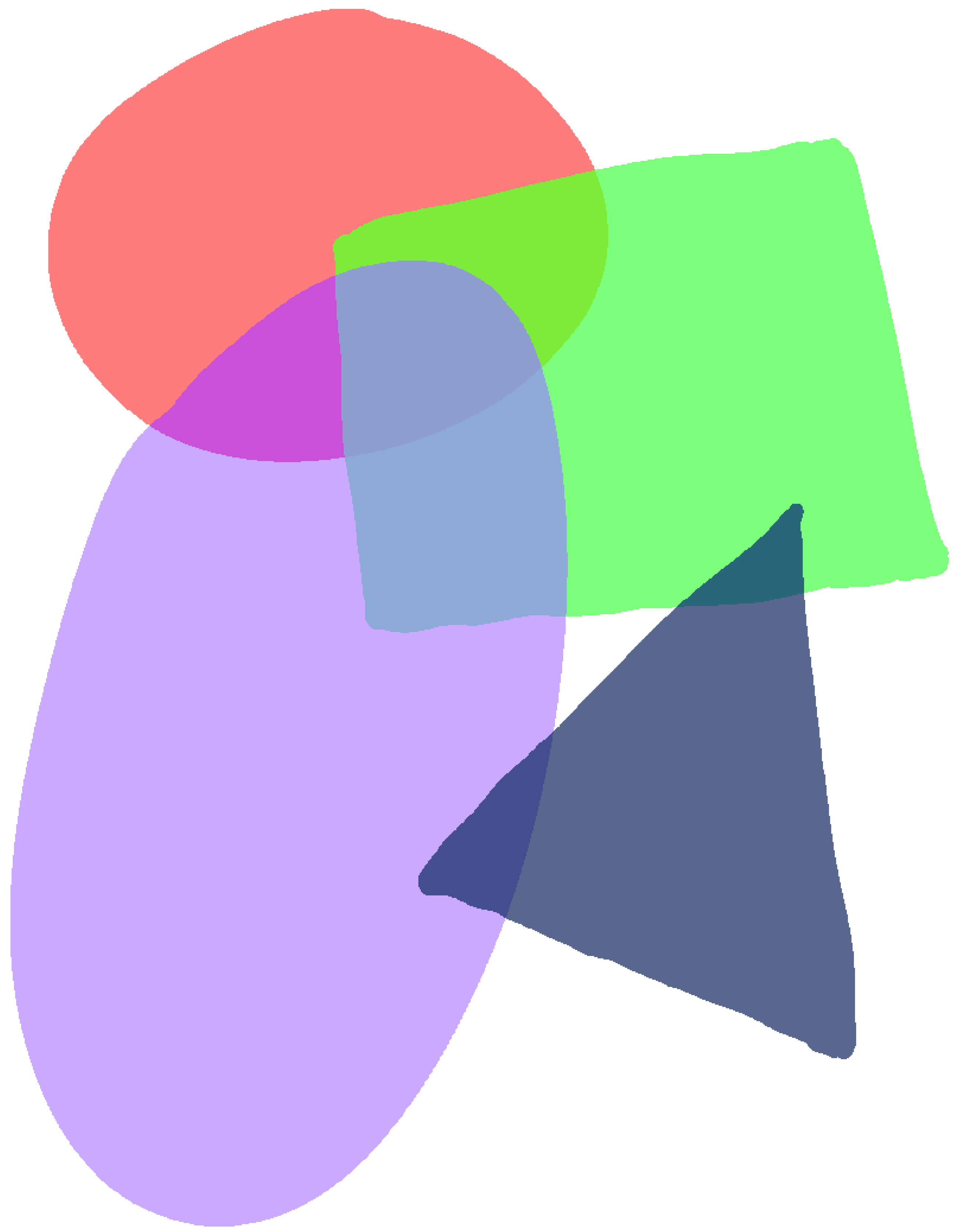
- $N(\mathcal{F})$ = nerve of $\mathcal{F}$:
  - A vertex is assigned to each set in the family.
  - Simplices correspond to subfamilies with non empty intersection.

Representability

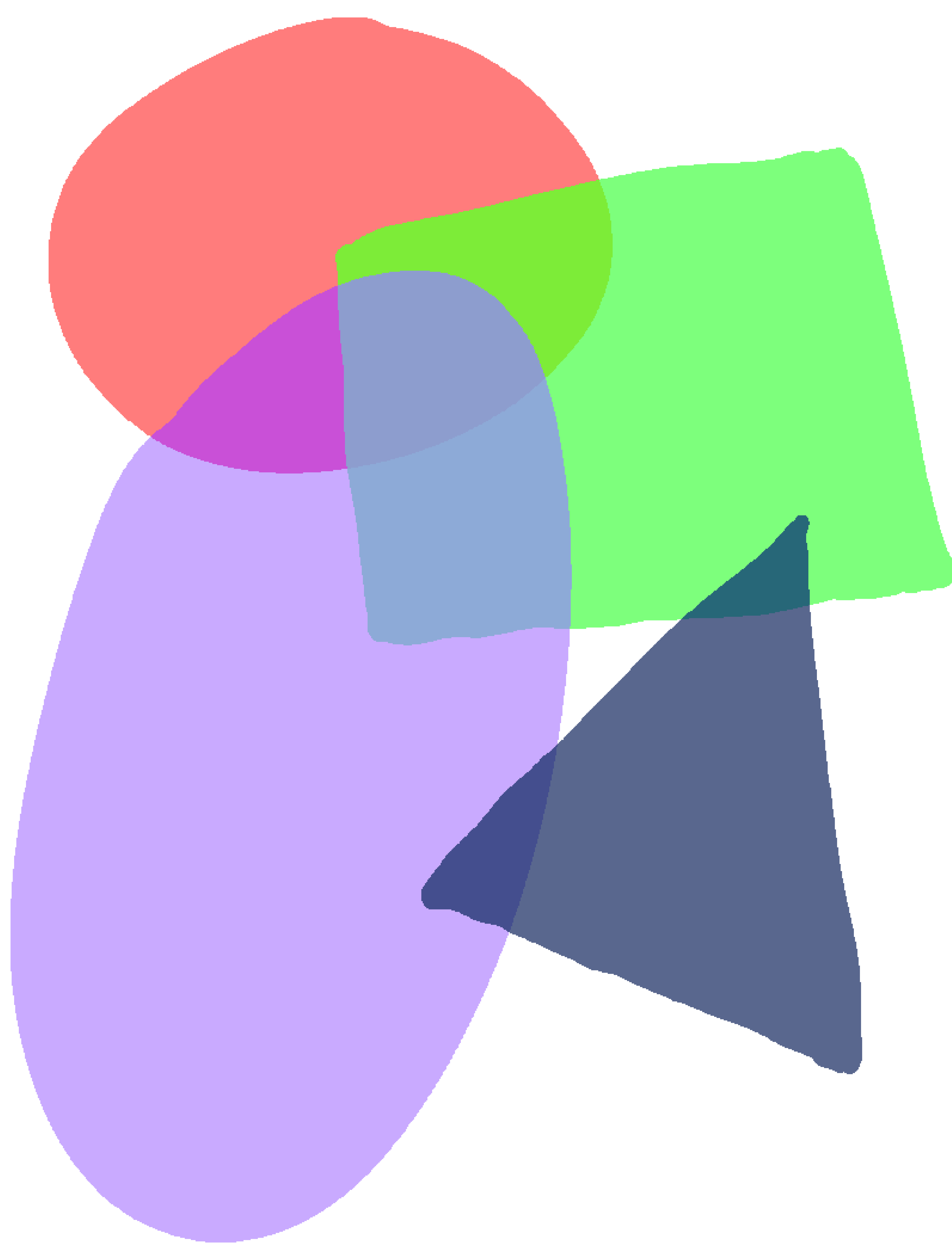
E.g.

$F =$ 

Representability

E.g.

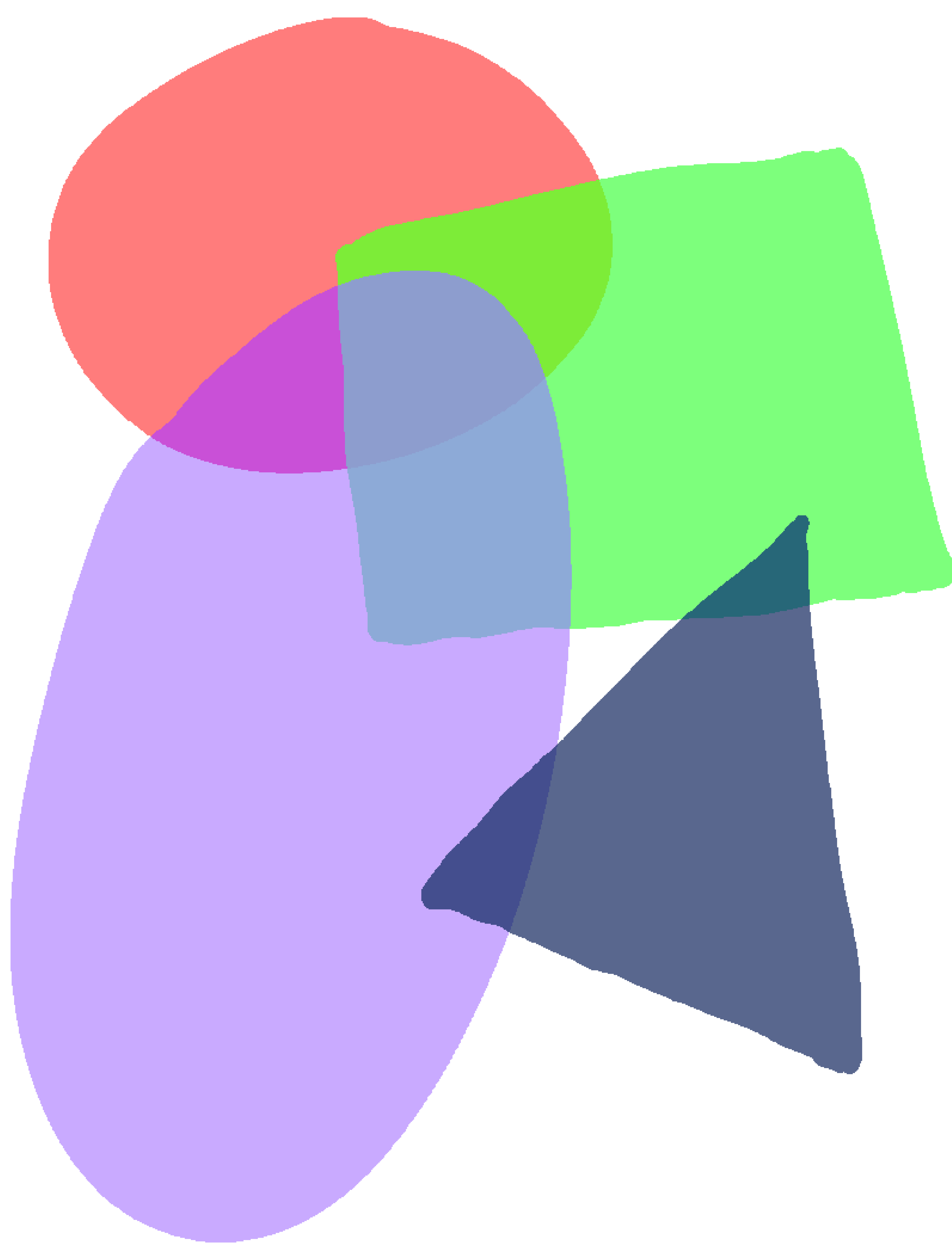
$F =$

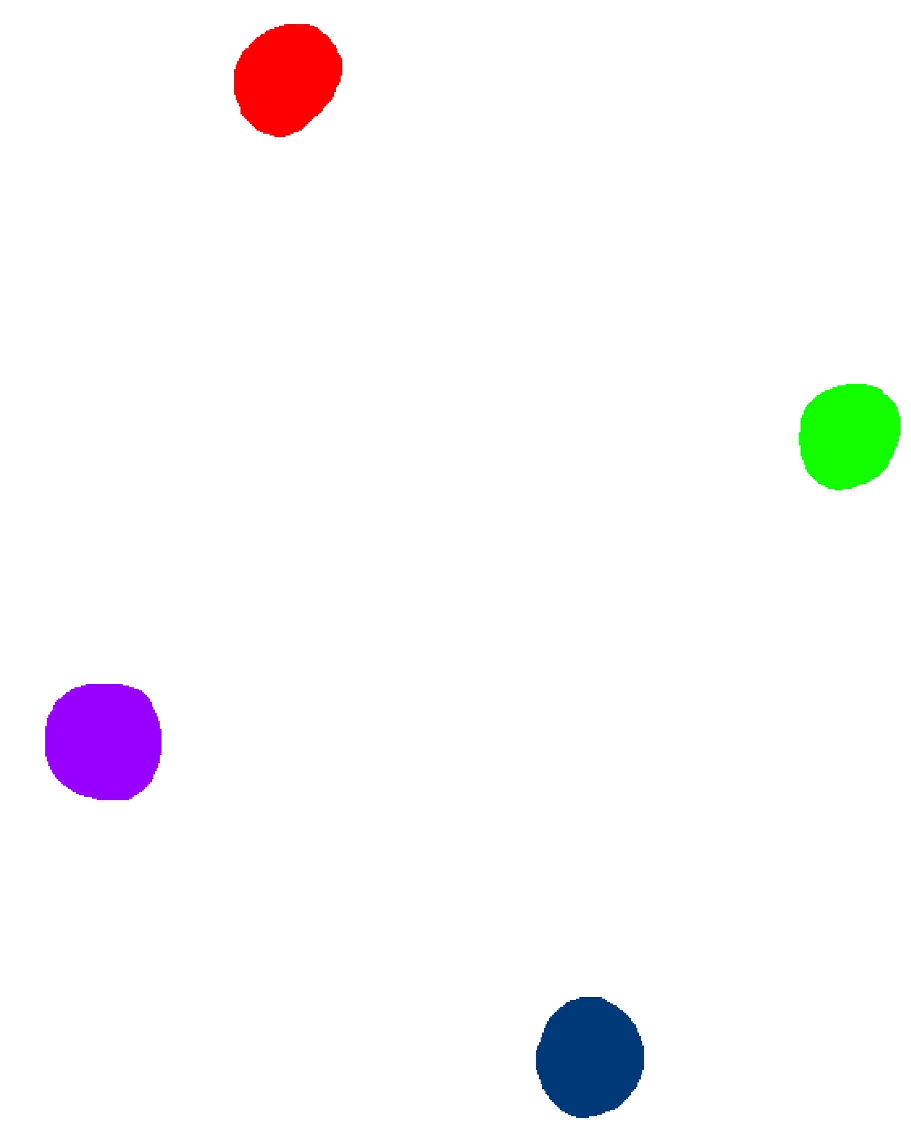


$N(F) =$

Representability

E.g.

$F =$ 

$N(F) =$ 

Representability

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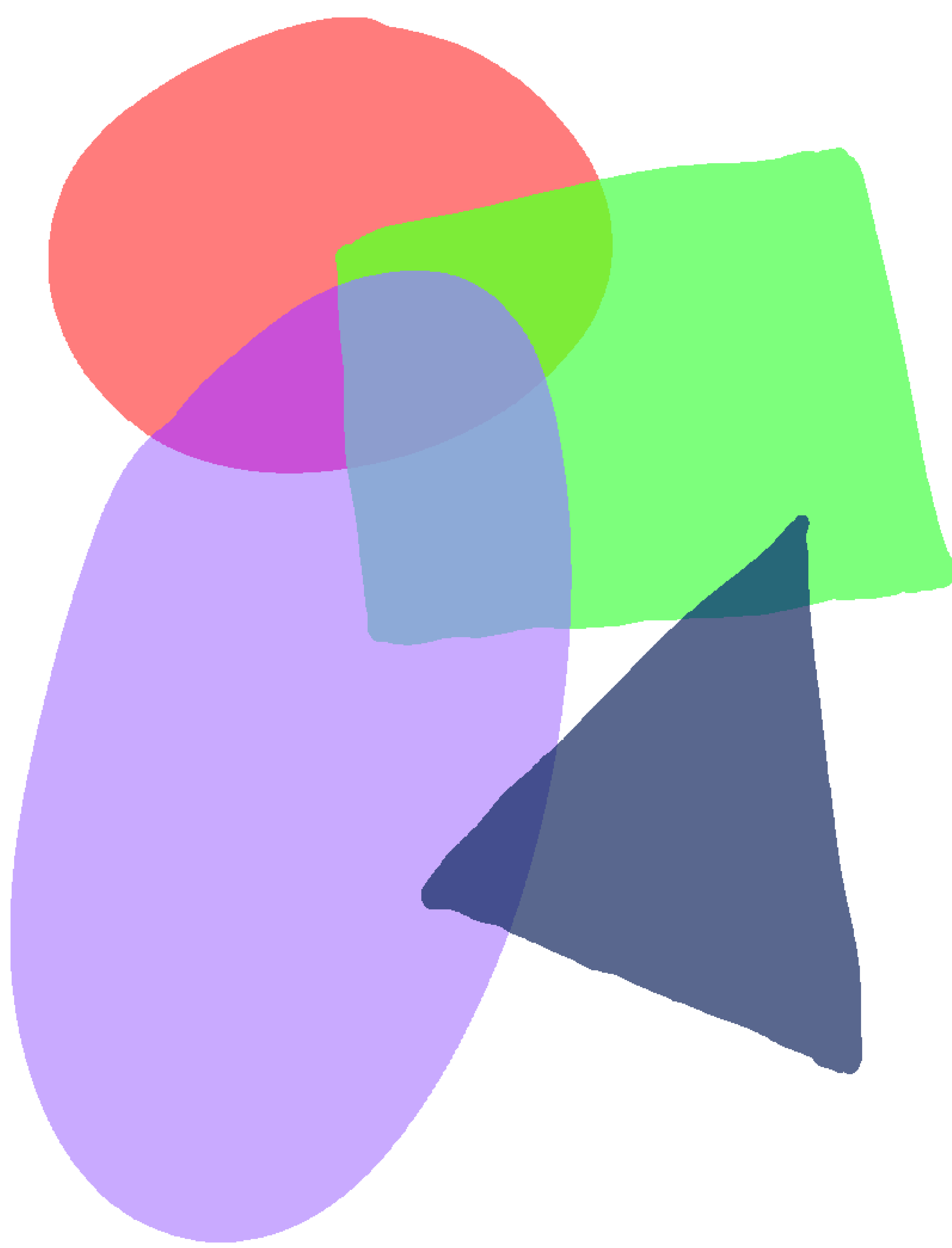
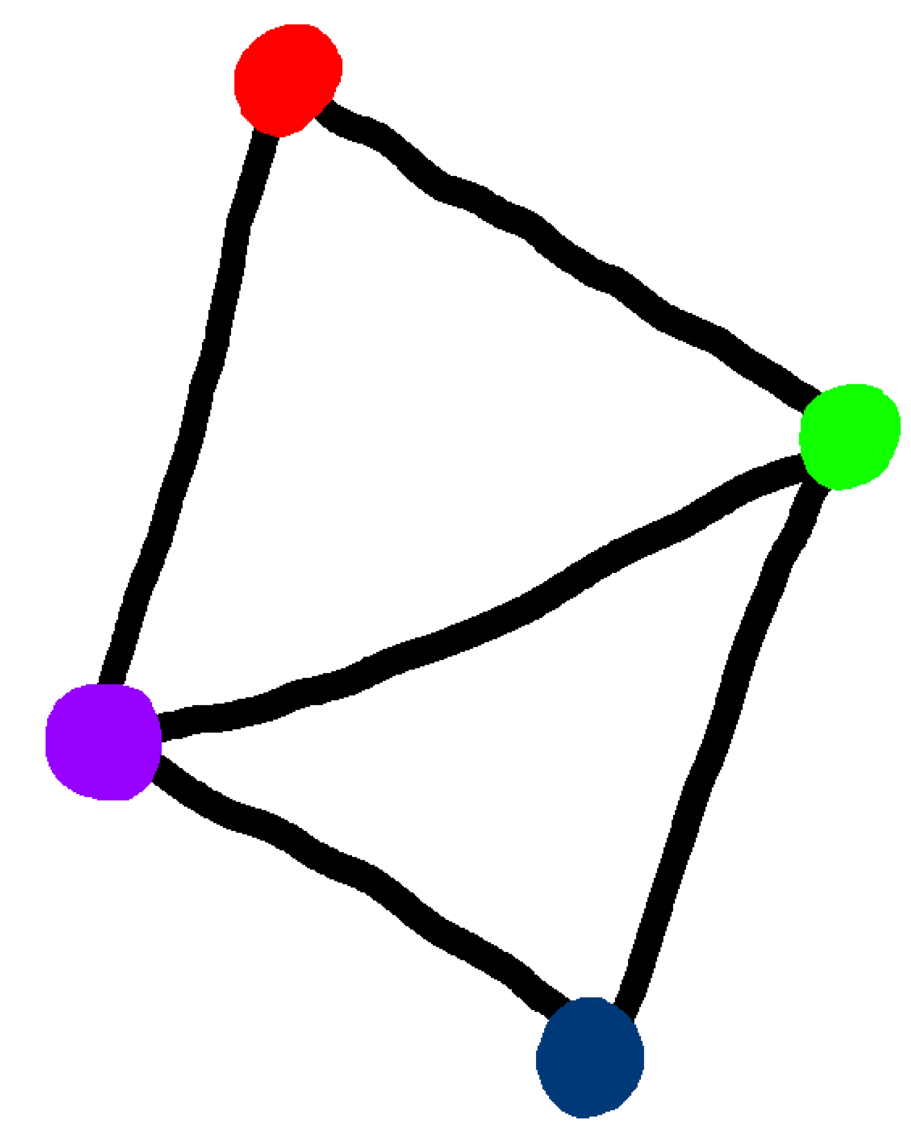
$F =$ 

$N(F) =$ 

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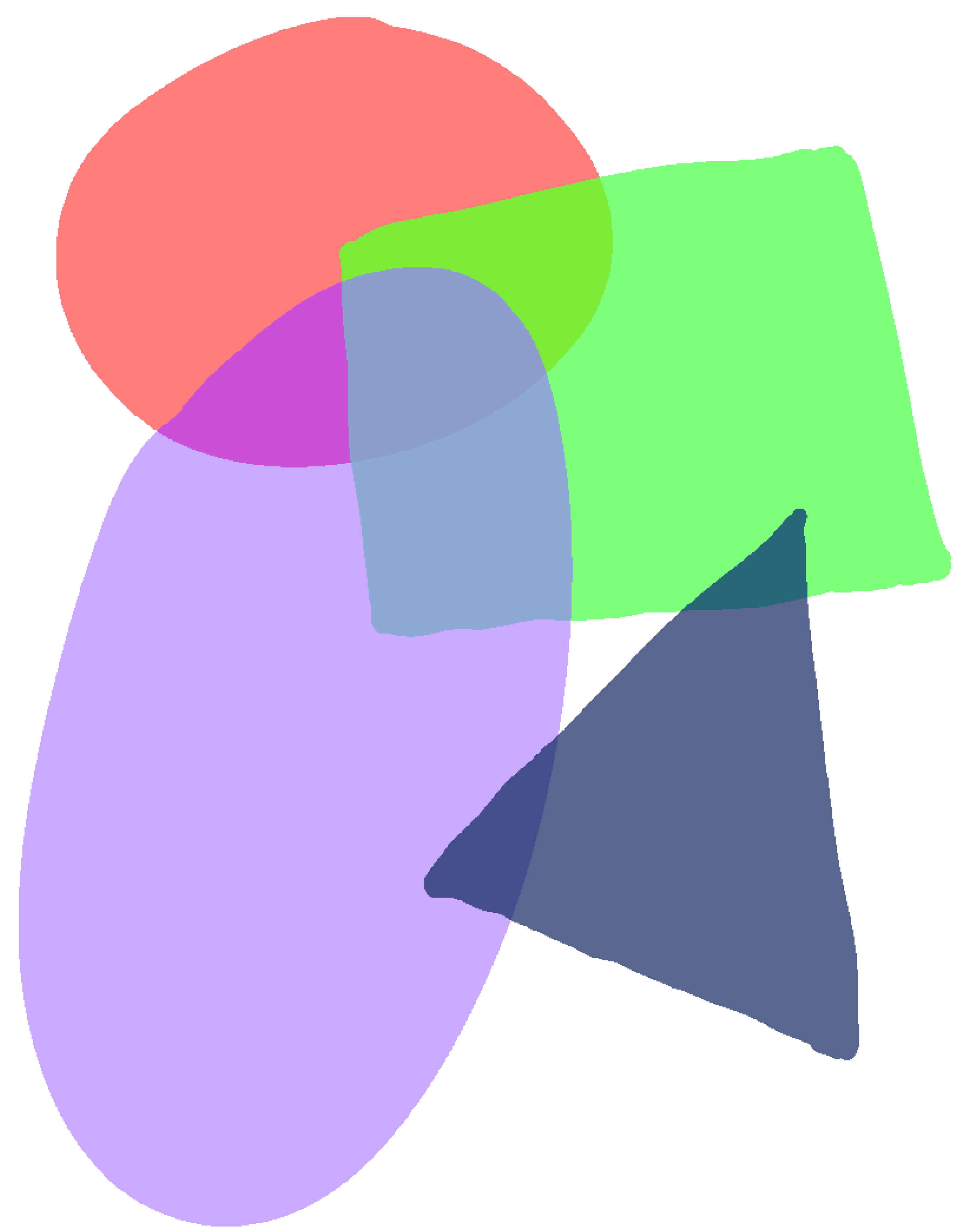
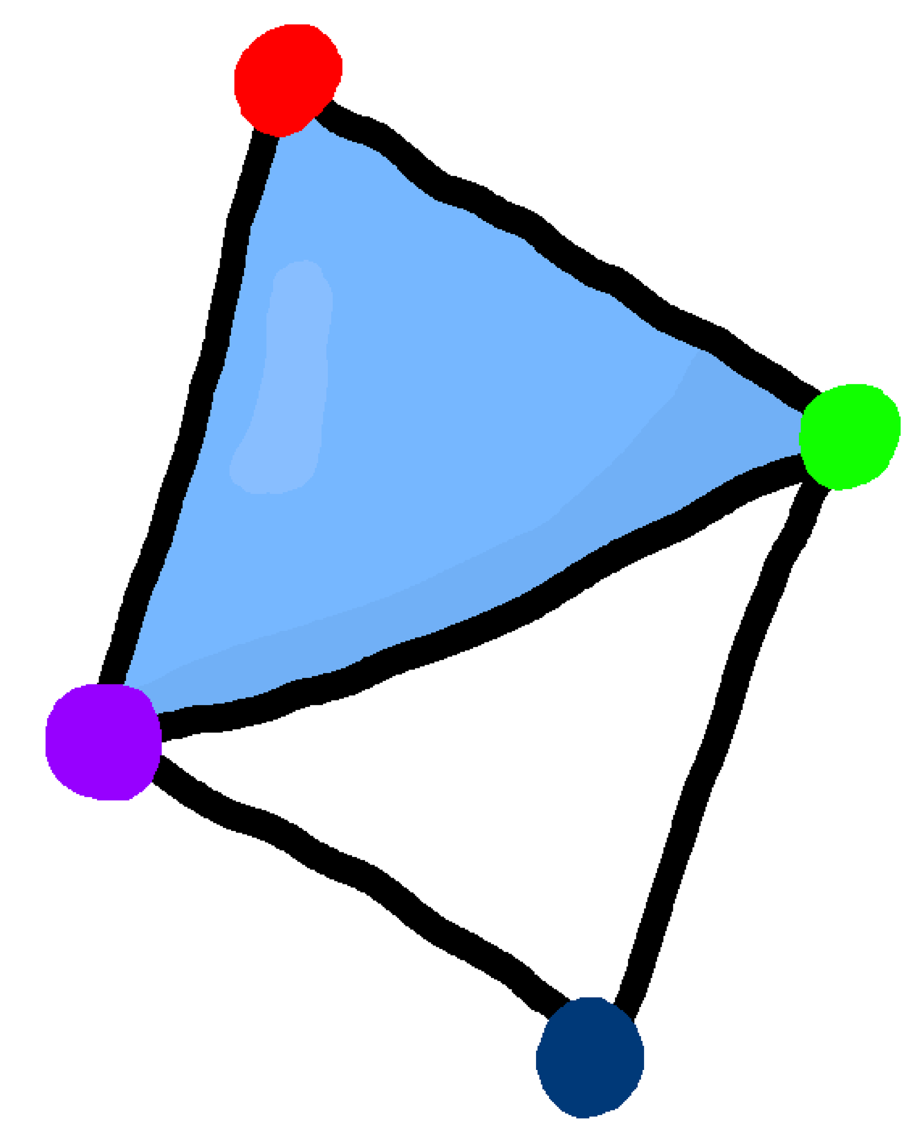
$F =$ 

$N(F) =$ 

Representability

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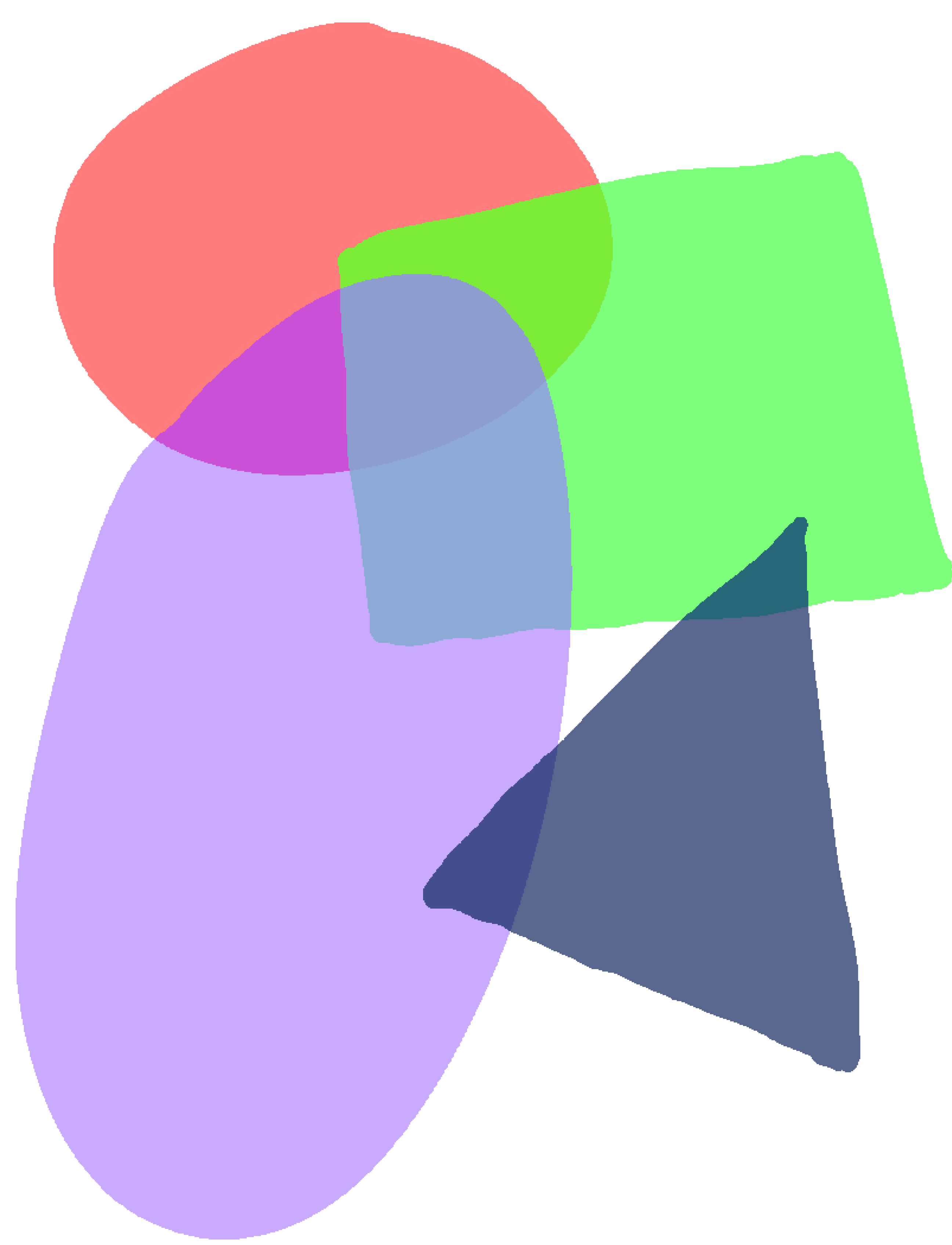
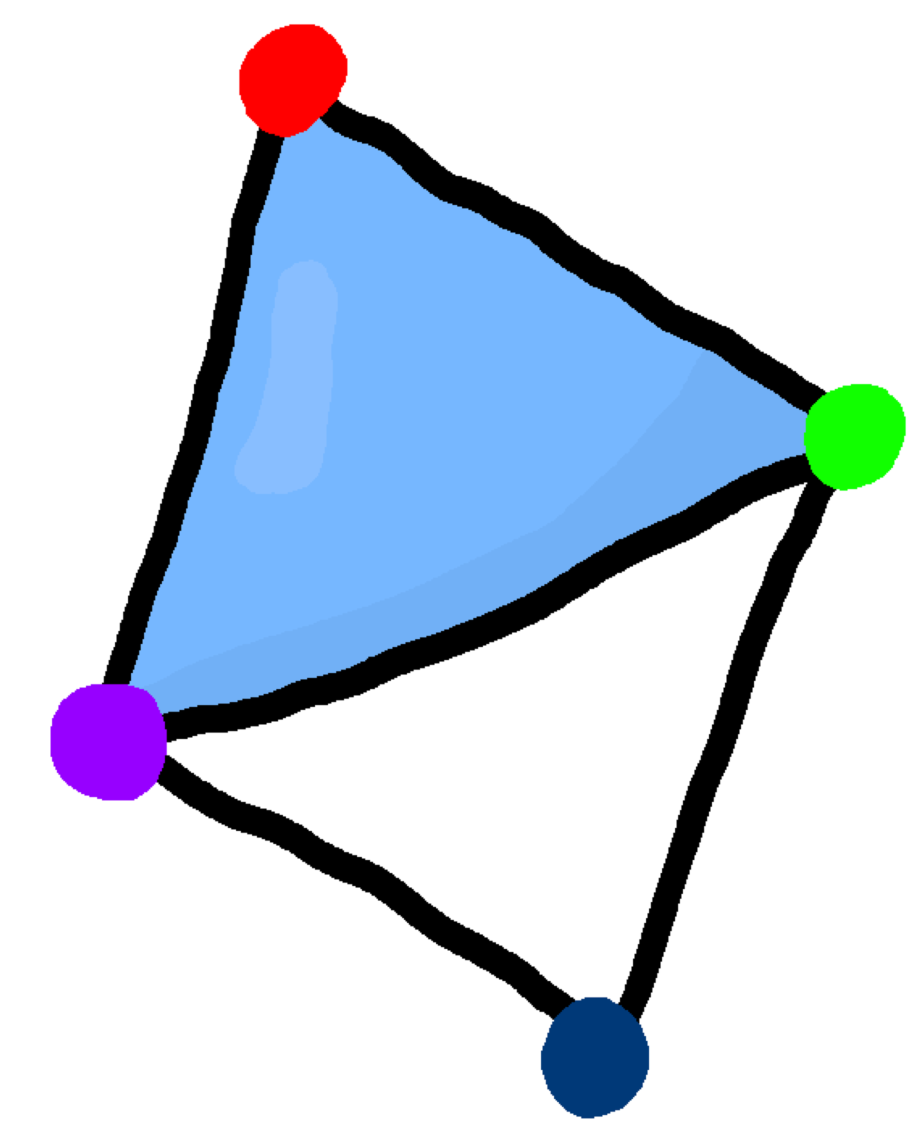
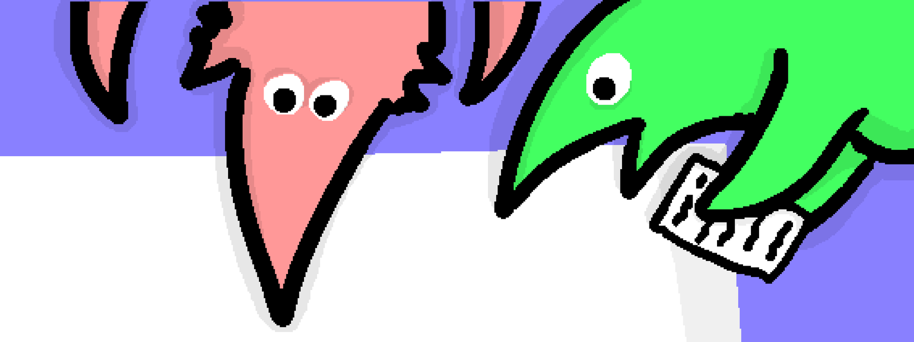
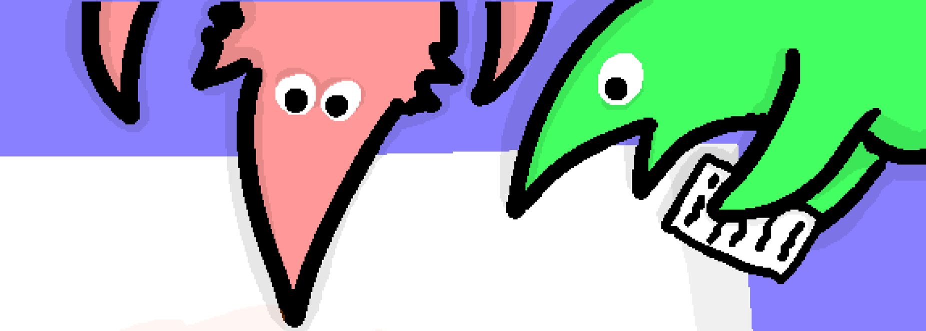
$F =$ 

$N(F) =$ 

- d-representable complex = nerve of a family of convex sets in $\mathbb{R}^d$

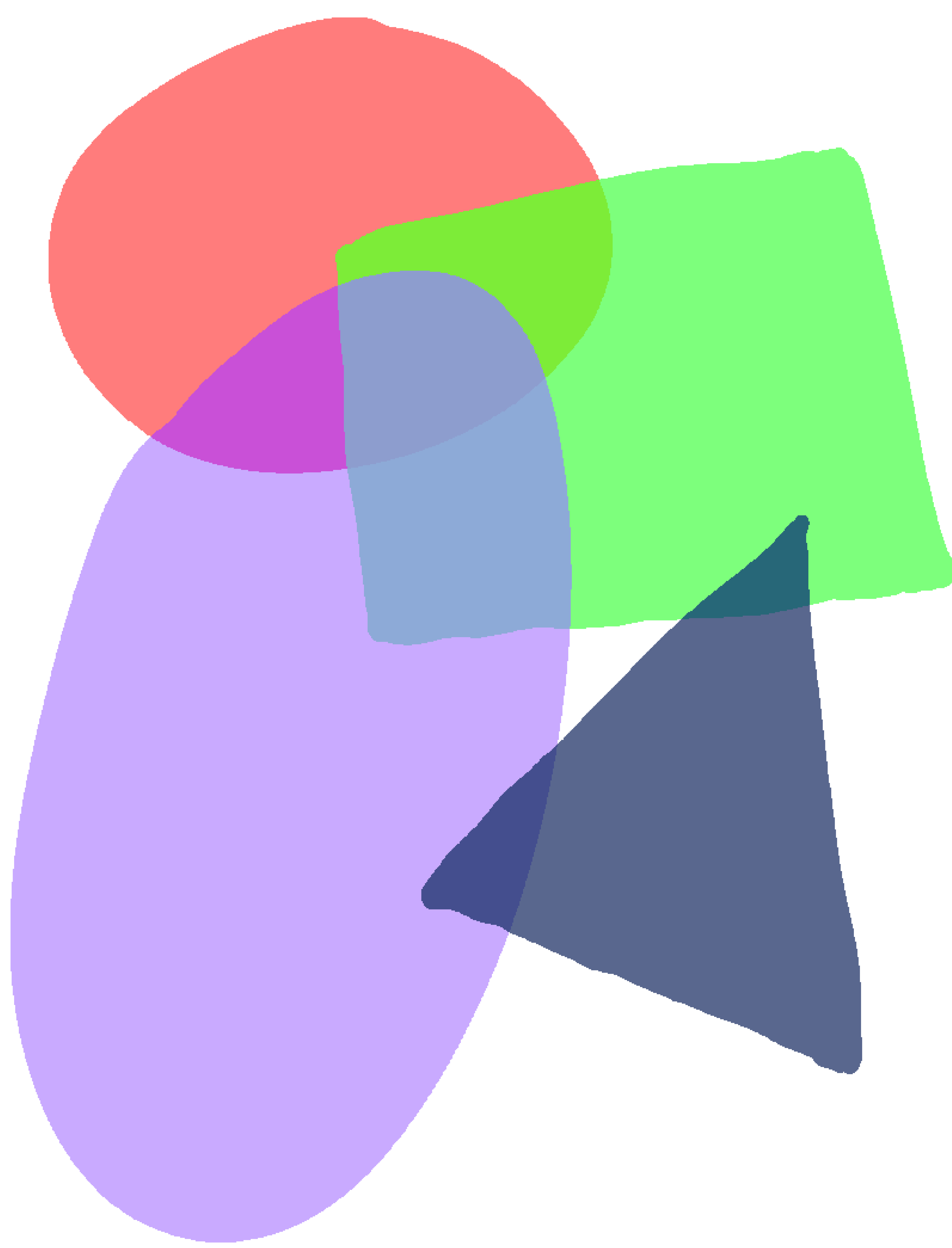


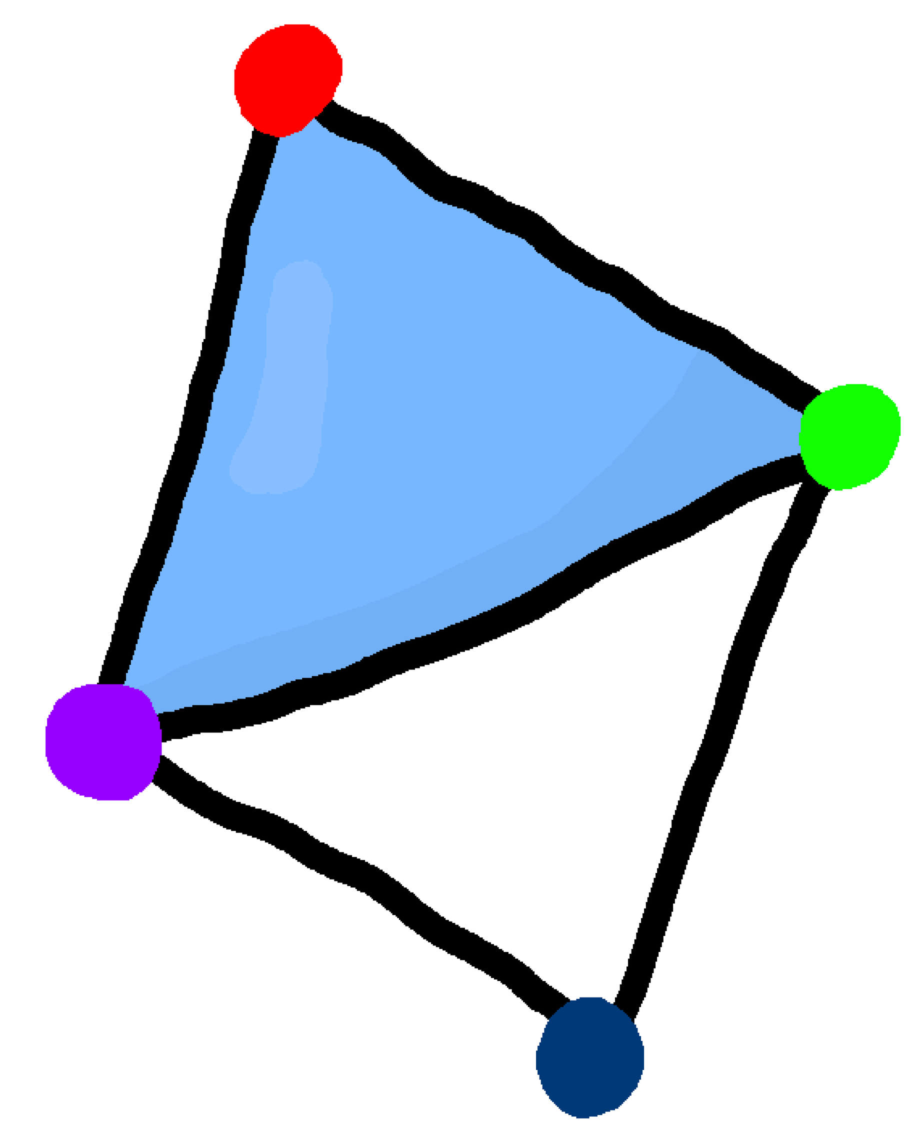


Representability

E.g.

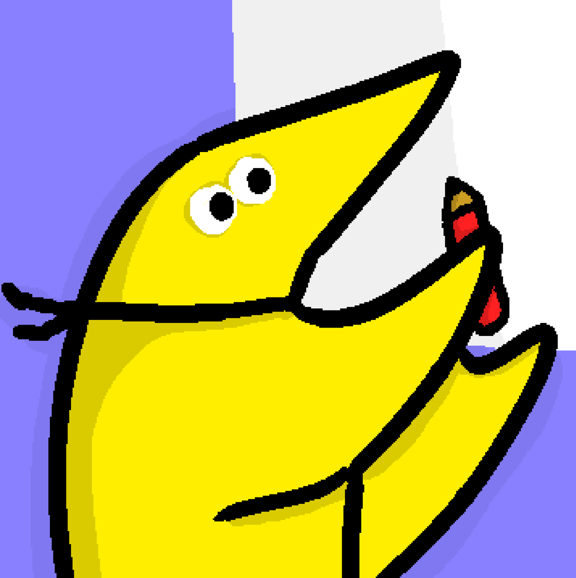
2-representable

$F =$ 

$N(F) =$ 

- d-representable complex = nerve of a family of convex sets in $\mathbb{R}^d$

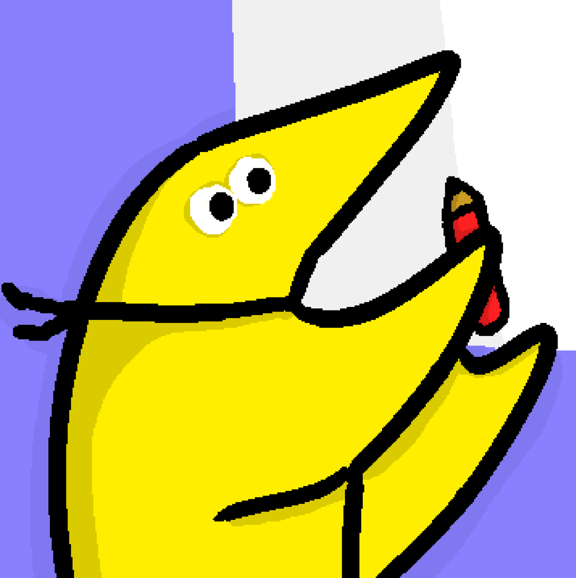
K is d-representable



K is d-collapsible

$\Nearrow$

K is d-representable



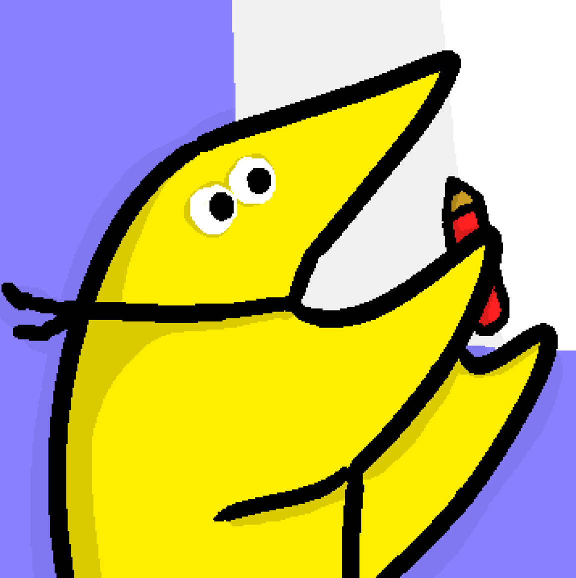
K is d-representable

$\Rightarrow$

K is d-collapsible

$\Rightarrow$

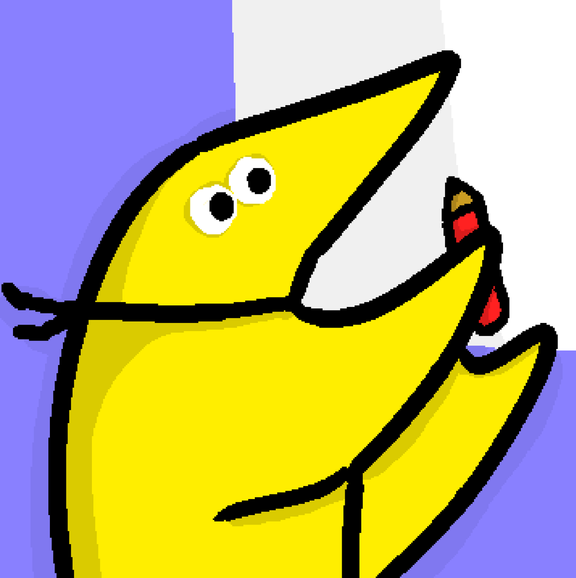
K is d-Leray



Wegner's Theorem ('75)

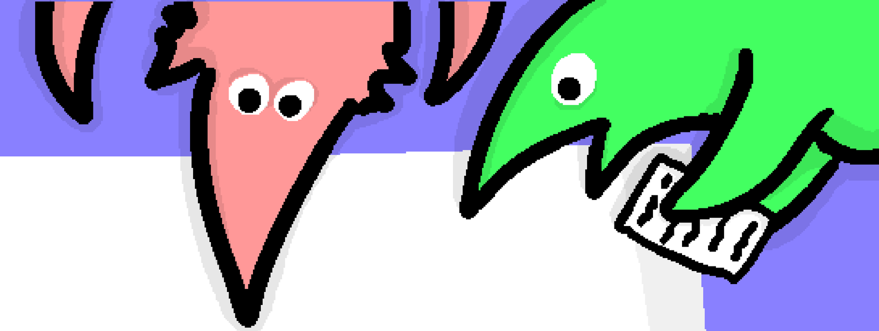
K is d-collapsible

K is d-representable $\Rightarrow$ K is d-collapsible $\Rightarrow$ K is d-Leray



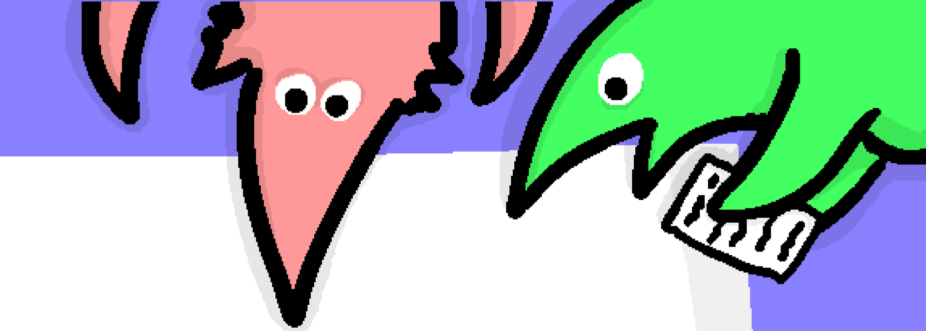
Helly's Theorem

Let $C_1, \dots, C_m \subseteq \mathbb{R}^d$ be convex sets.



Helly's Theorem

Let $C_1, \dots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common,



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Helly's Theorem

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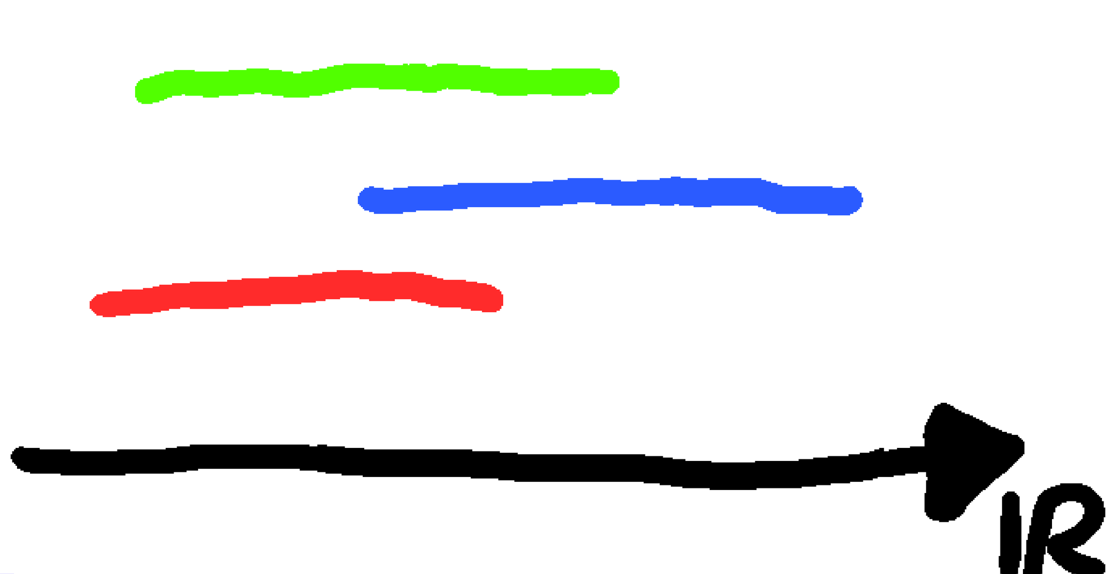
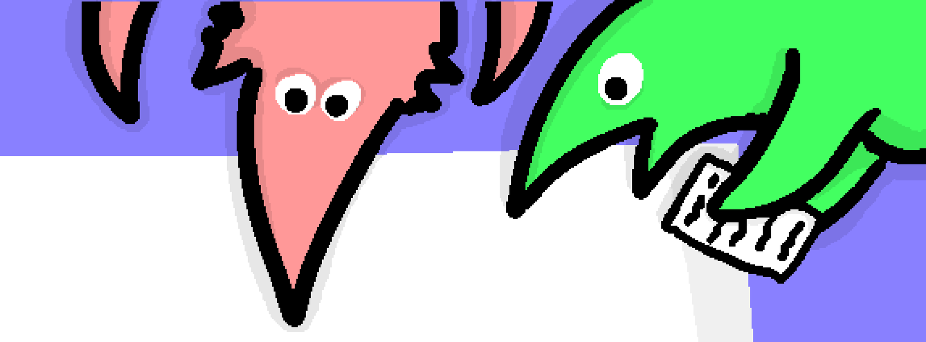
E.g.

$d=1$





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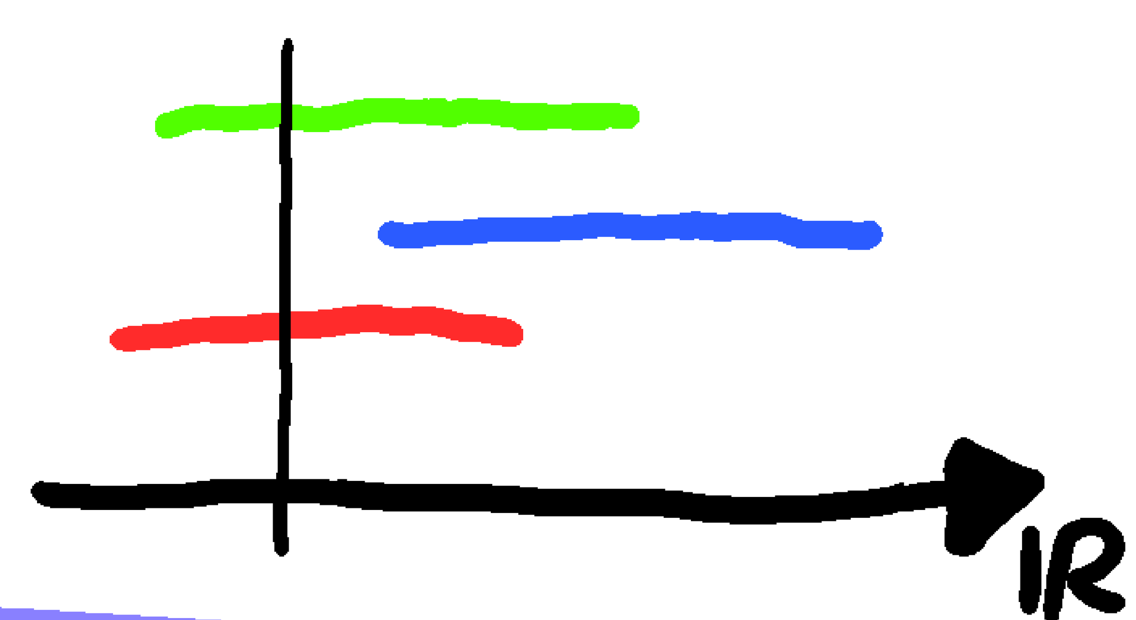

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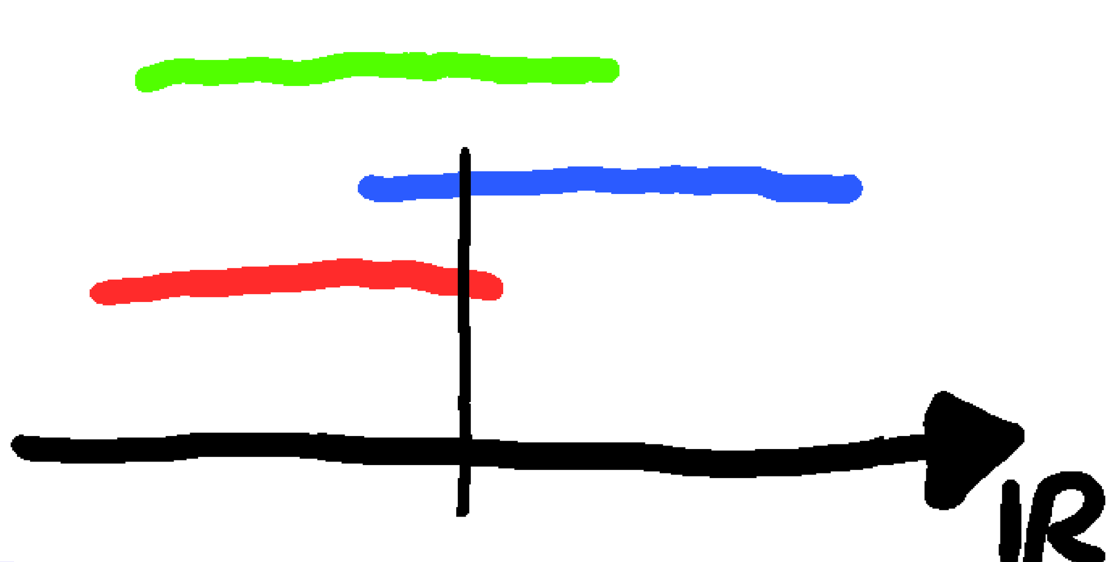

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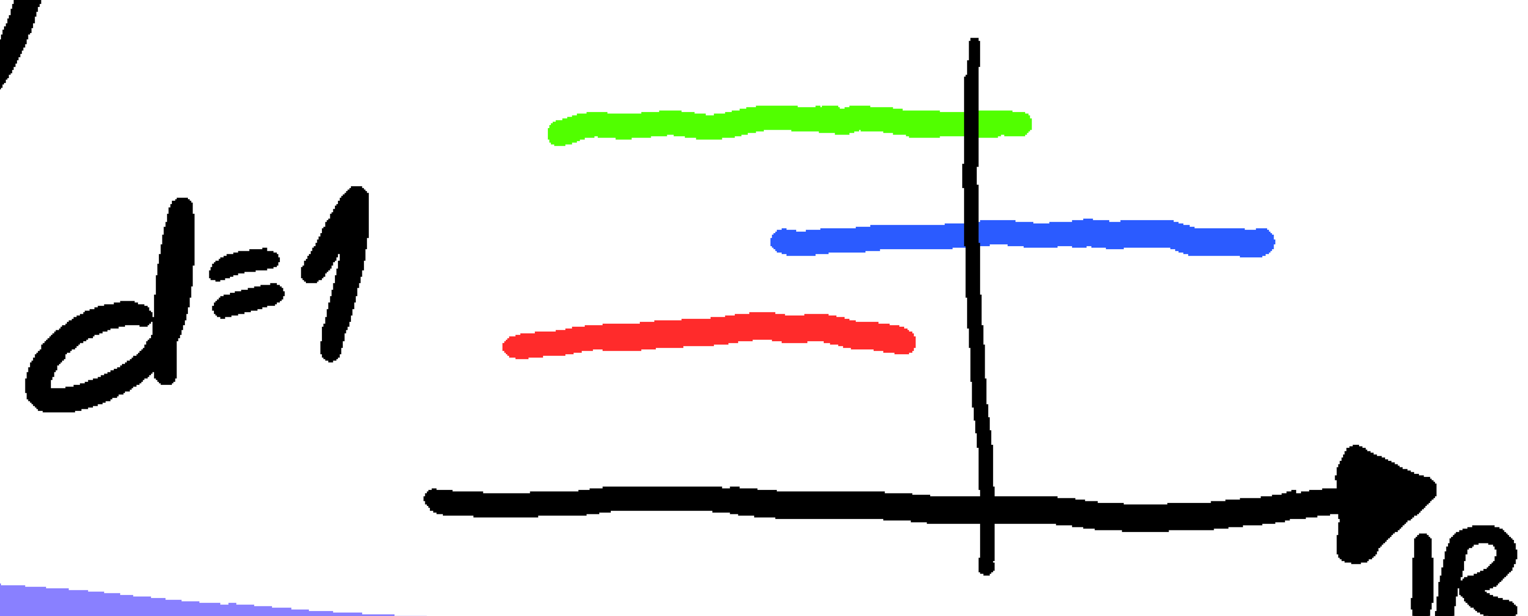

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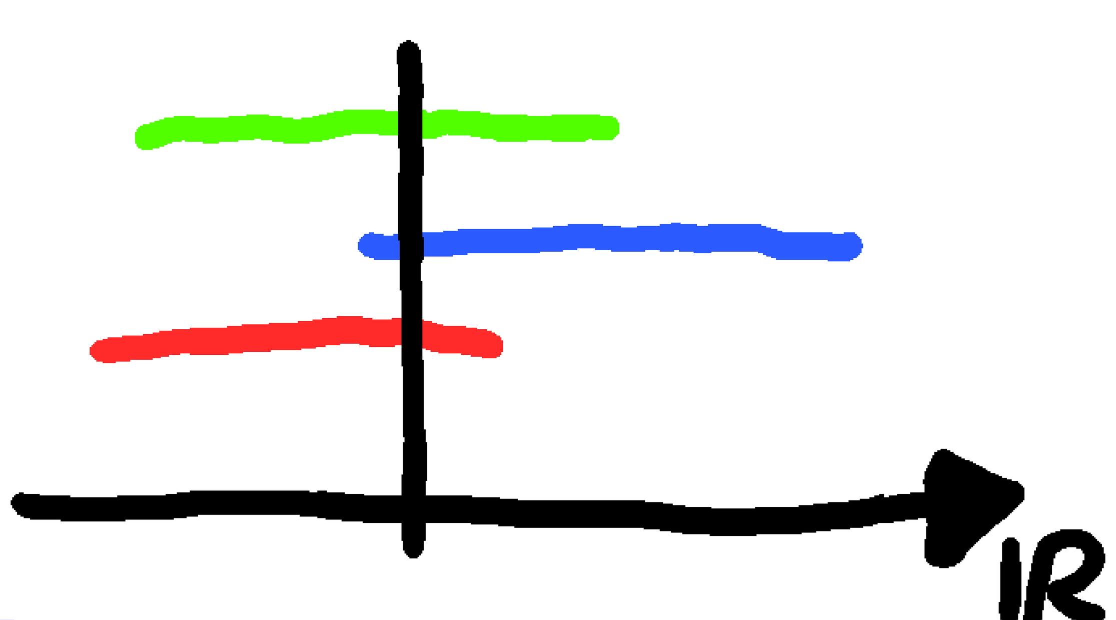

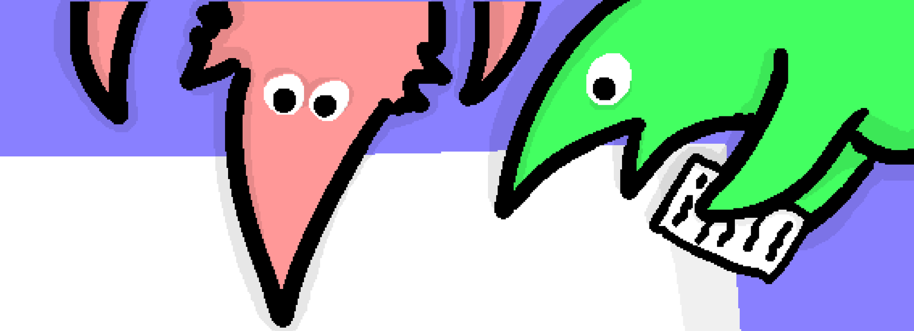

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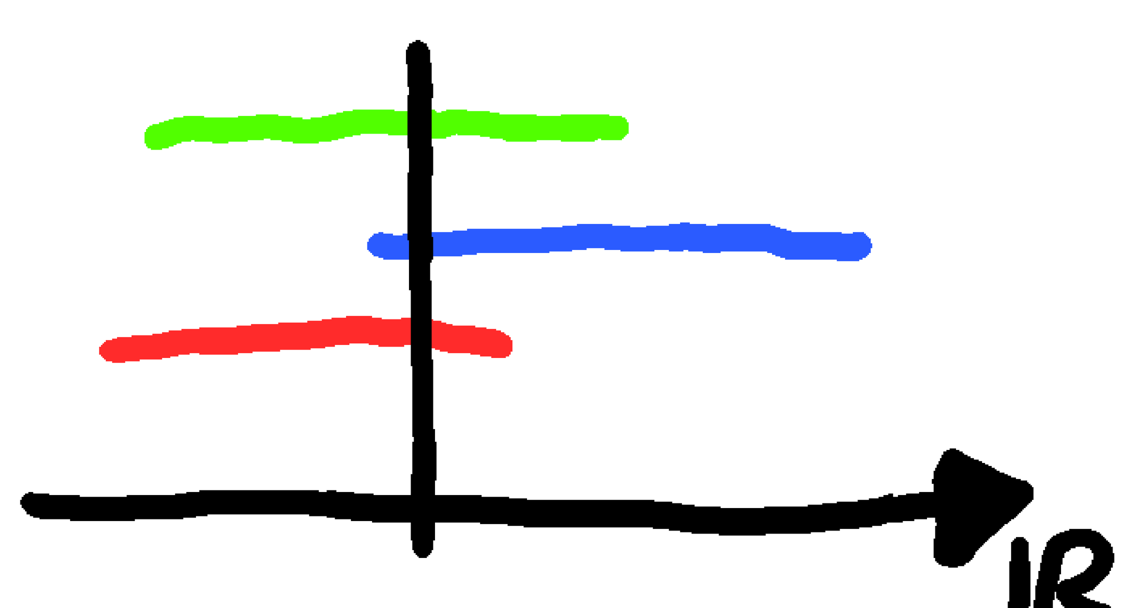

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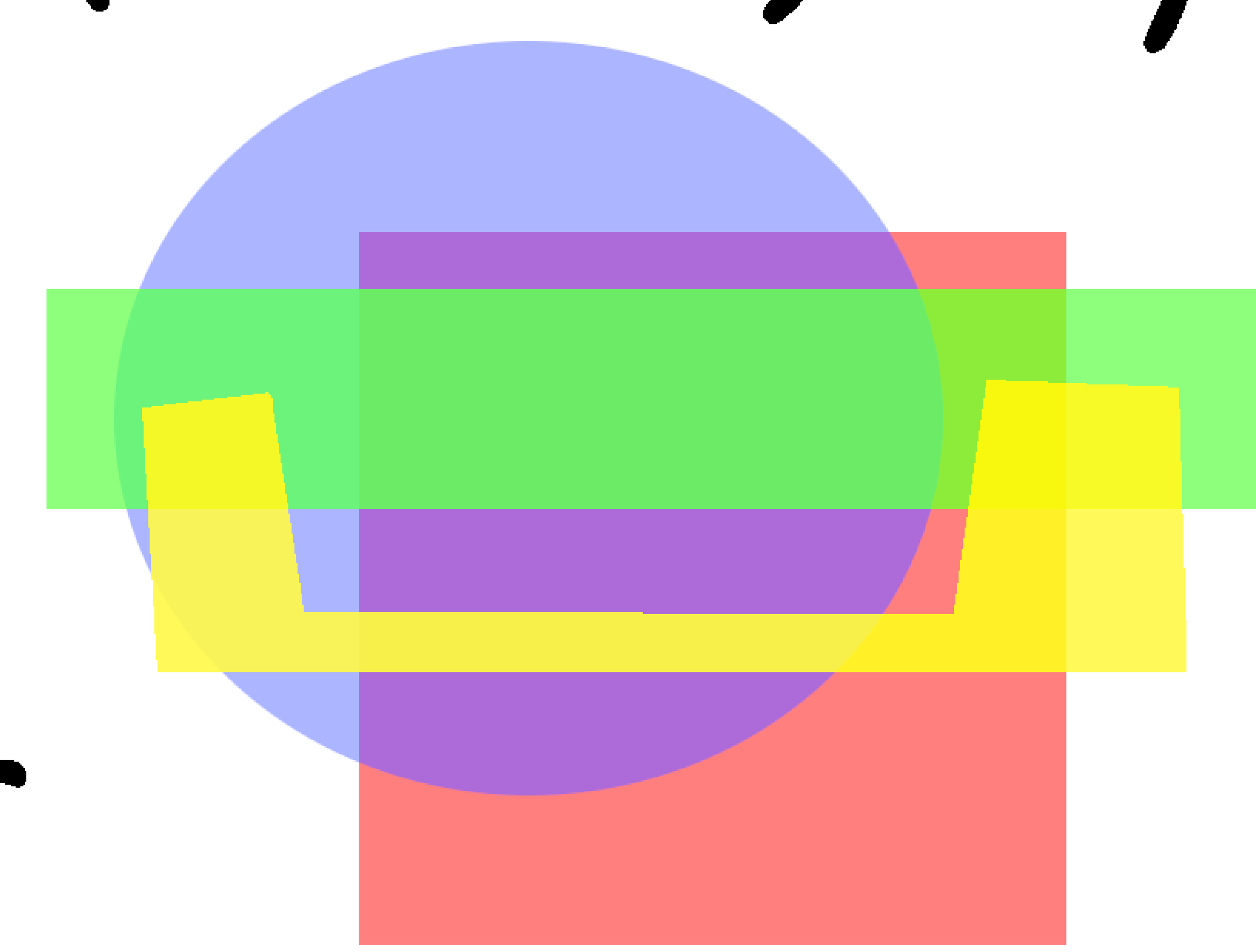

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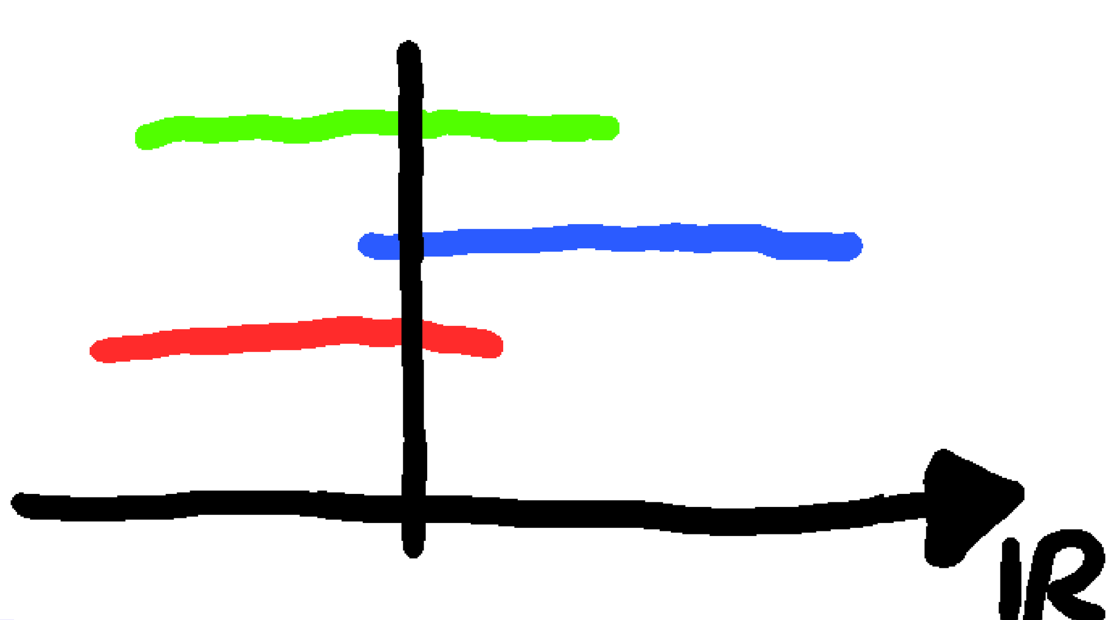
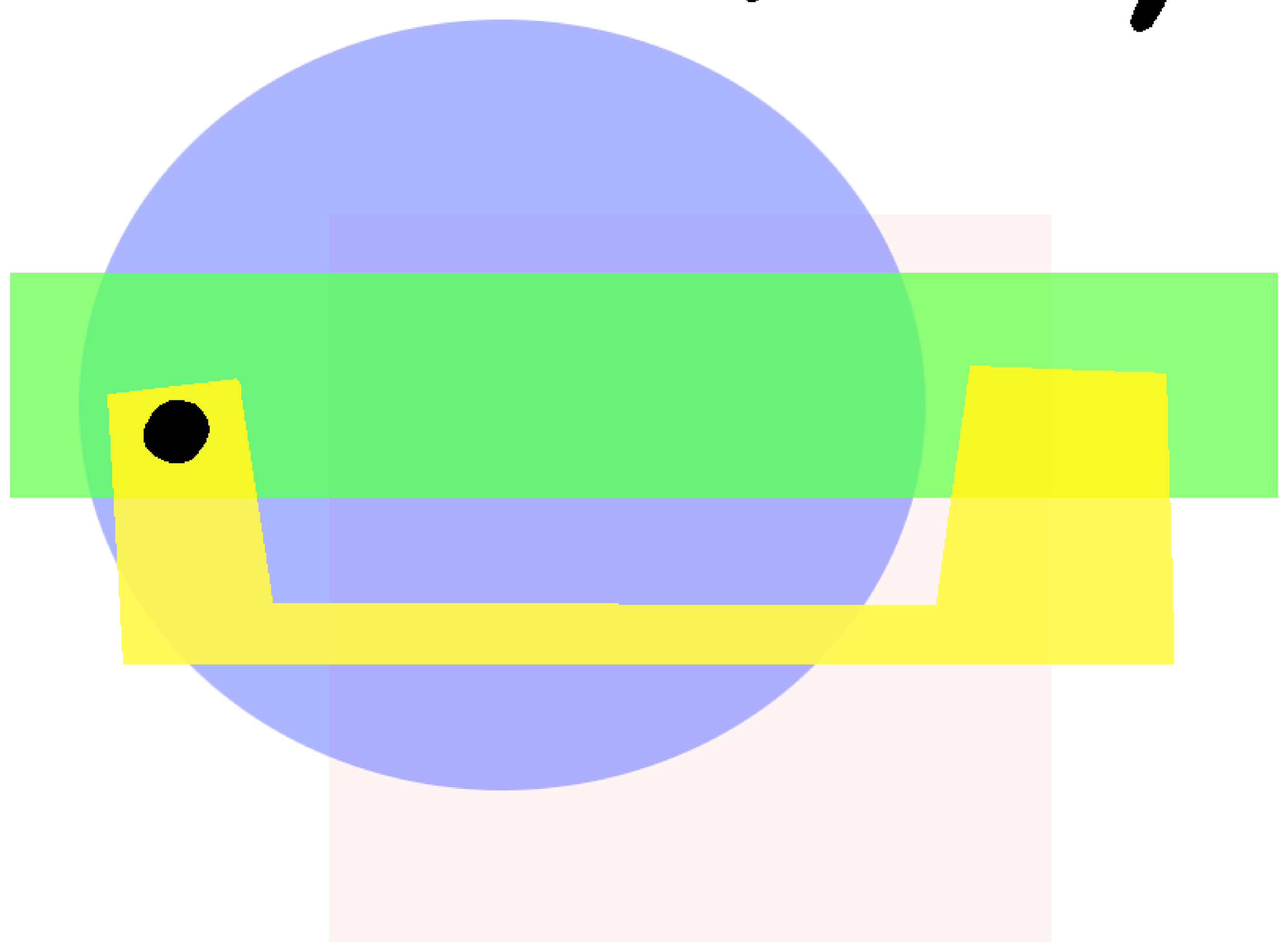
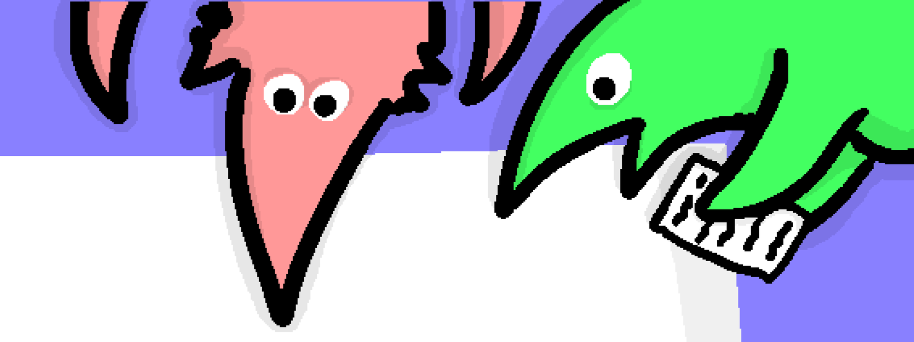
d=1 

d=2 



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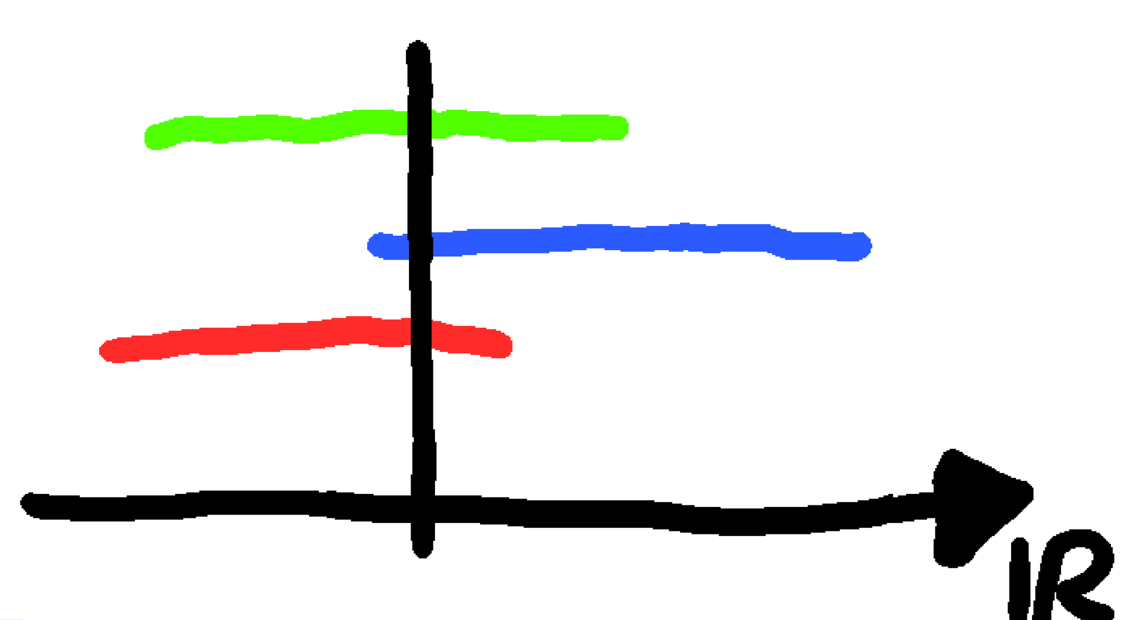

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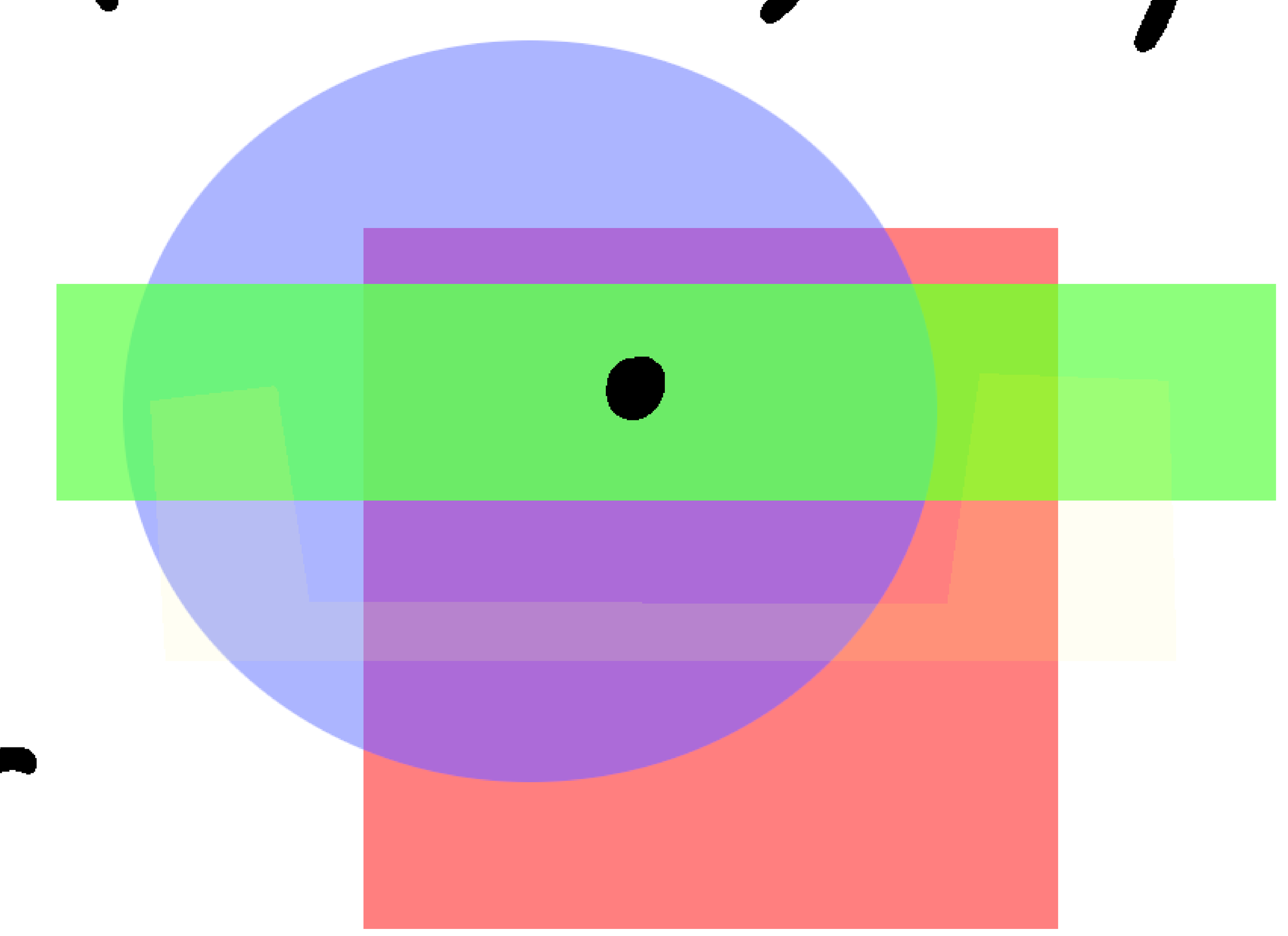

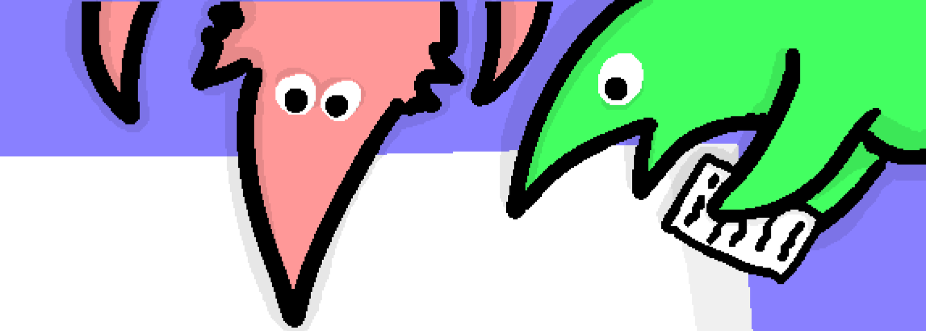

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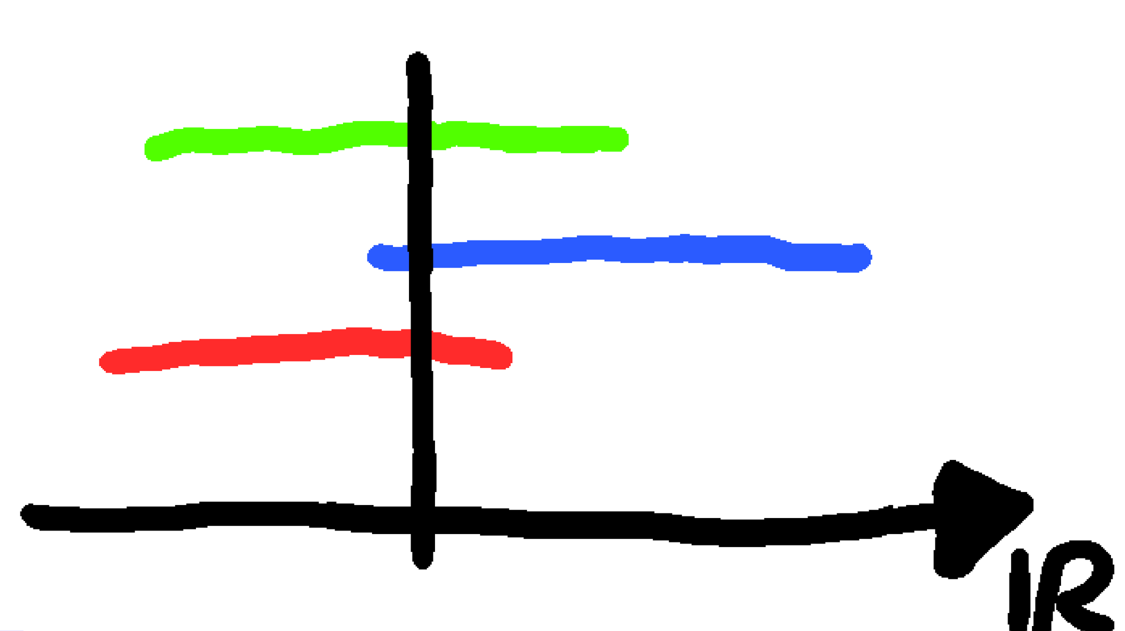

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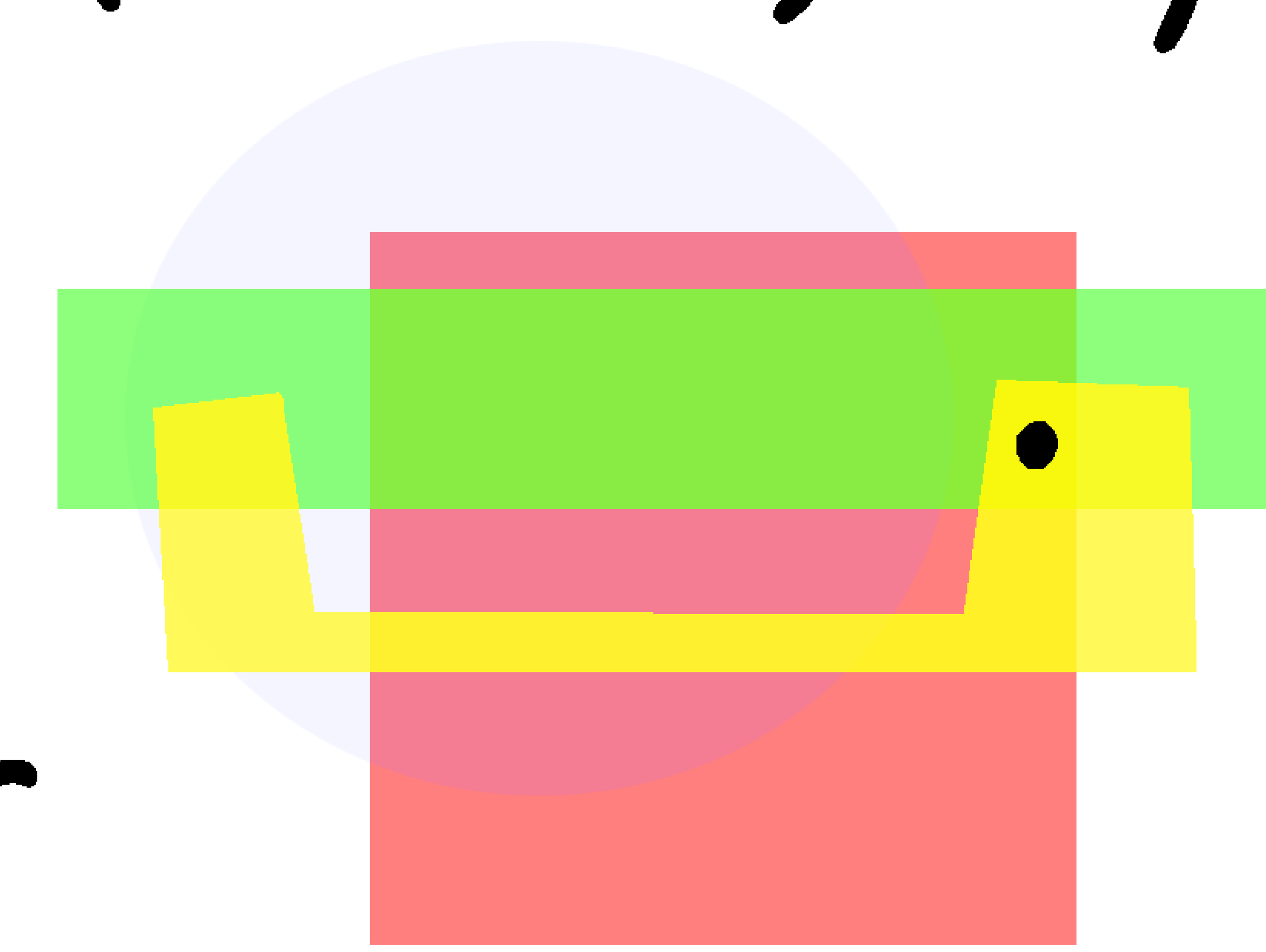

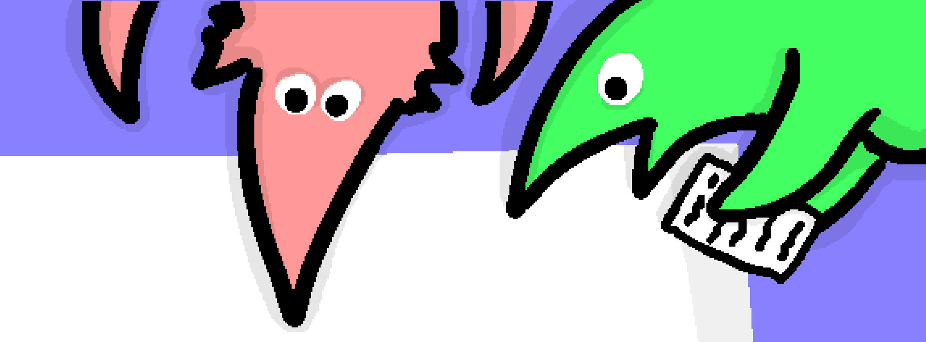

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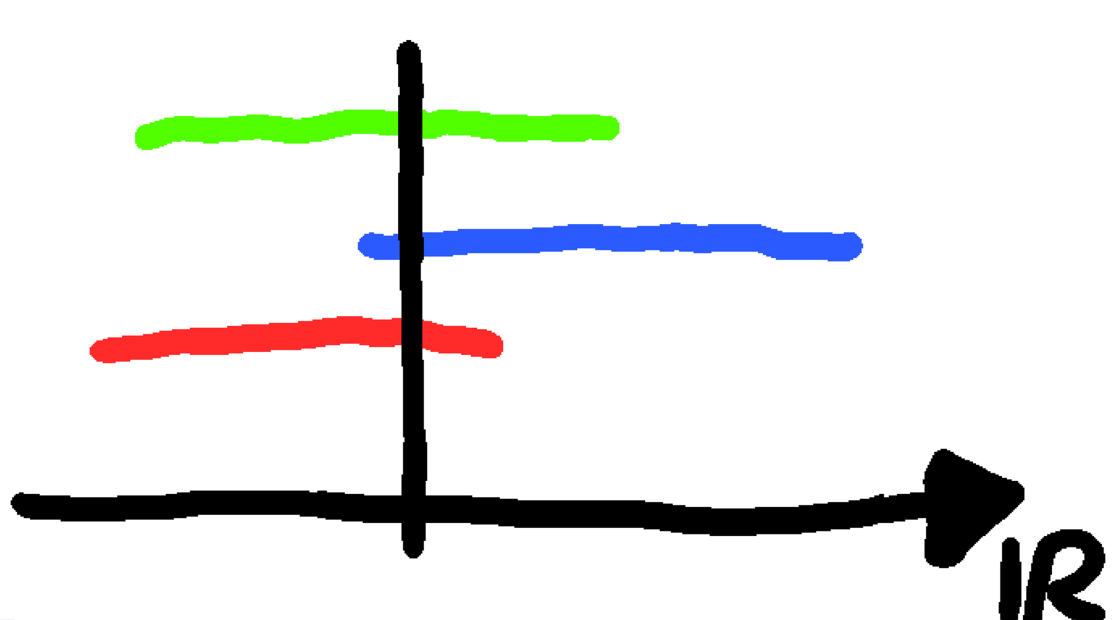

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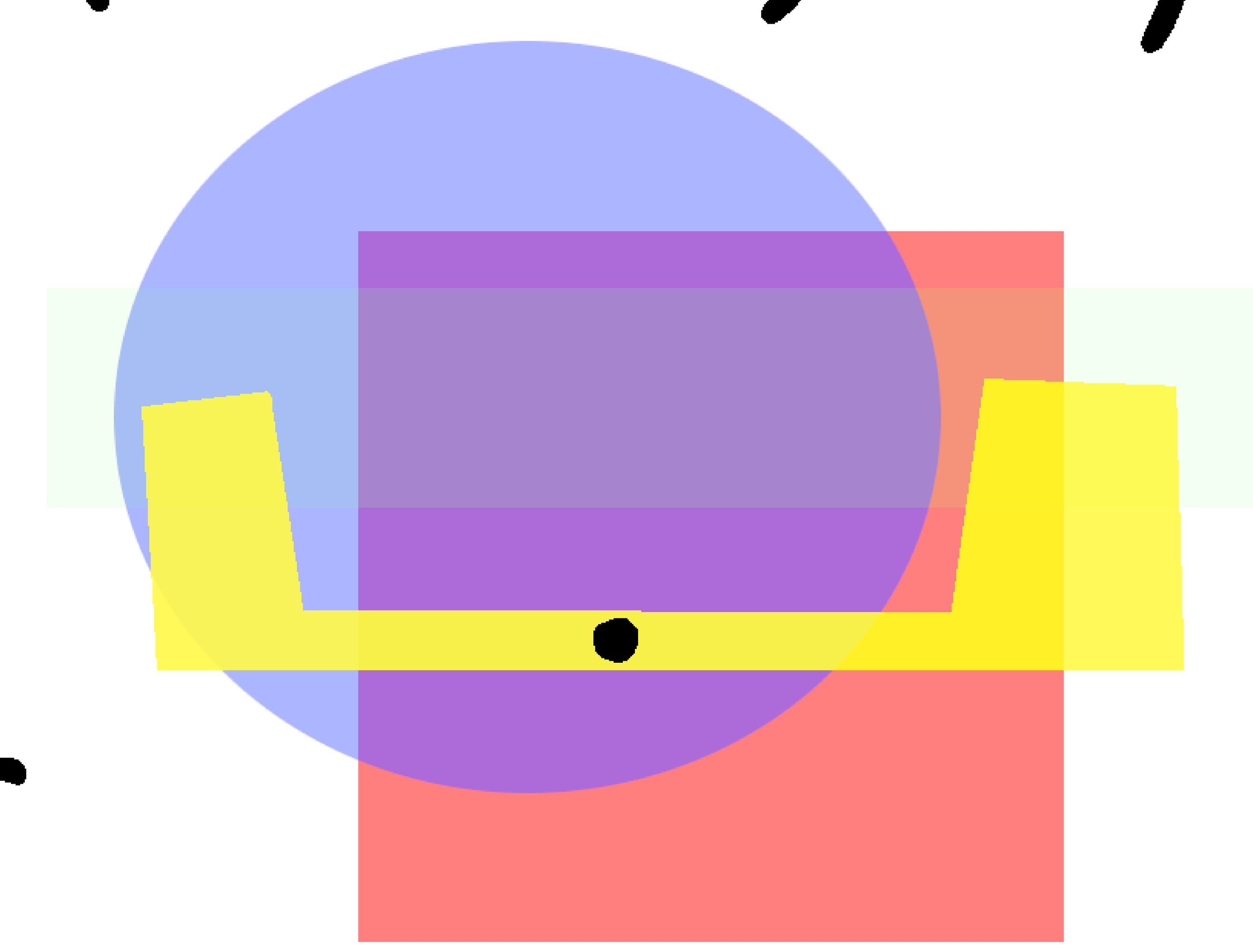

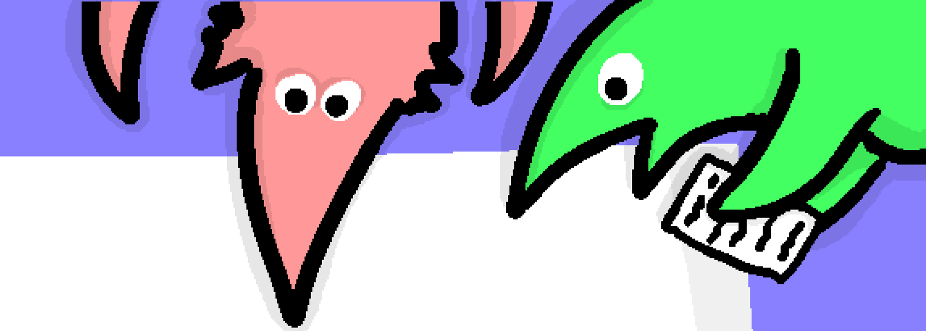

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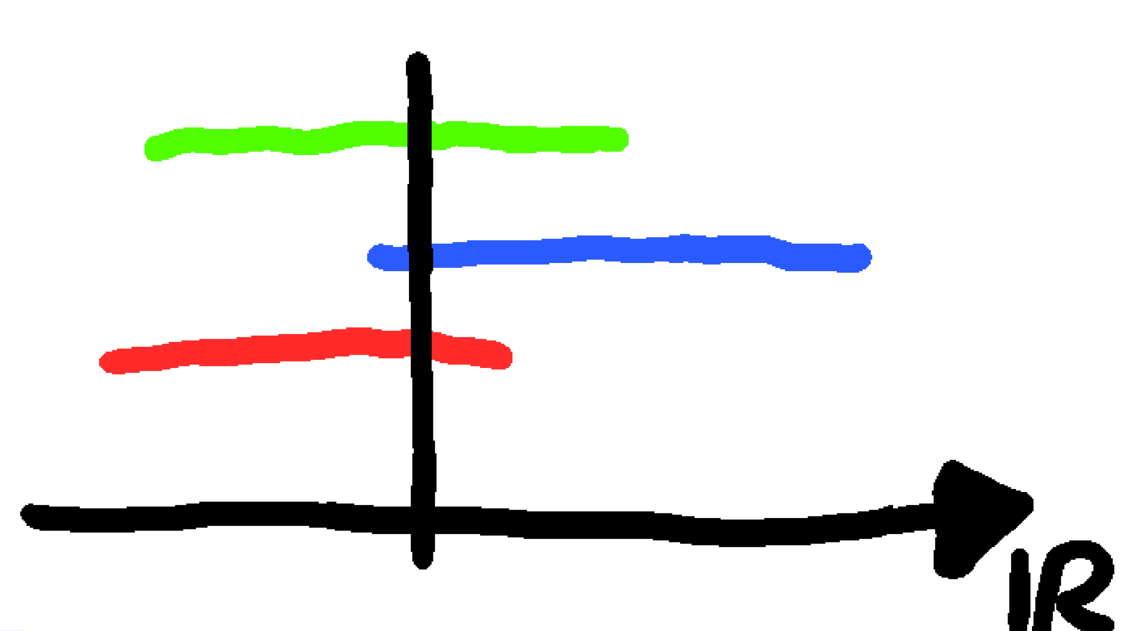
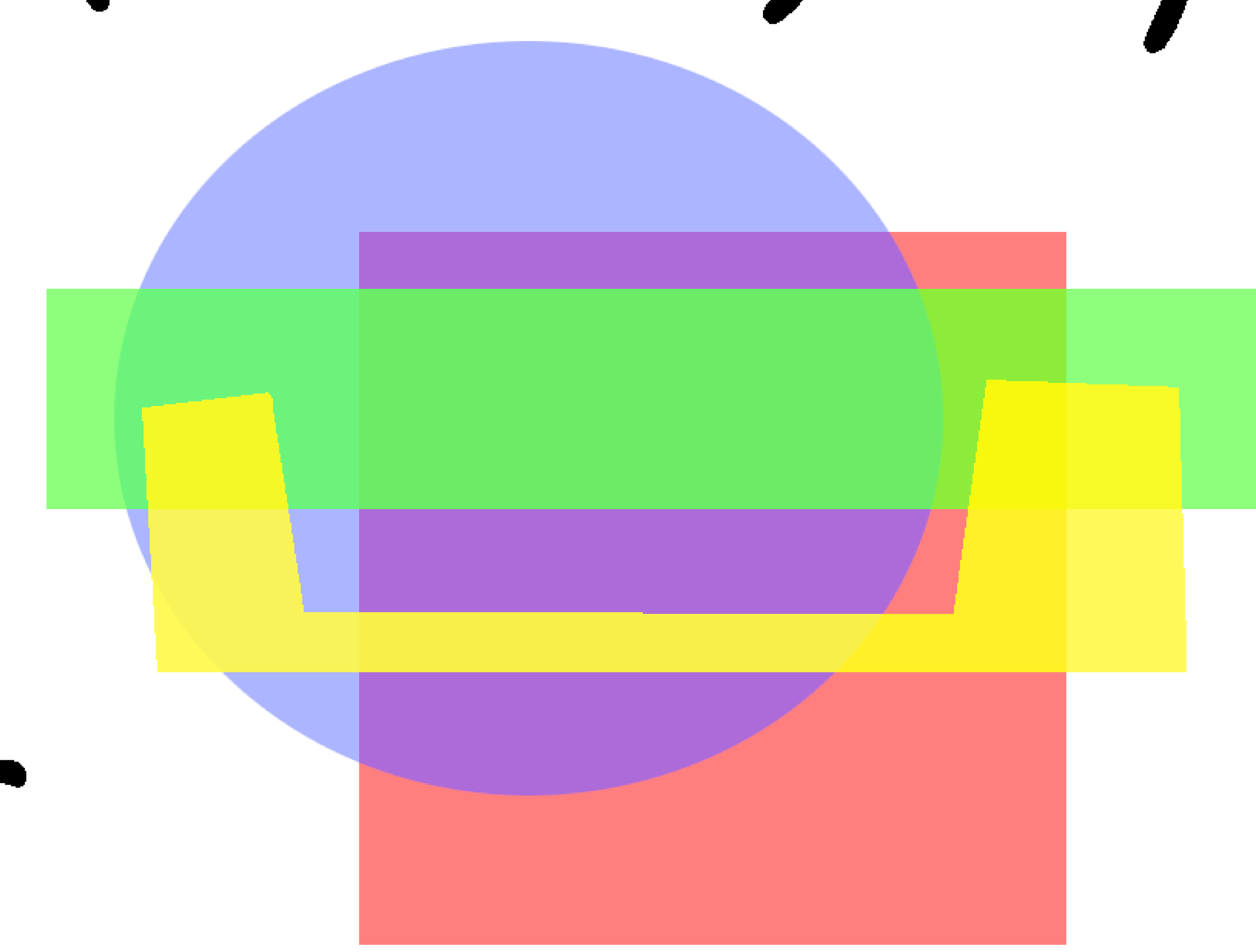
$d=1$ 

$d=2$ 

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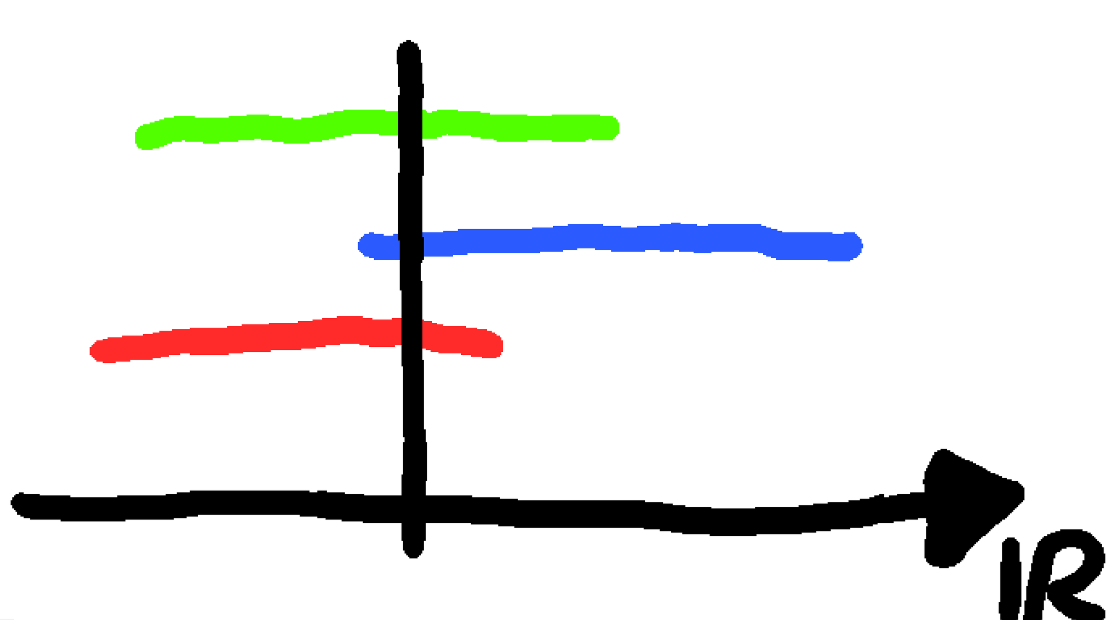

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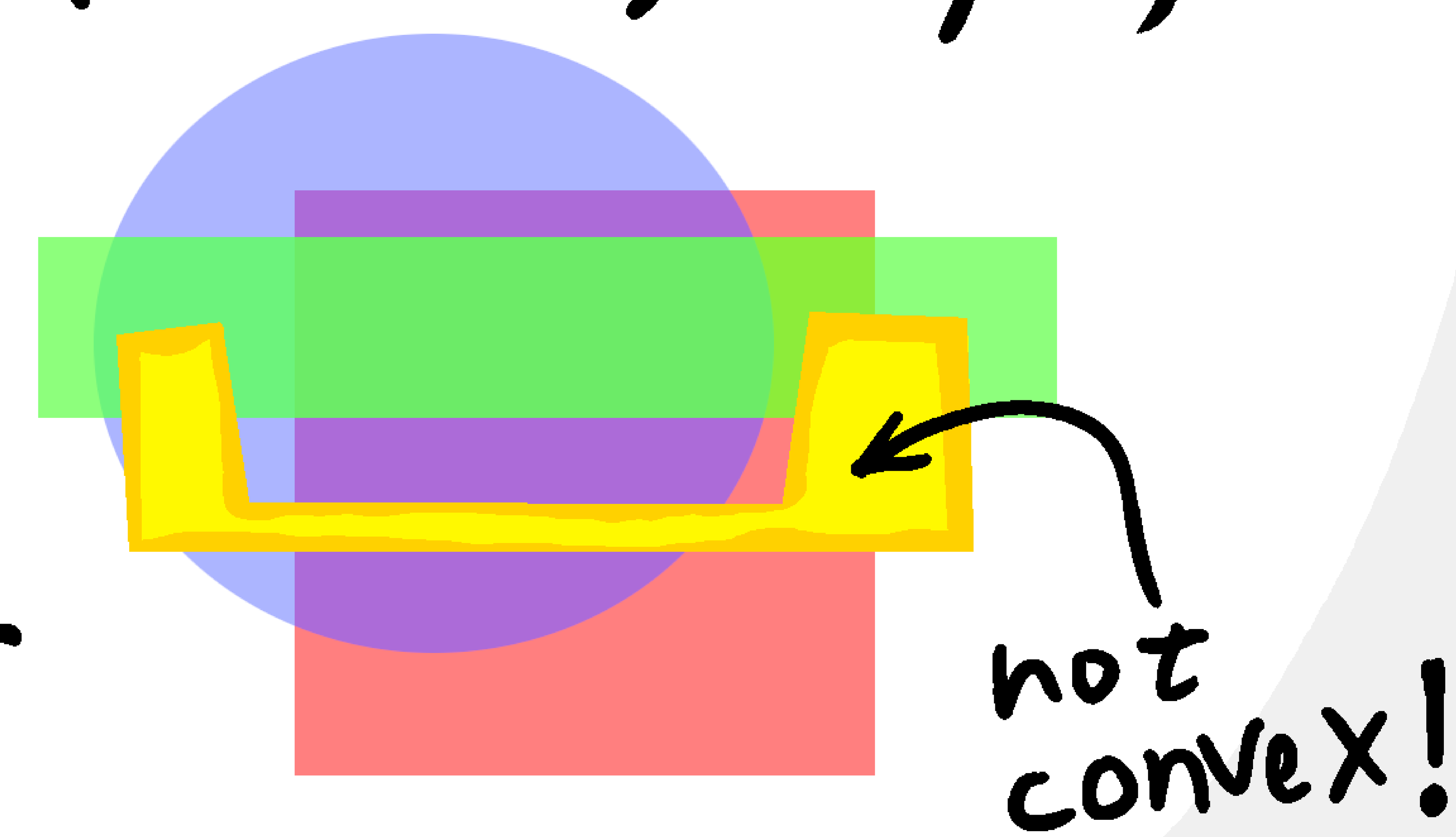

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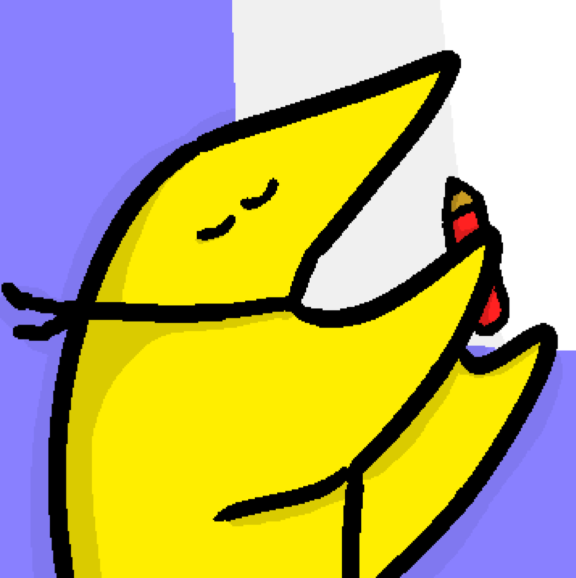

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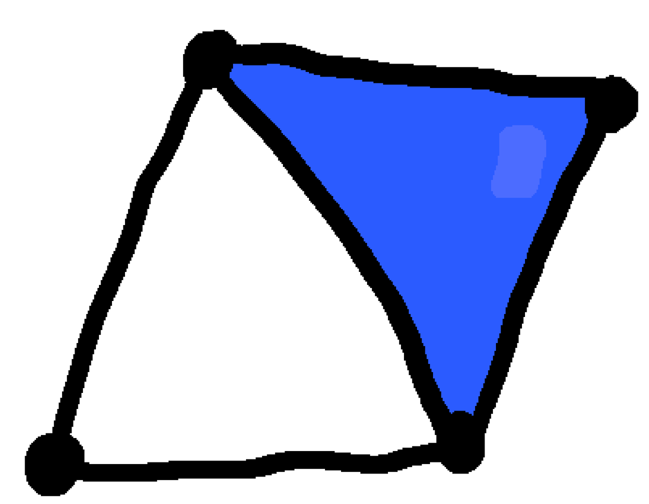
$K =$ 

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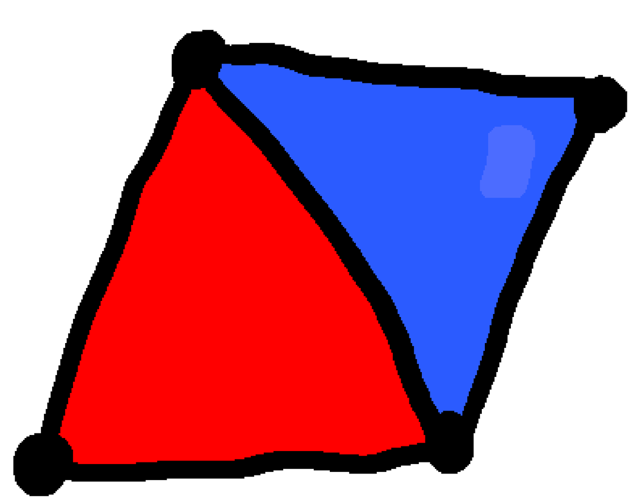
$K =$ 

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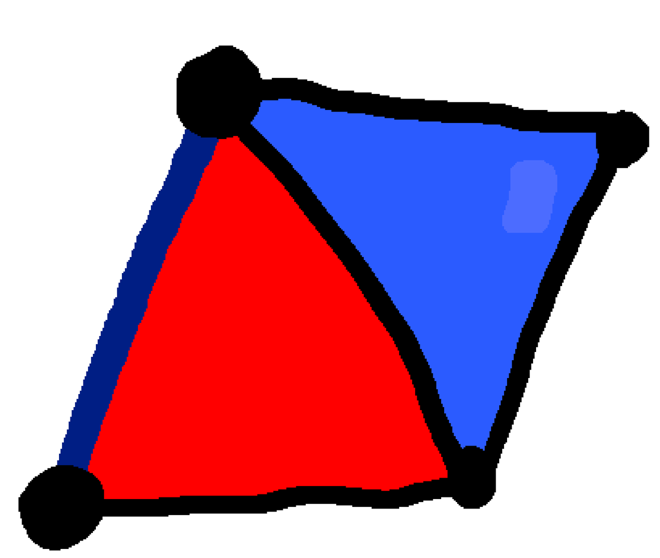
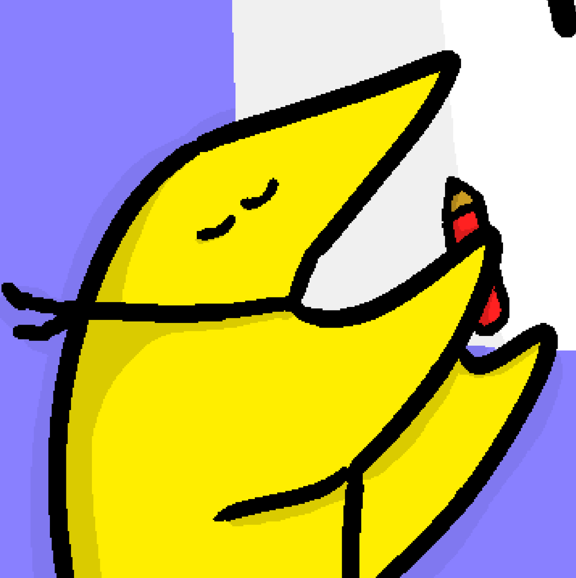
$K =$ 



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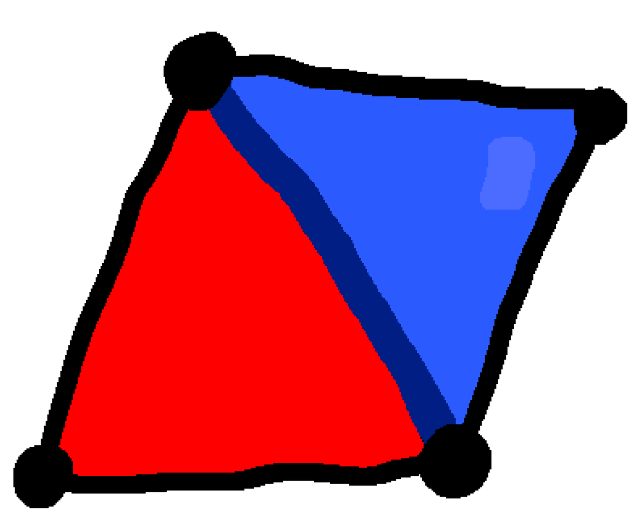
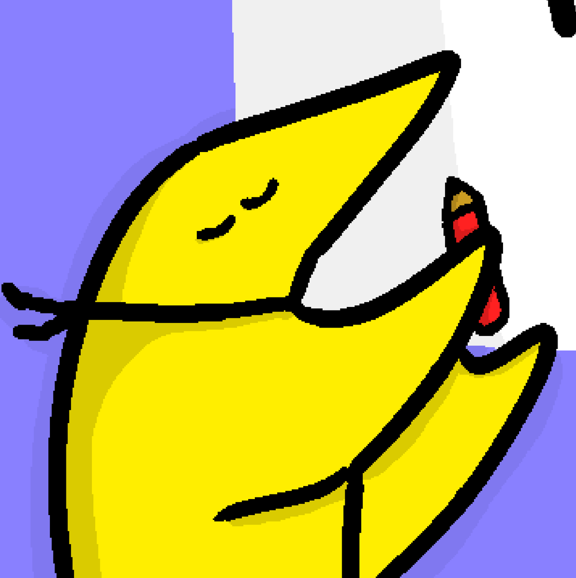
$K =$ 



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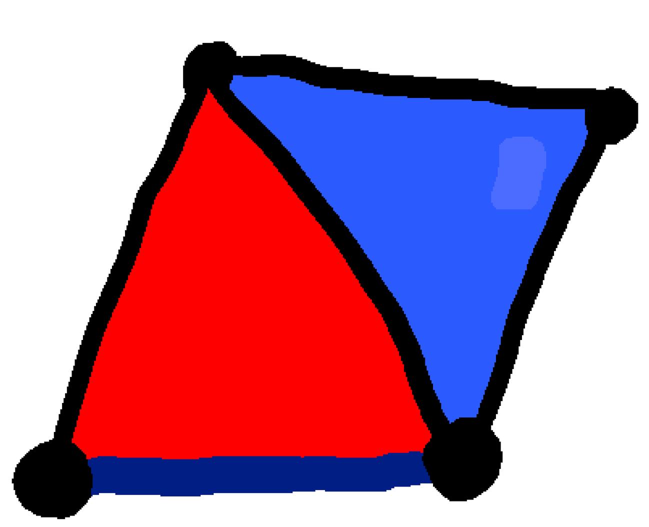
$K =$ 

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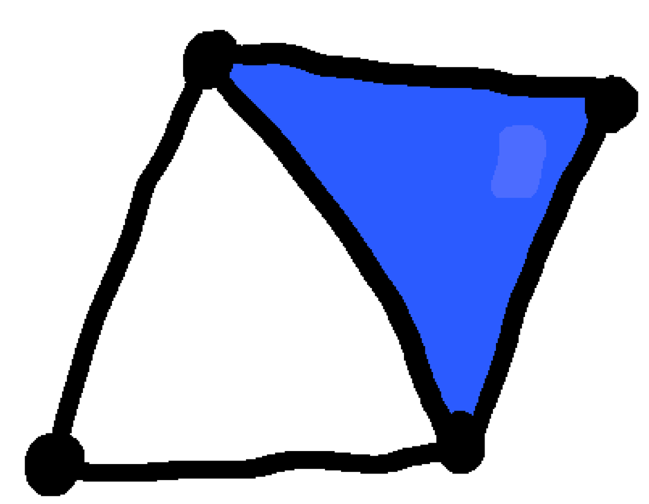
$K =$ 

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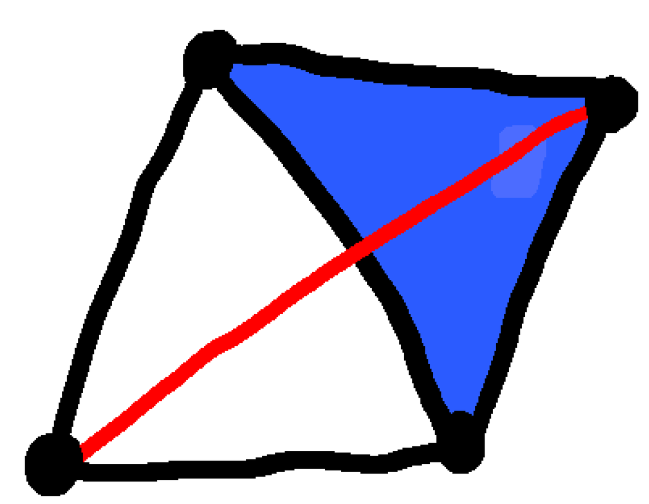
$K =$ 

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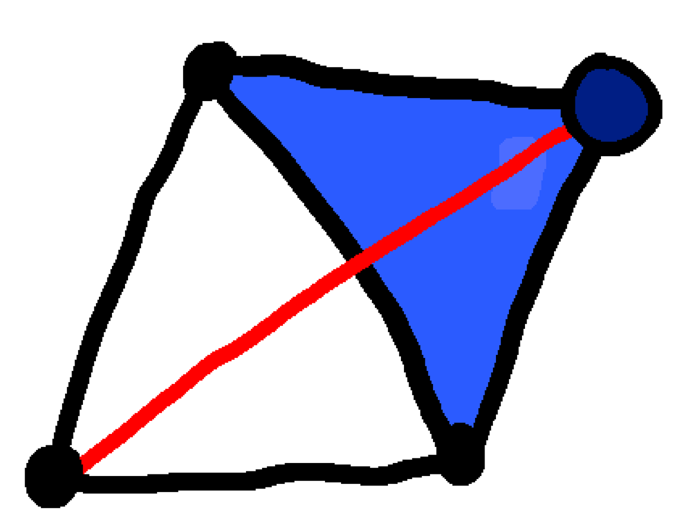
$K =$ 

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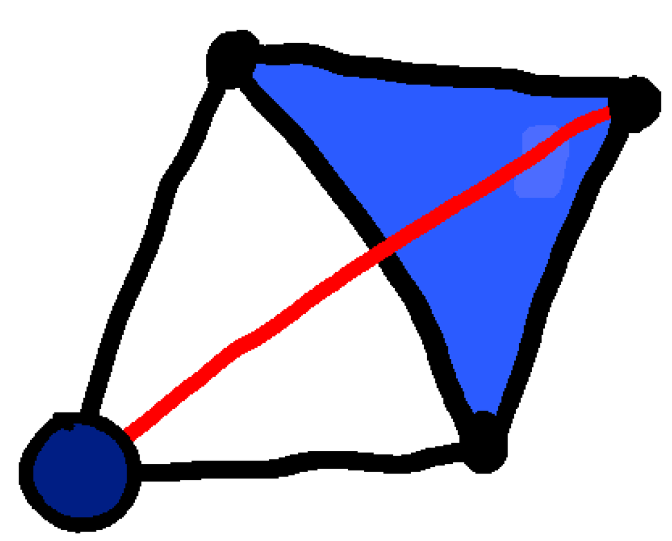
$K =$ 

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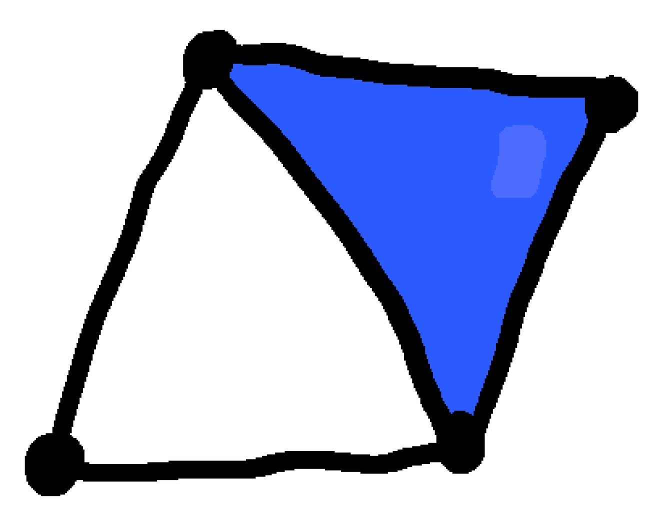
$K =$ 

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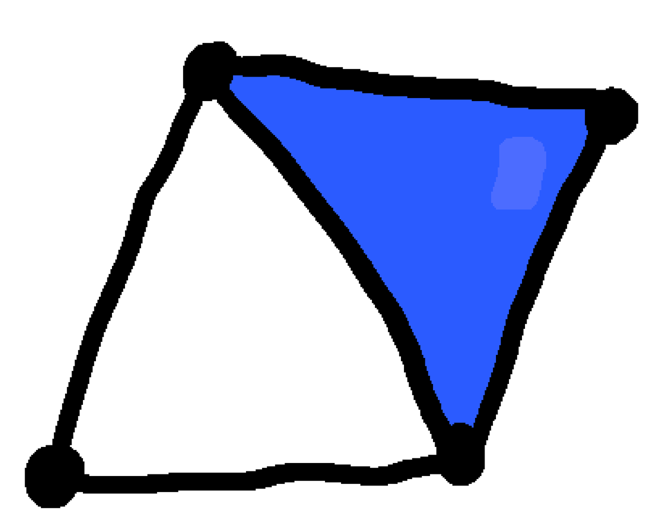
$K =$ 

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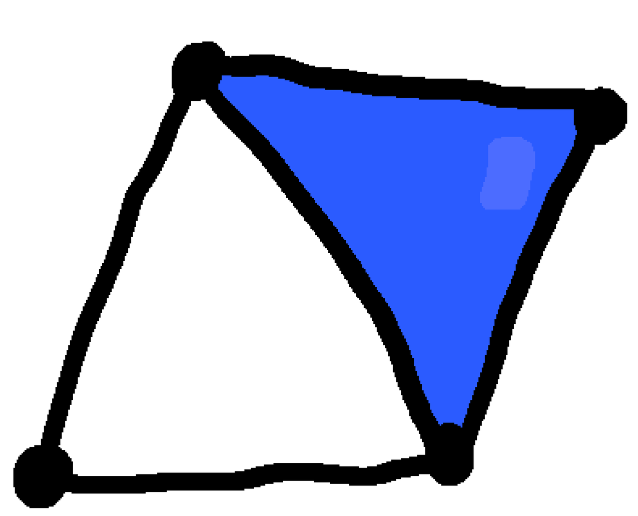
$K =$ 

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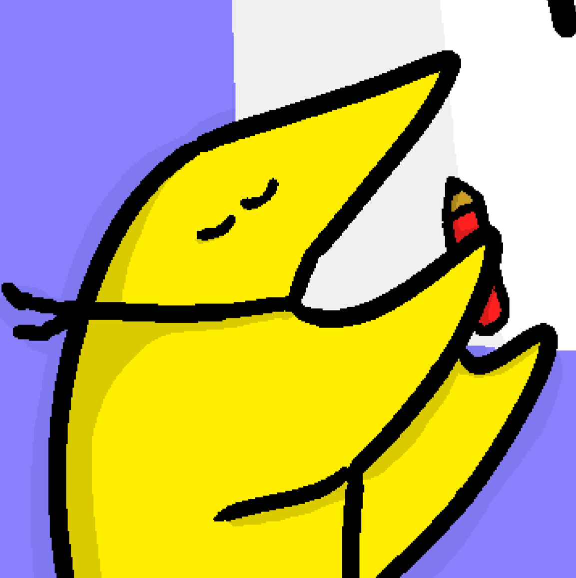

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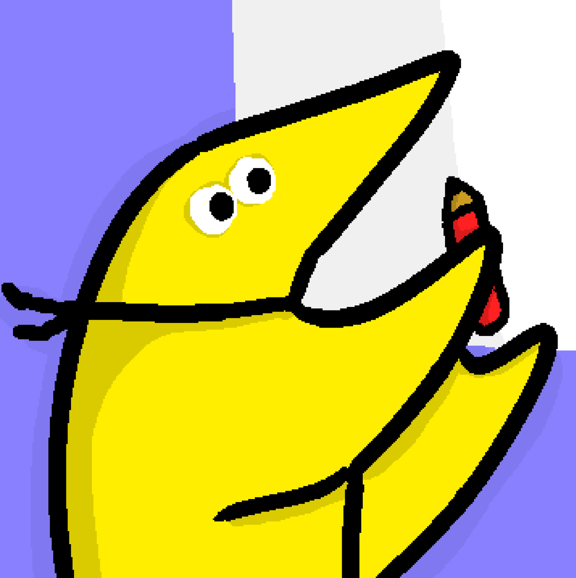

Helly for d-Leray complexes

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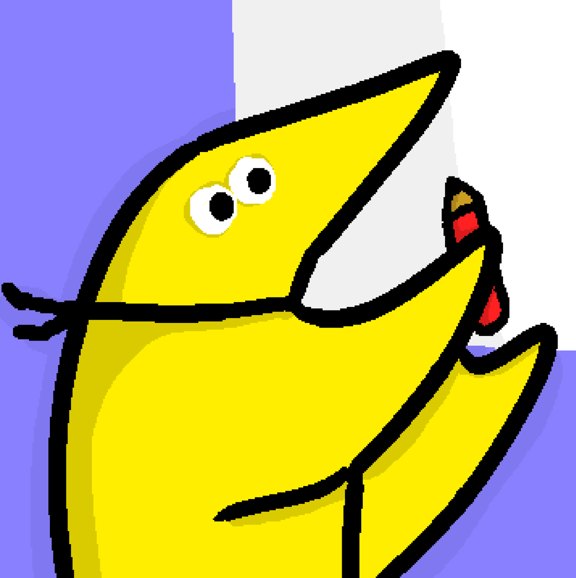

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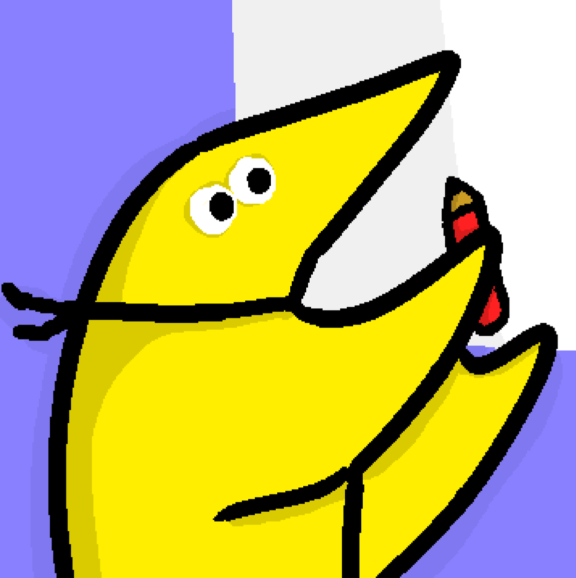

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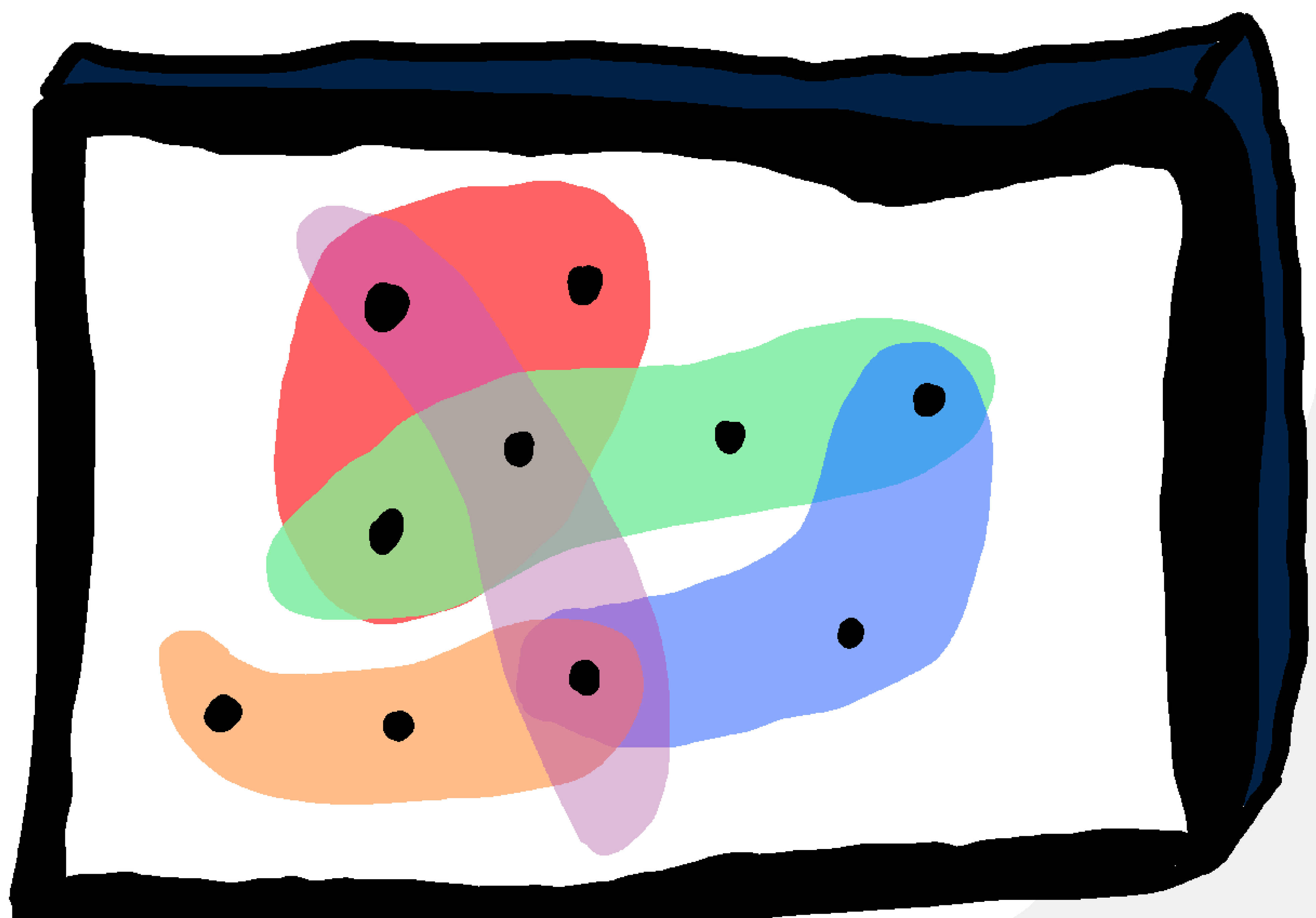

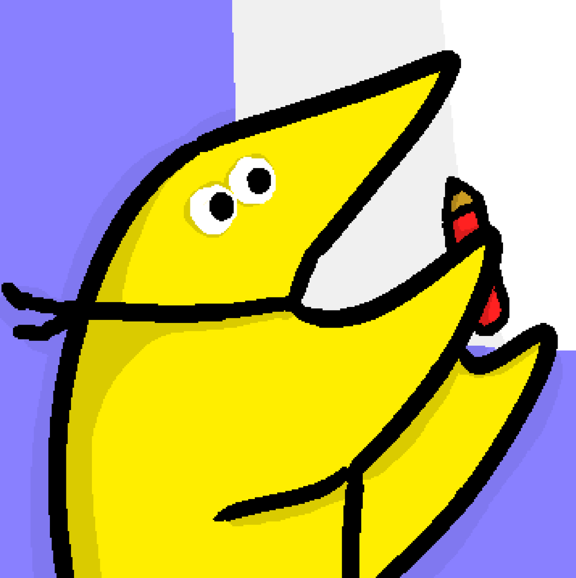

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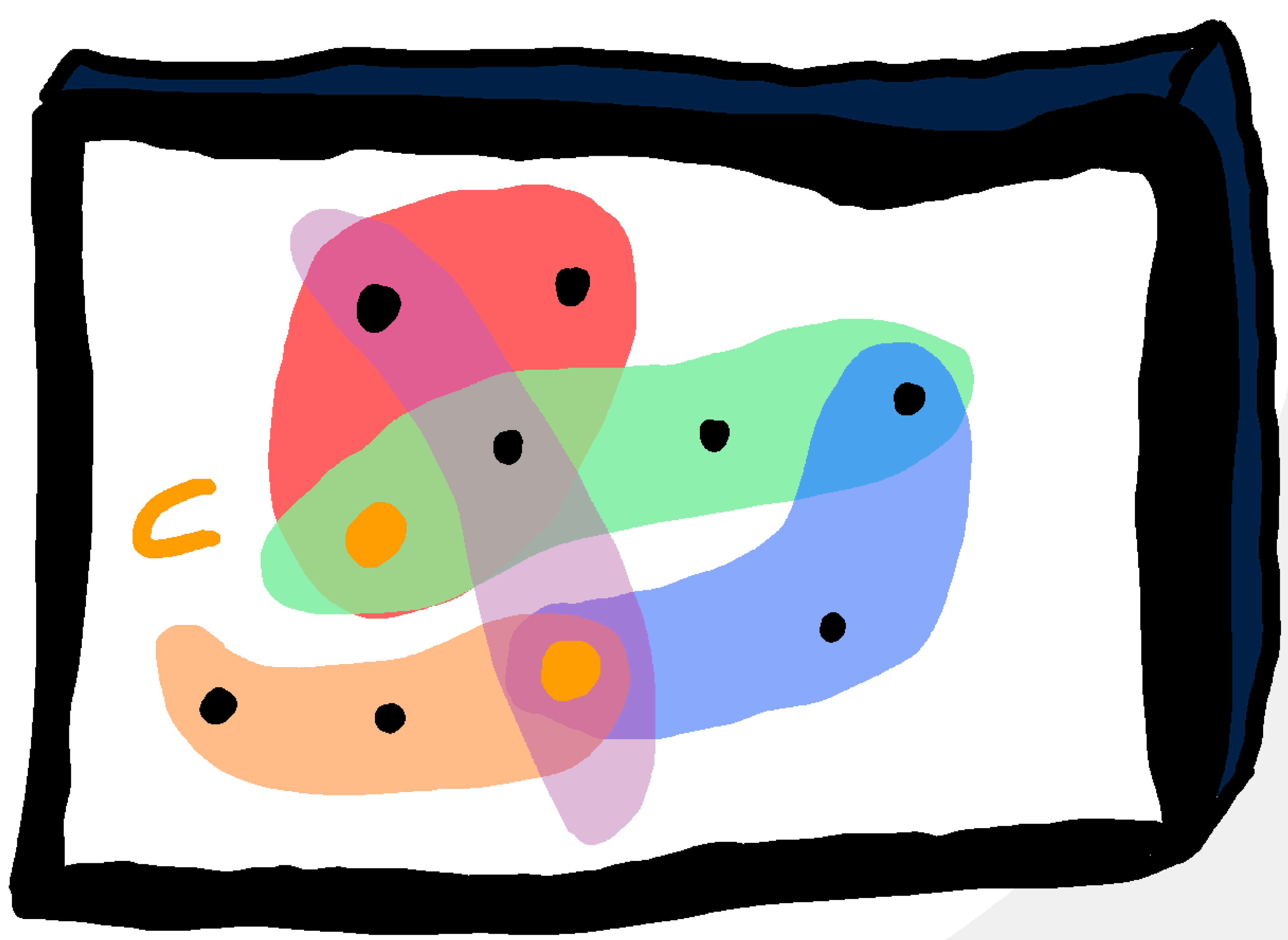

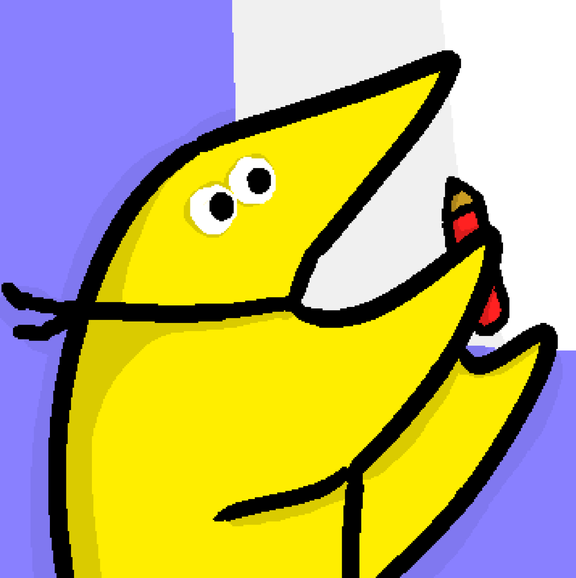

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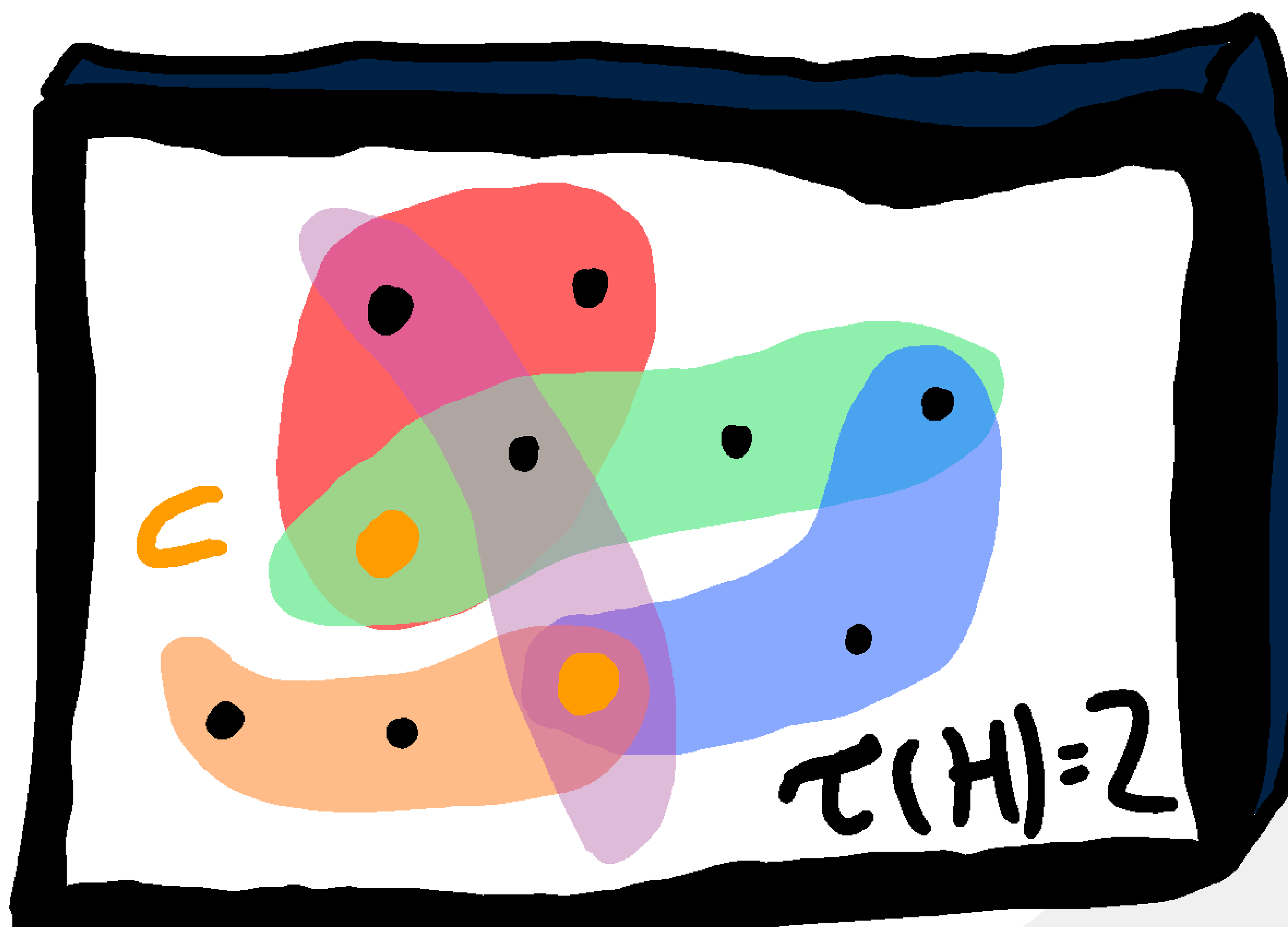

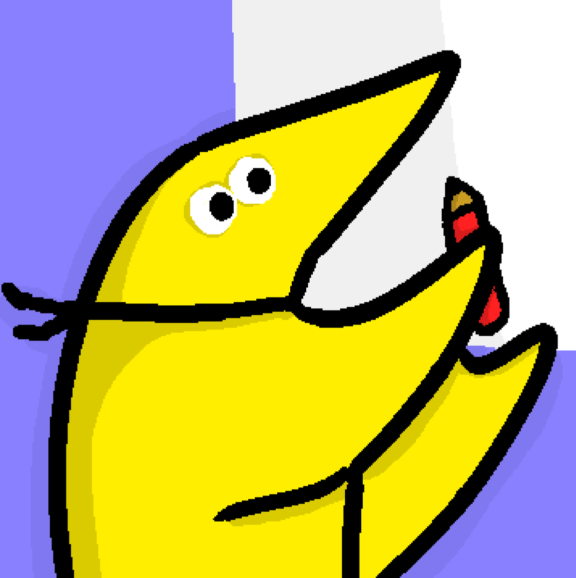

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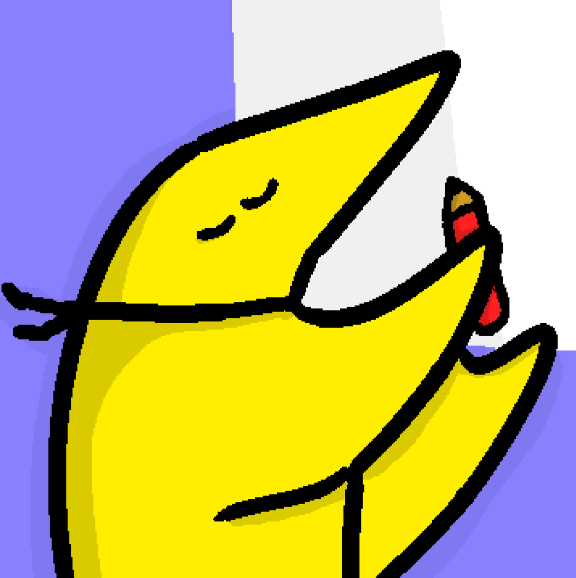

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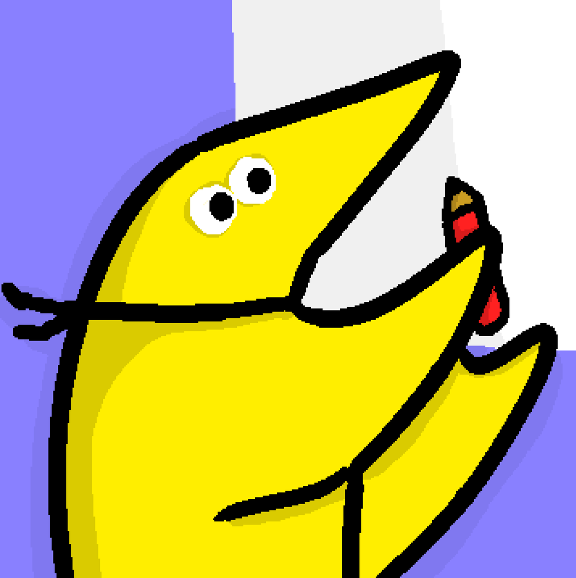
$\eta(r,t) = O(r^t)$ for fixed t

$\eta(r,t) = O(t^{r-1})$ for fixed r

Helly with tolerance

- $\mathcal{F}$ = family of sets.



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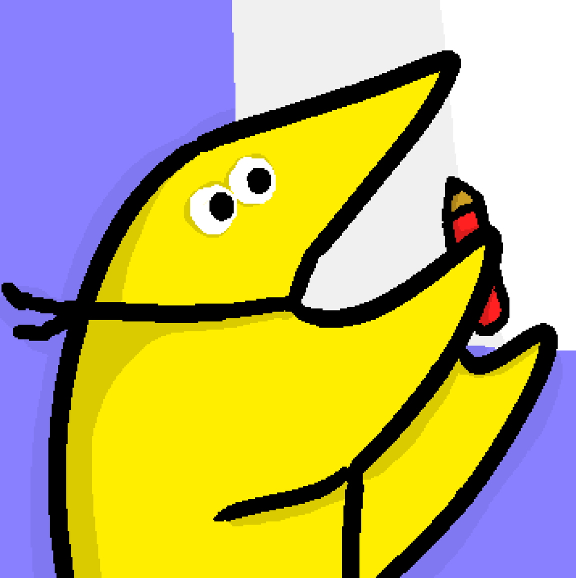
- F = family of sets.
- F has a point in common with tolerance t if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.



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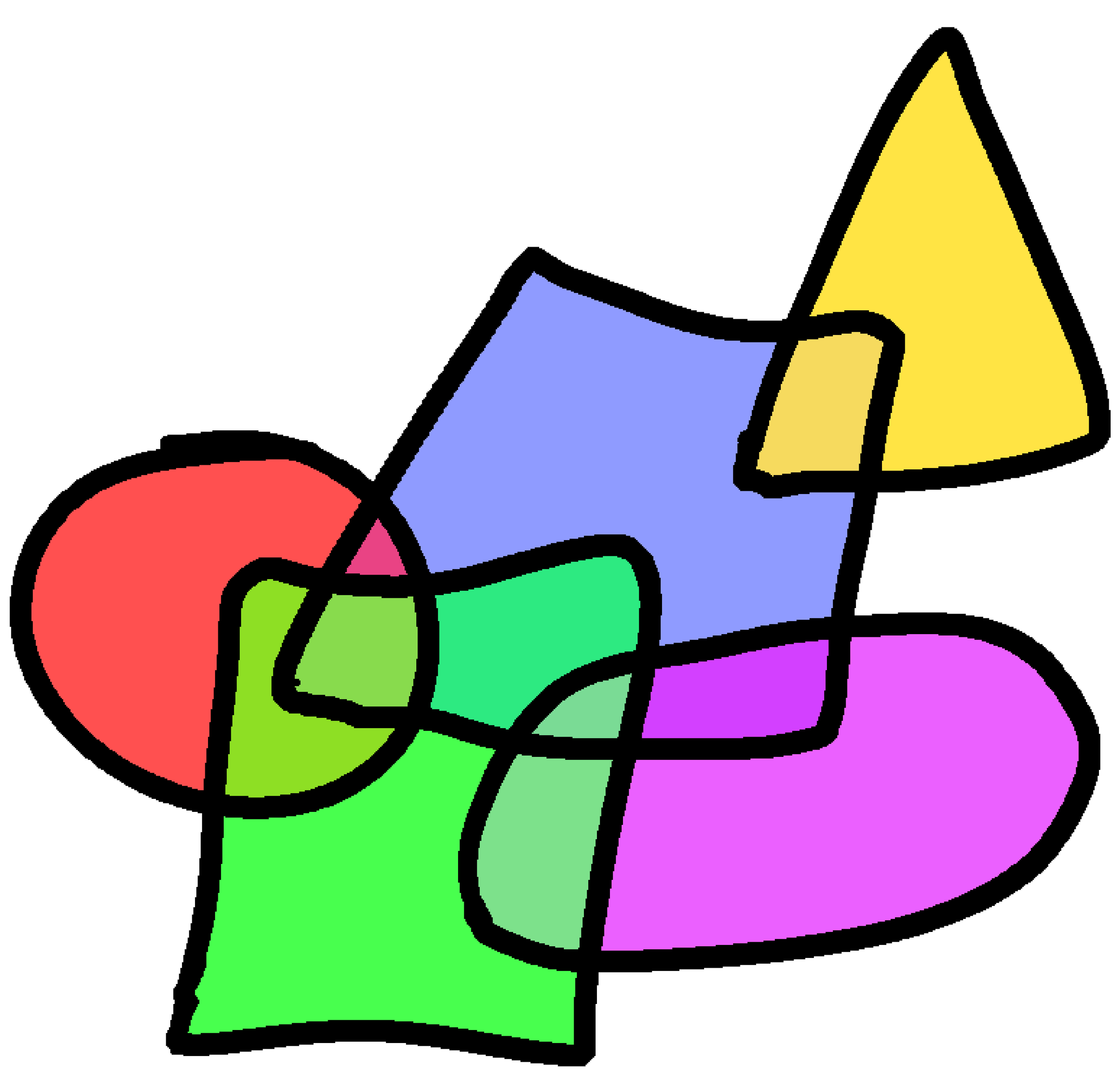
$F =$ 

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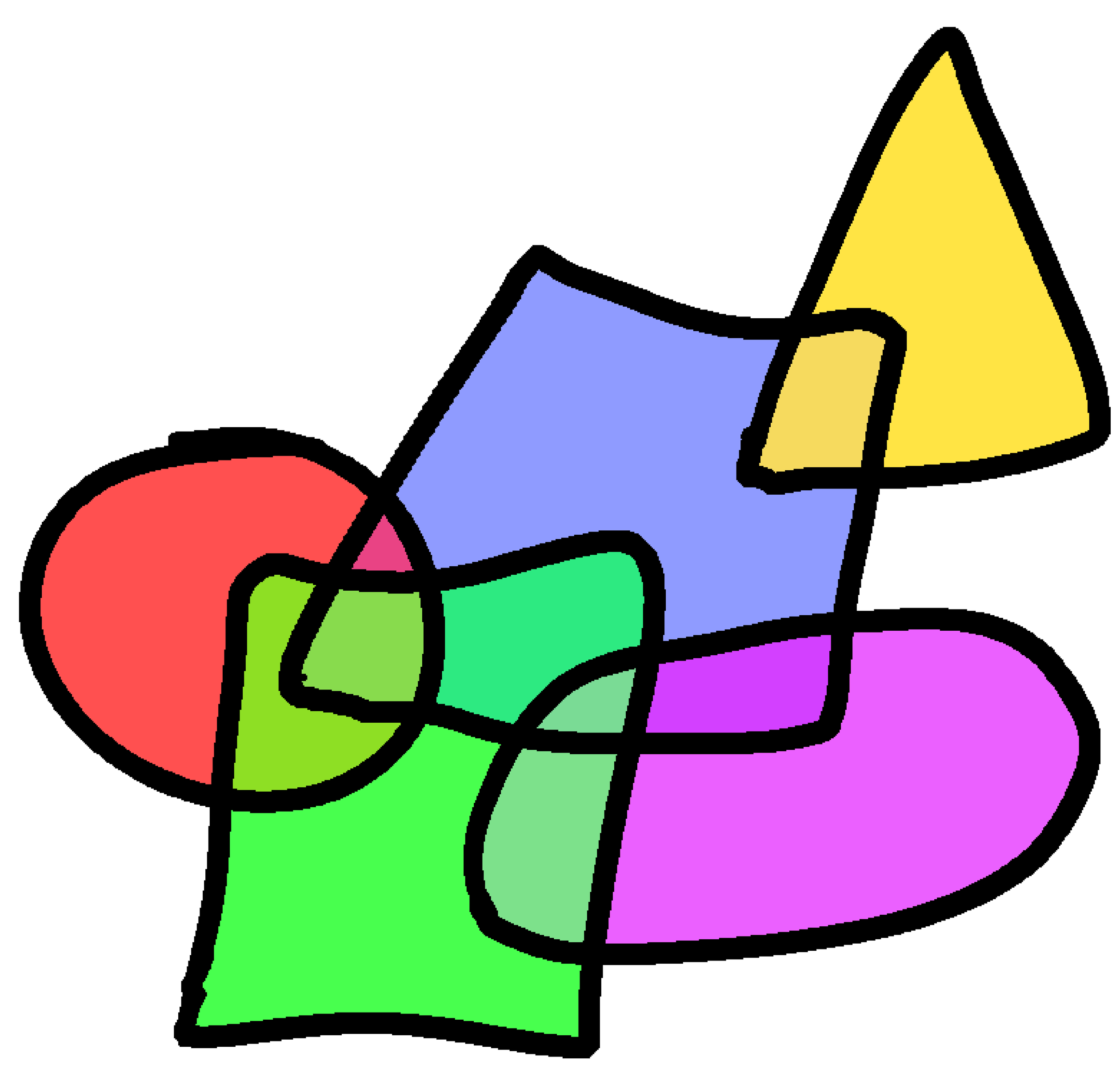
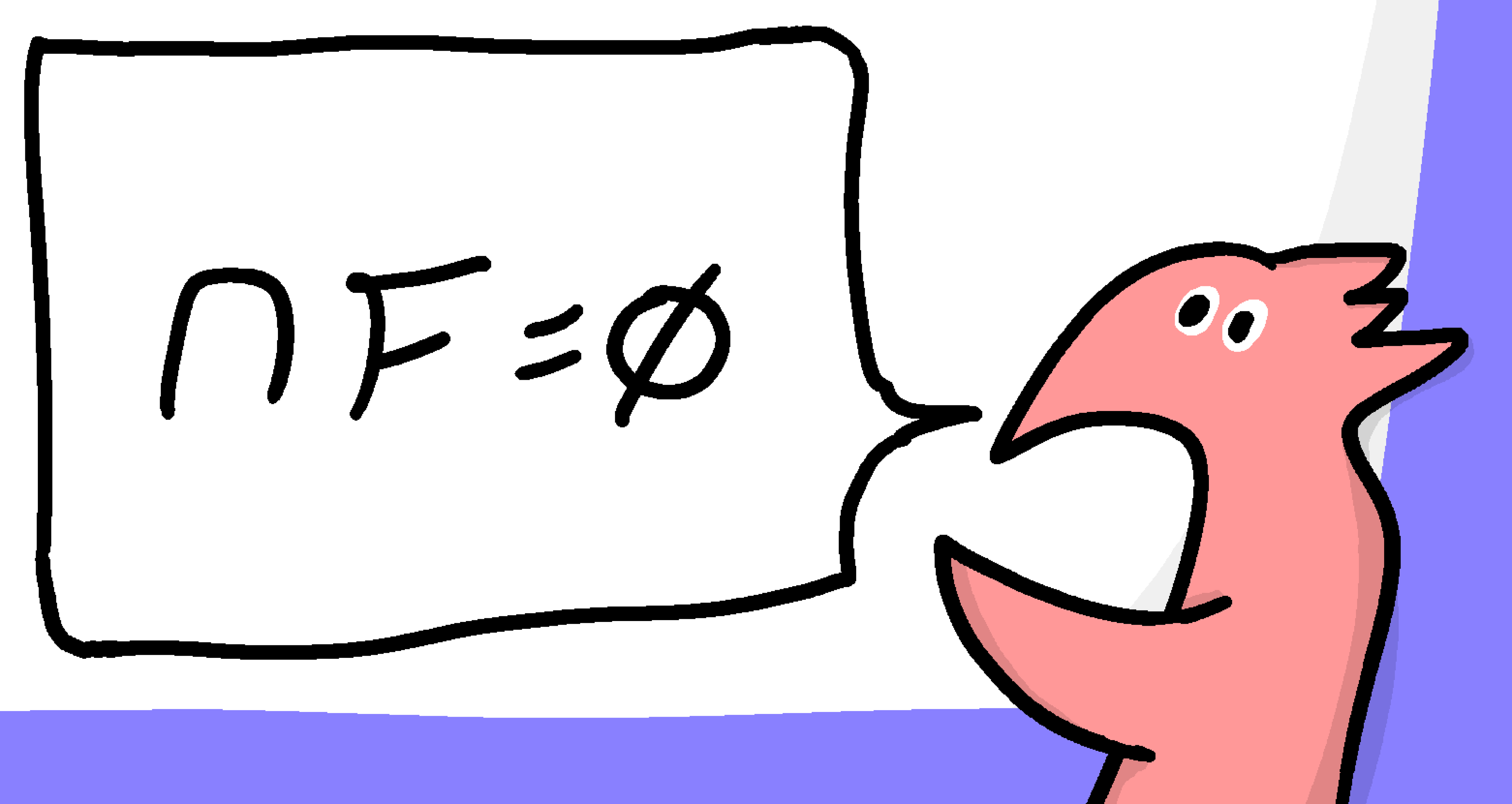
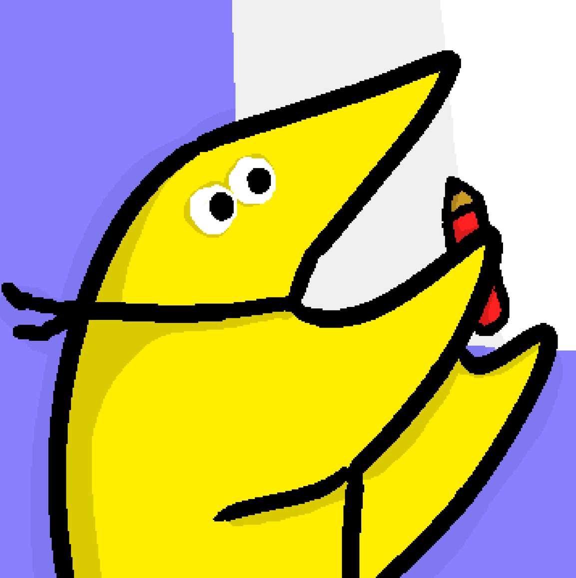
$F =$ 





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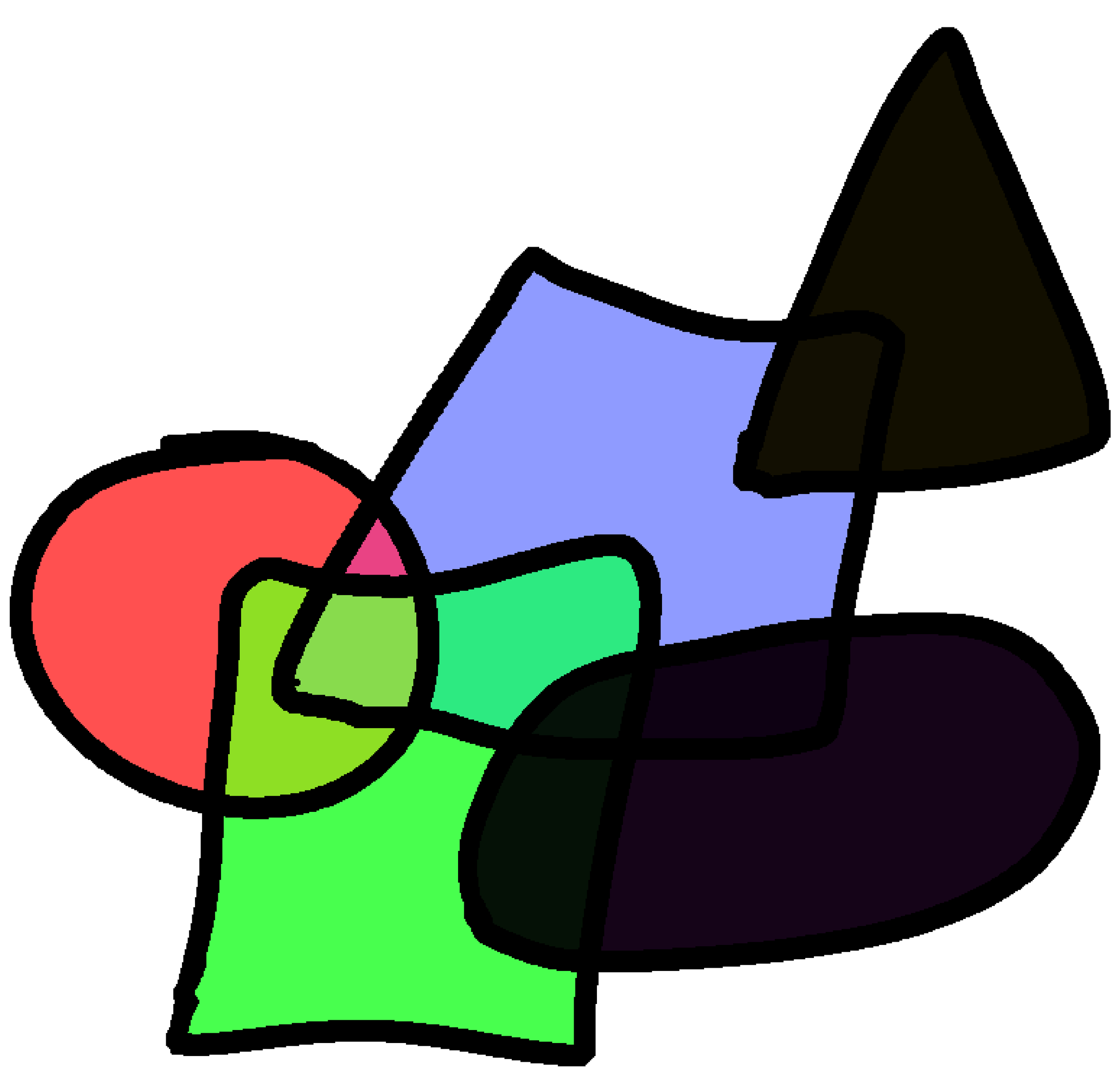
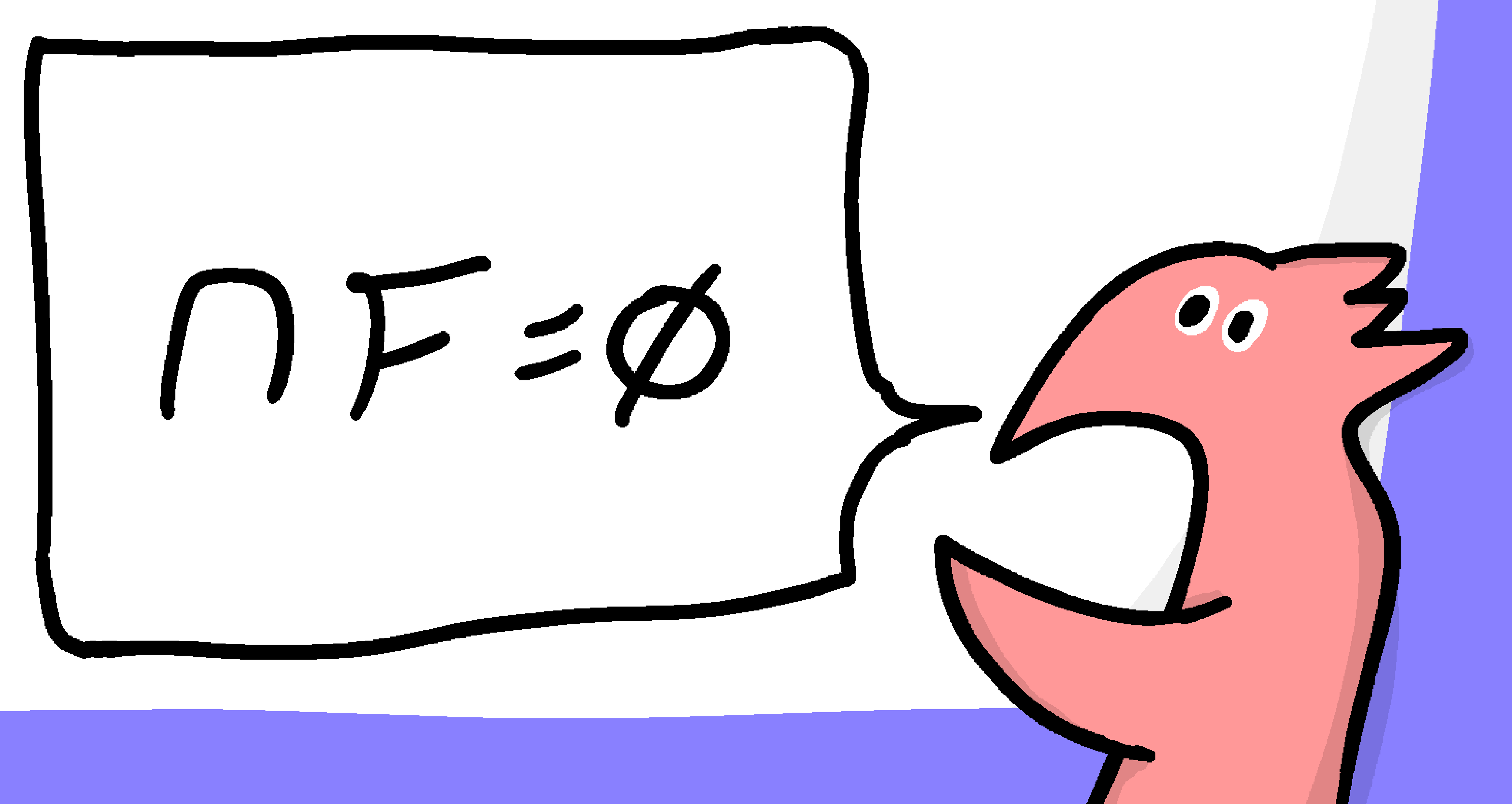
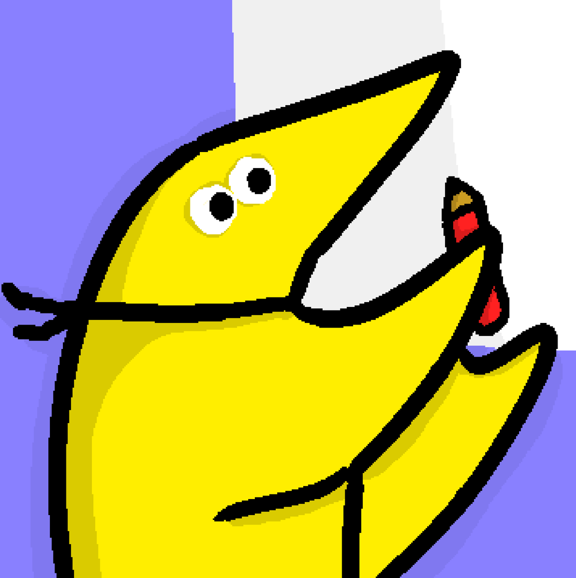
$\mathcal{F} =$ 





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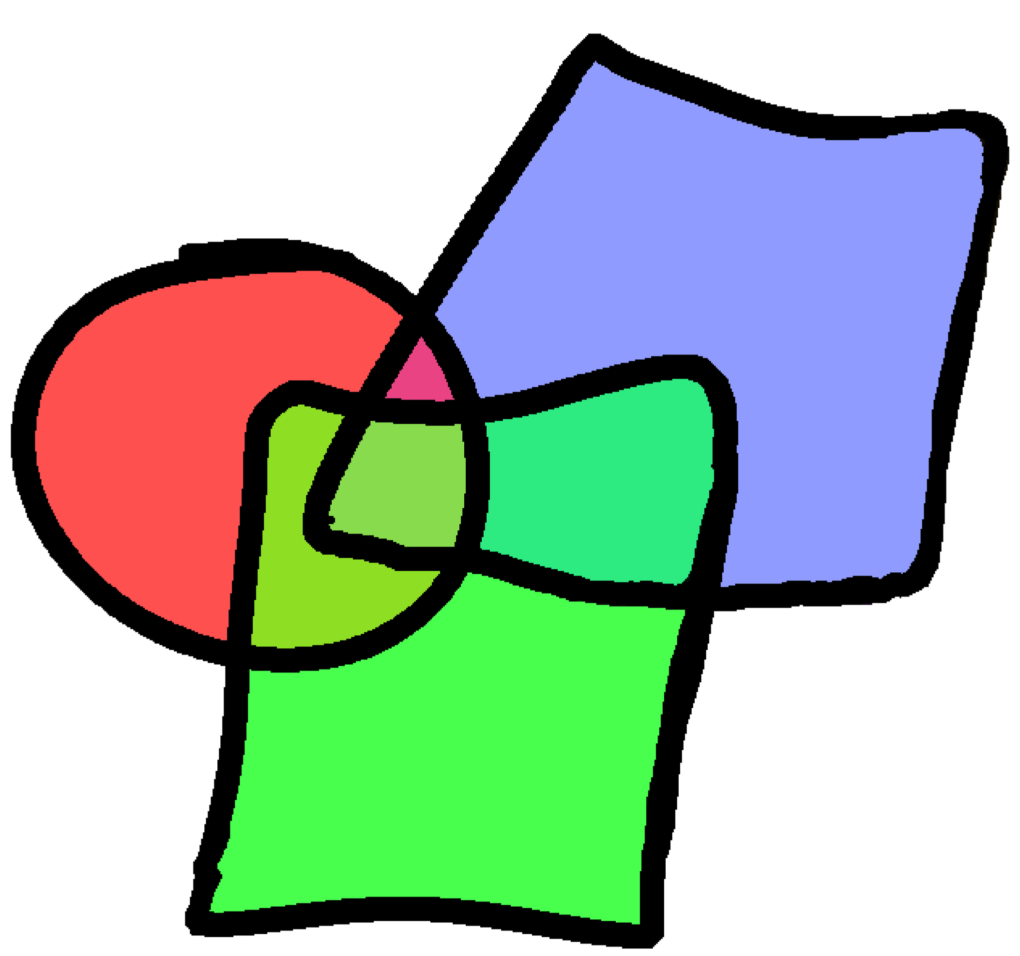
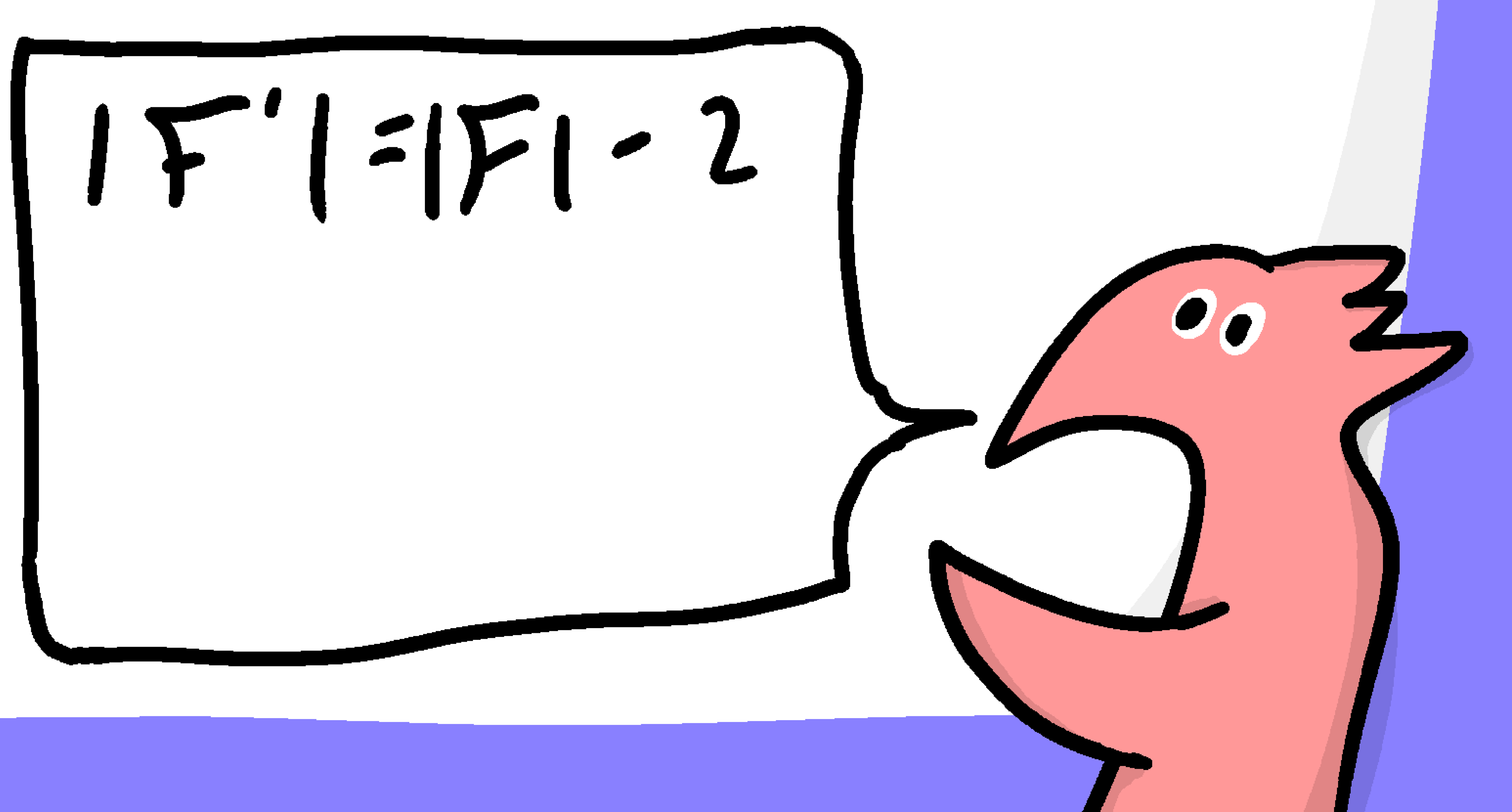
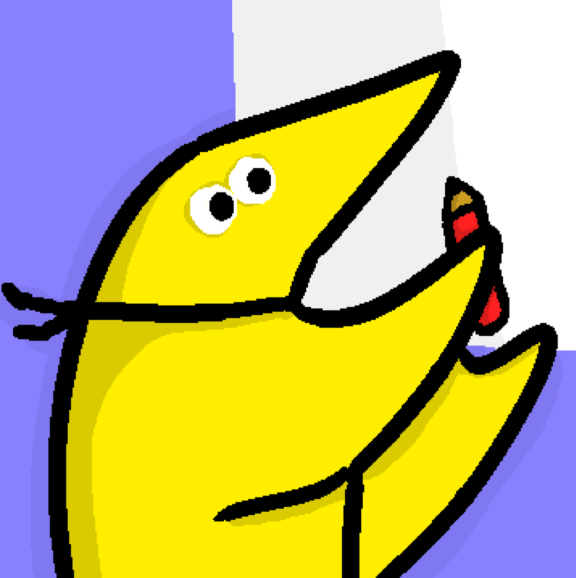
$\mathcal{F}' =$ 





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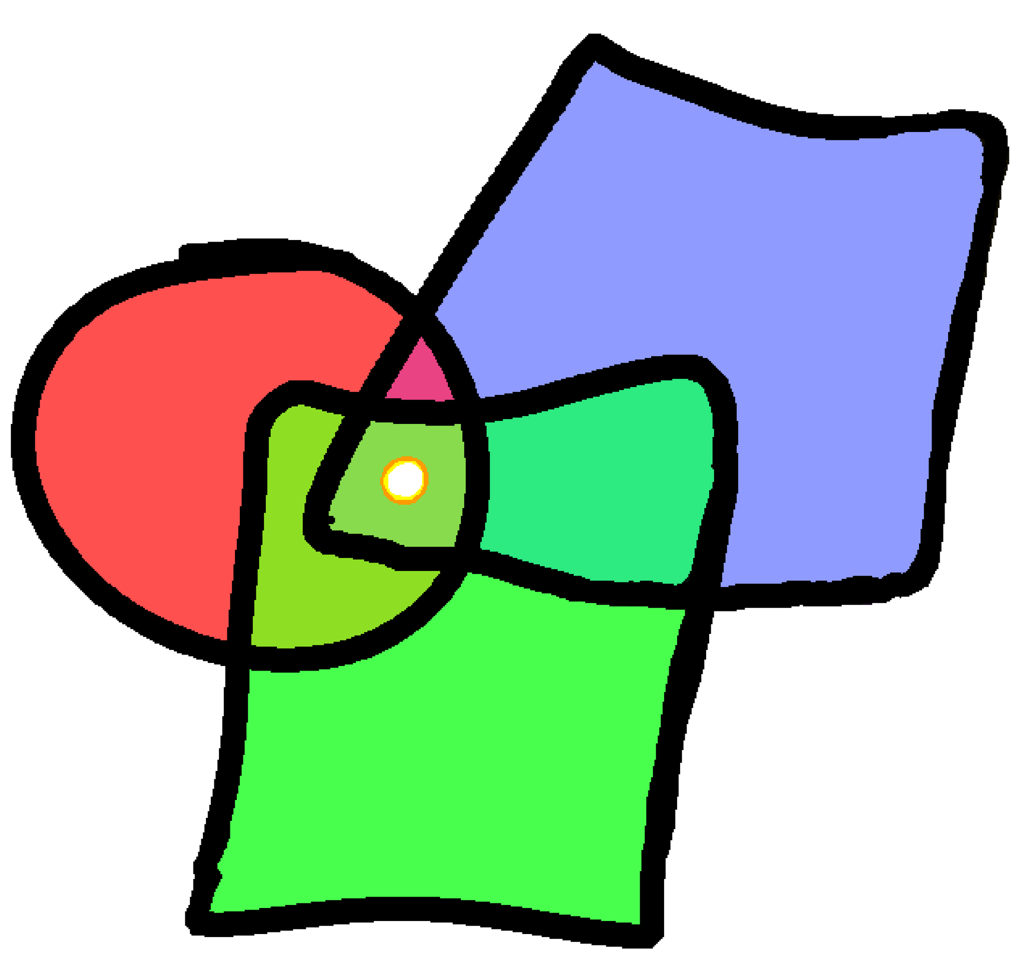
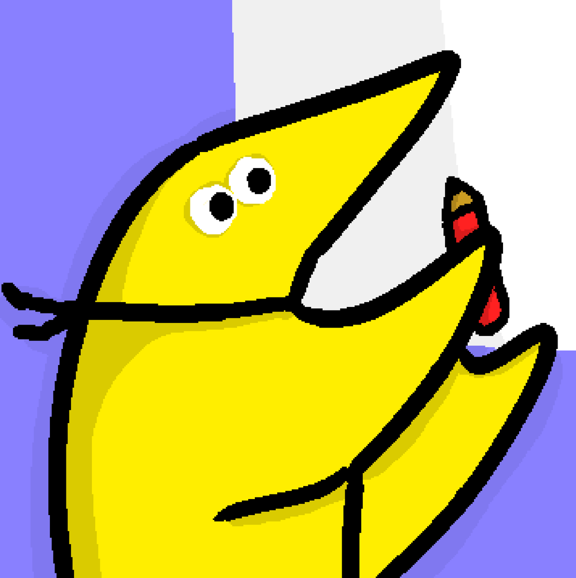
$\mathcal{F}' =$ 

$|\mathcal{F}'| = |\mathcal{F}| - 2$

$\bigcap \mathcal{F}' \neq \emptyset$



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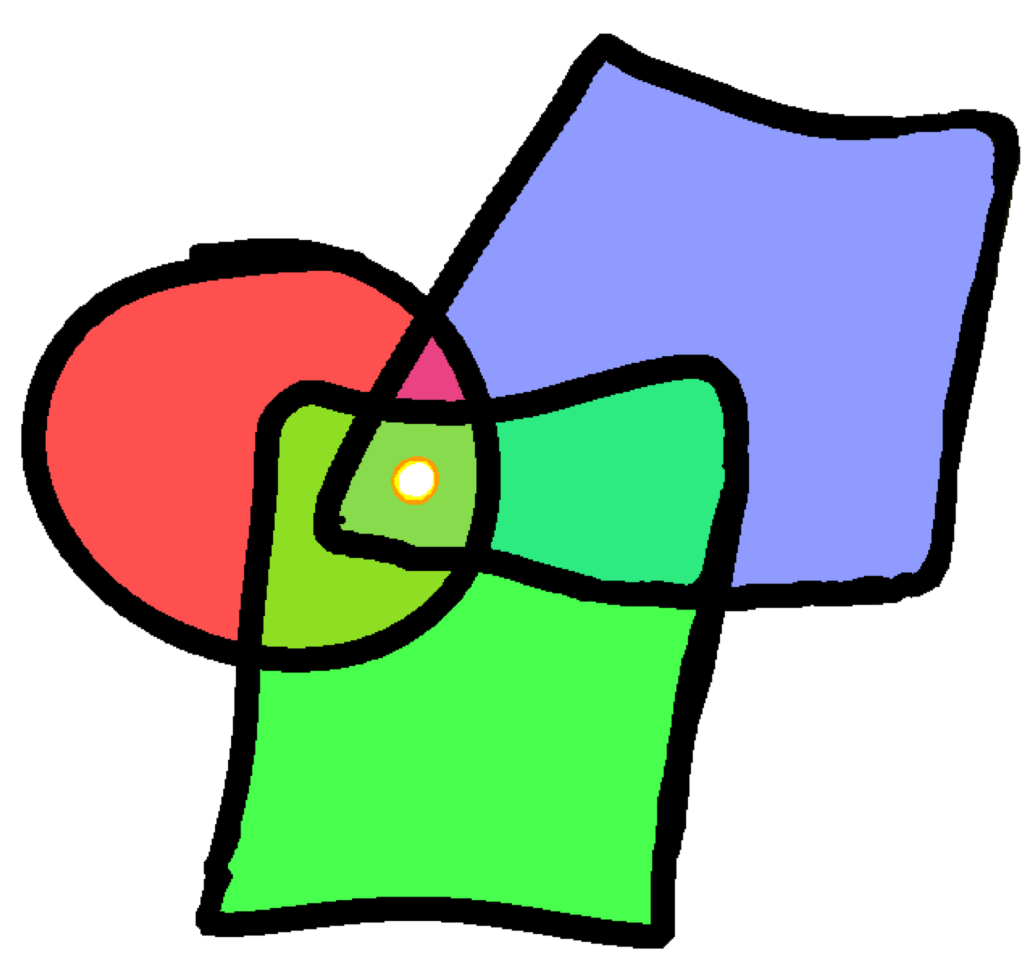
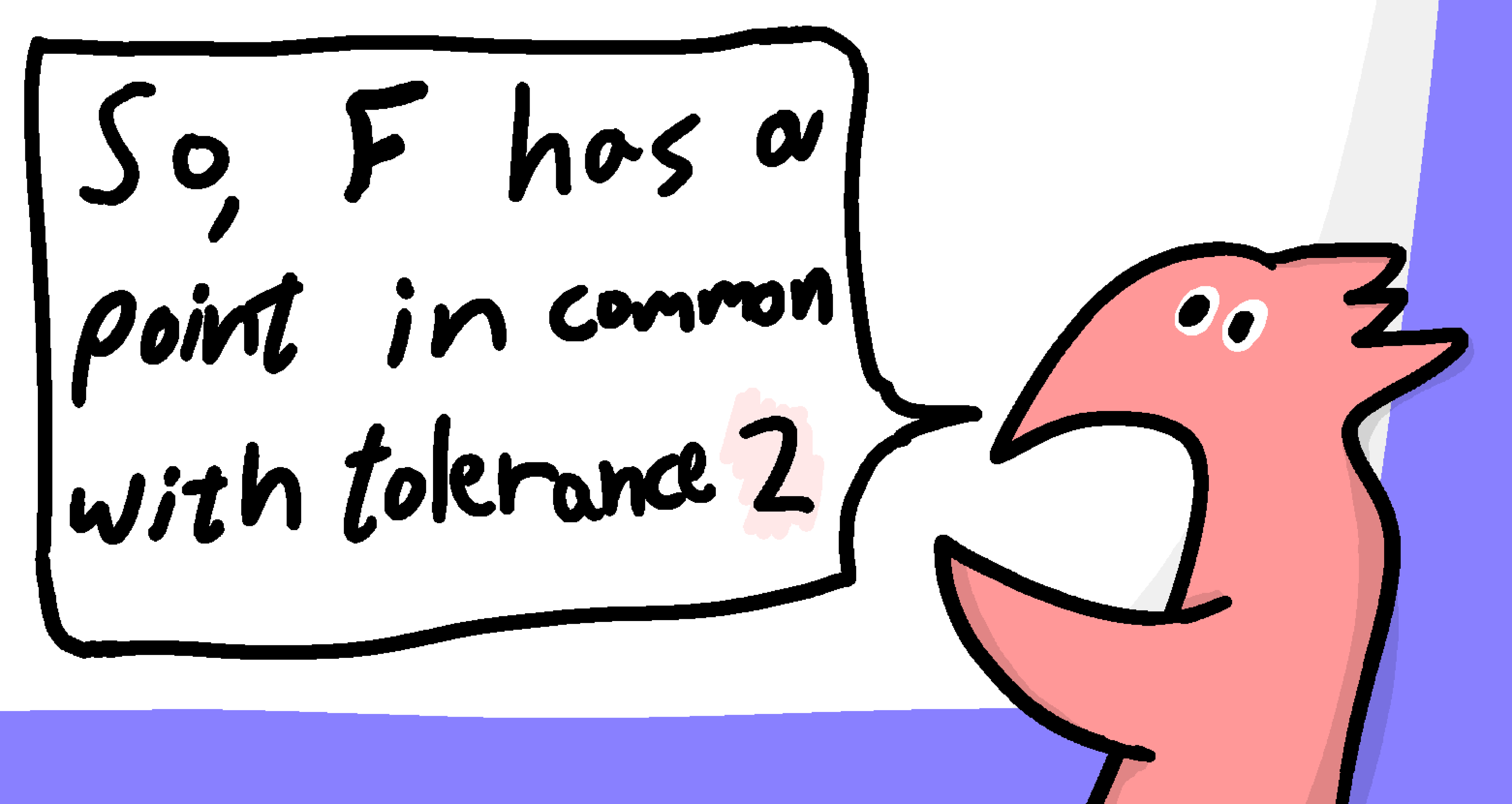
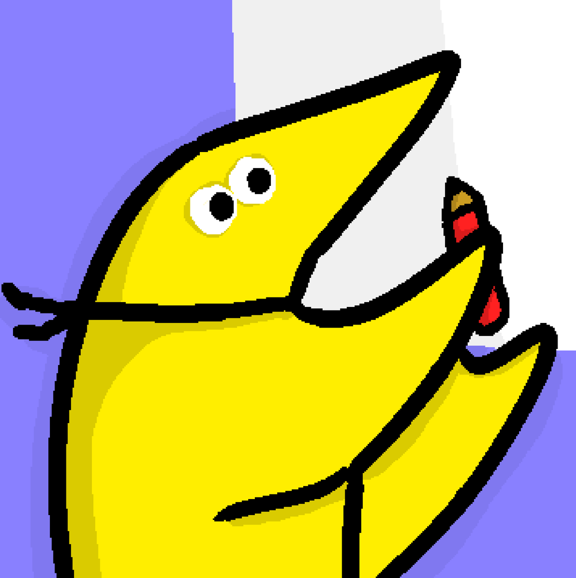
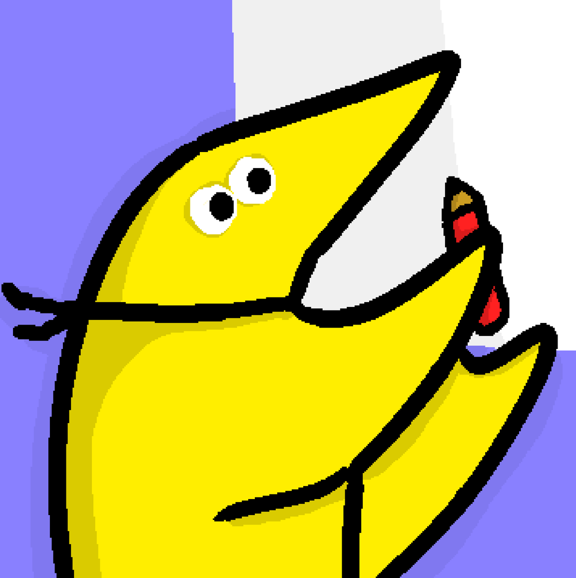
$F' =$ 





Helly with tolerance

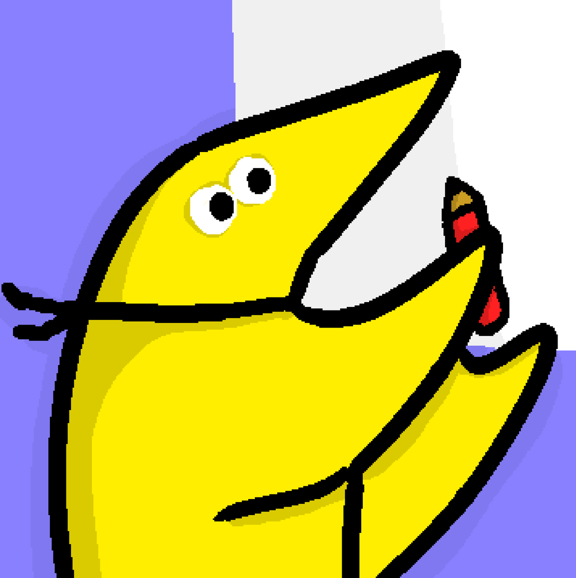
Thm (Montejano–Oliveros '10):



Helly with tolerance

Thm (Montejano–Oliveros '10):

Let F be a finite family of convex sets in $\mathbb{R}^d$.



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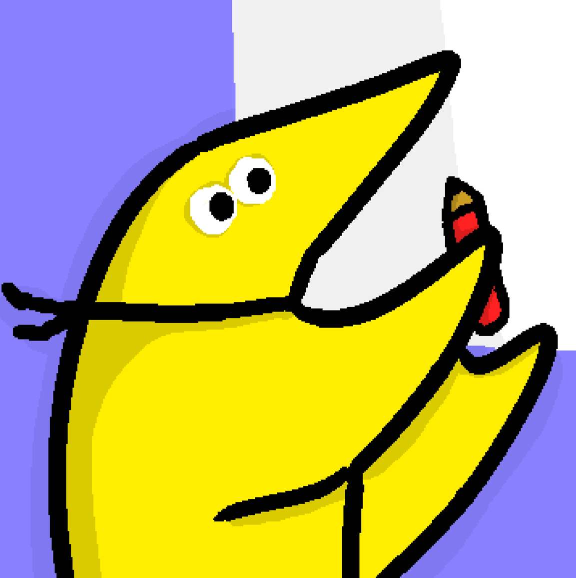
If any $F' \subseteq F$ of size $h(d+1,t+1)$ has a point in common with tolerance t,



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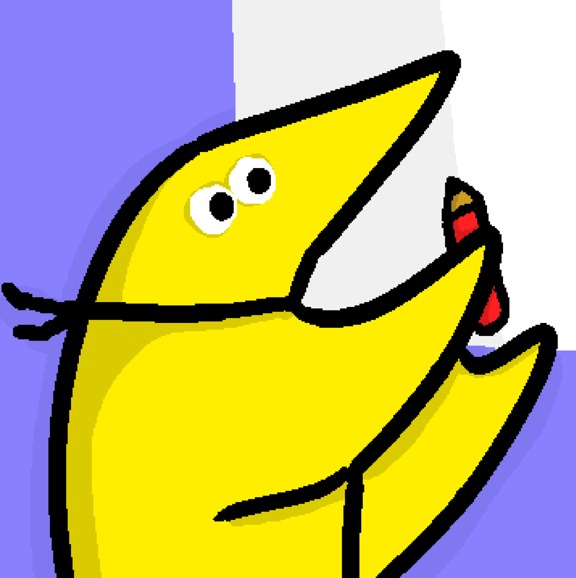

Tolerance Complexes

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$$T_t(K) = \left\{ \sigma \subseteq V \;\middle|\; \begin{array}{c} \exists\, \sigma' \subseteq \sigma \\ |\sigma'| \geq |\sigma| - t \\ \sigma' \in K \end{array} \right\}$$

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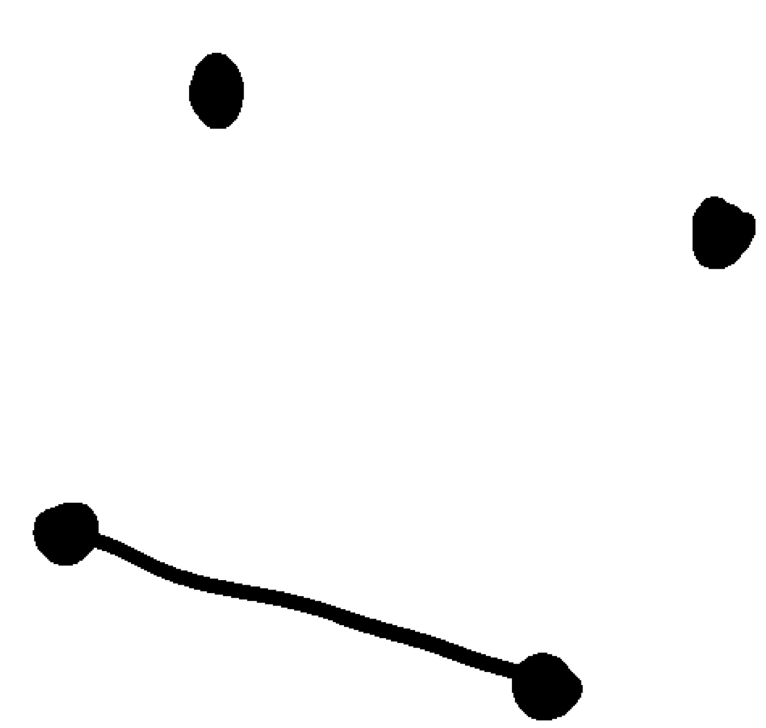
$K =$ 

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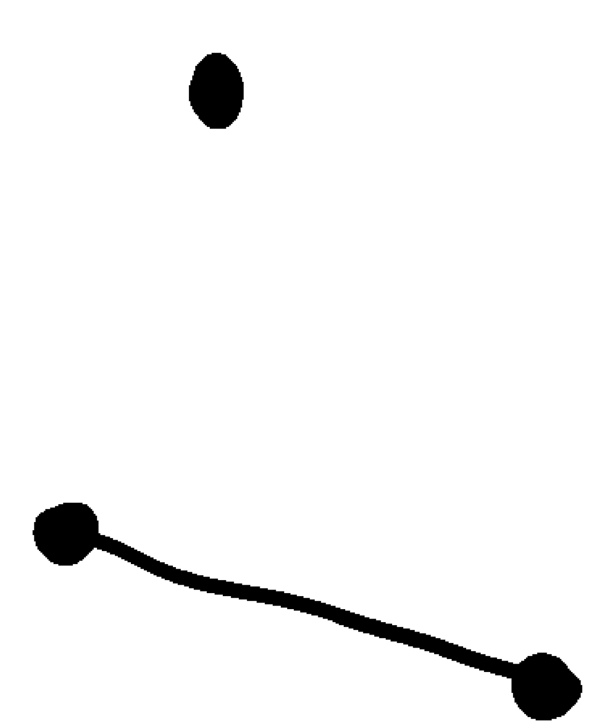
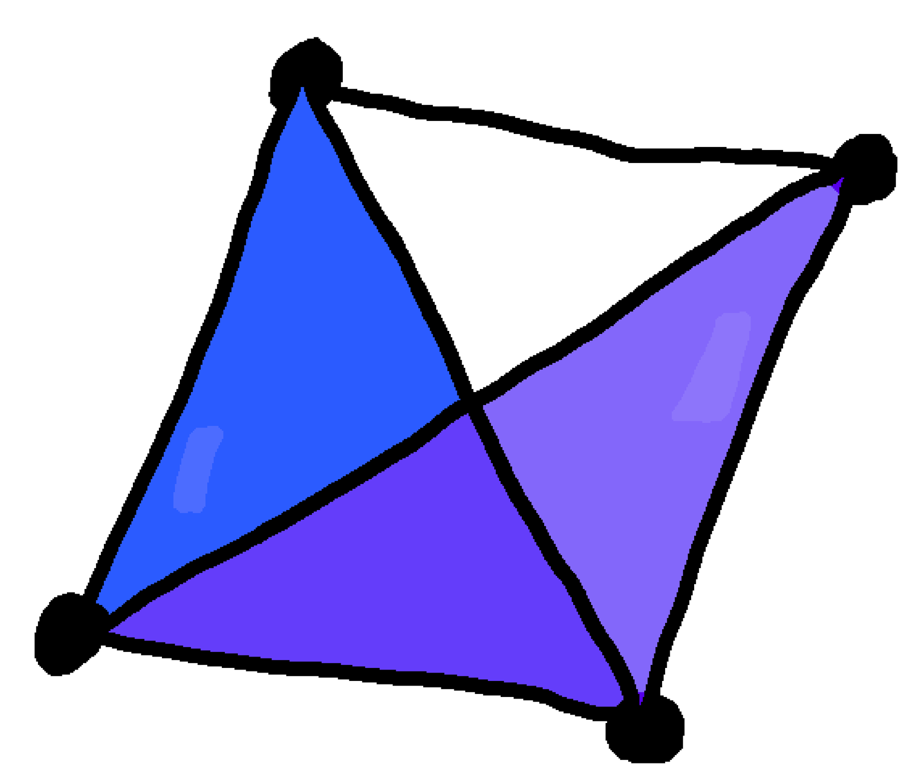
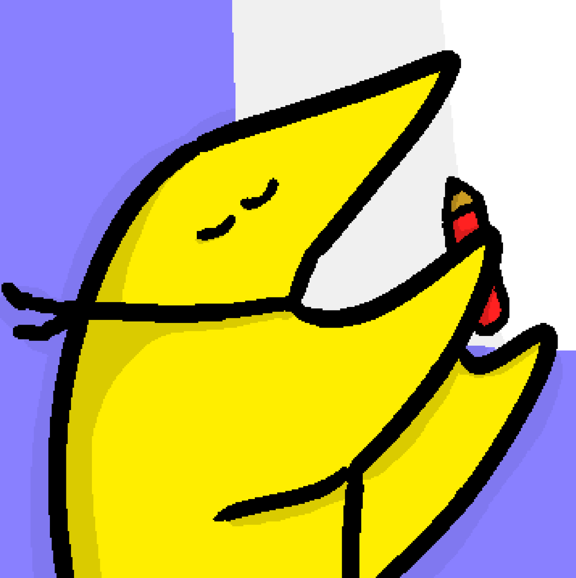
E.g. $K =$  $T_1(K) =$ 




Tolerance Complexes
• K =
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$\mathcal{F}$ = family of sets

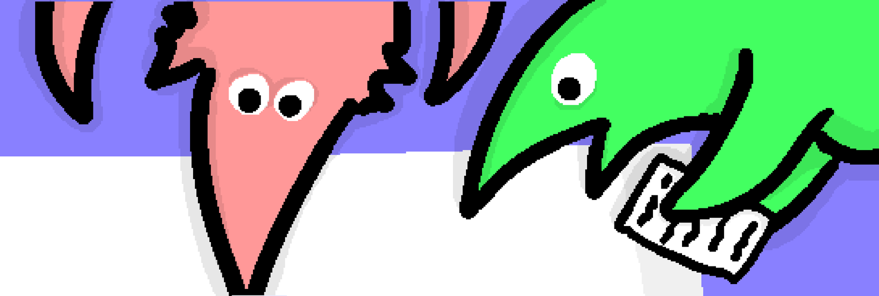

Tolerance Complexes
$\mathcal{F}$ = family of sets
$\mathcal{T}_t(N(\mathcal{F})) = \{ \mathcal{F}' \subseteq \mathcal{F} \mid \mathcal{F}' \text{ has pt. in common with tolerance } t \}$

Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If K is d-representable, then

$$h\left(T_t(K)\right) \le h(d+1, t+1) - 1$$



Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If $h(K) \leq d$, then

$$h(T_t(K)) \leq h(d+1, t+1) - 1$$

Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If $h(K) \le d$, then

$$h\big(T_t(K)\big) \le \eta(d+1, t+1) - 1$$

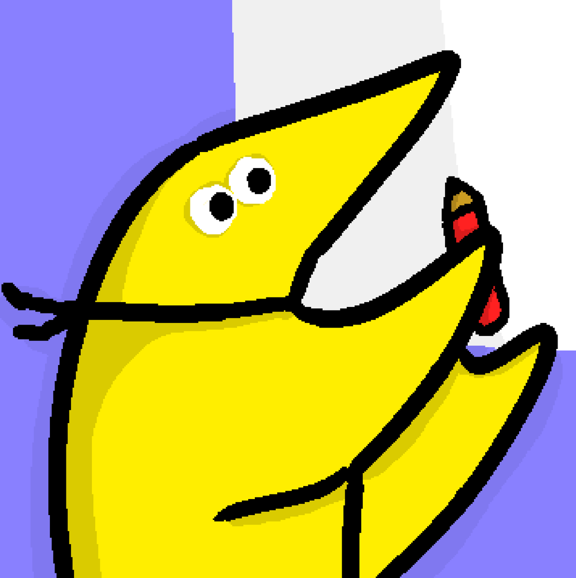
- If we assume K is d-collapsible / d-Leray, can

we obtain a stronger conclusion?

Collapsibility and Leray numbers of $T_t(K)$

Conjecture: If K is d-Leray, then

$T_t(K)$ is $(\eta(d+1, t+1) - 1)$-Leray.



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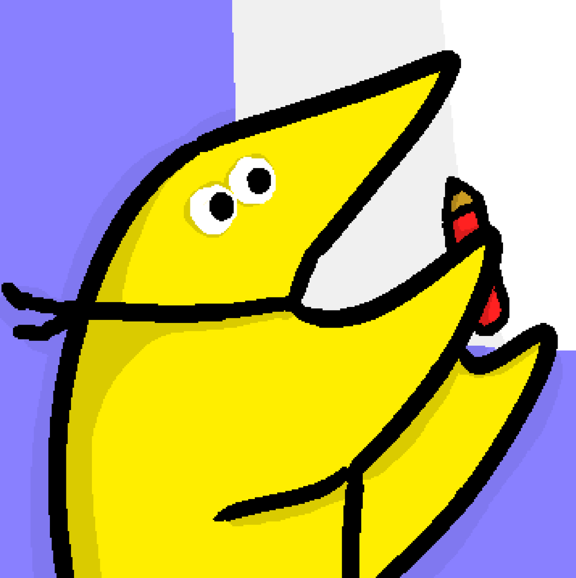

Extremal examples

$\mathcal{C}$ = facets of d-dim simplex,
$t+1$ copies of each.

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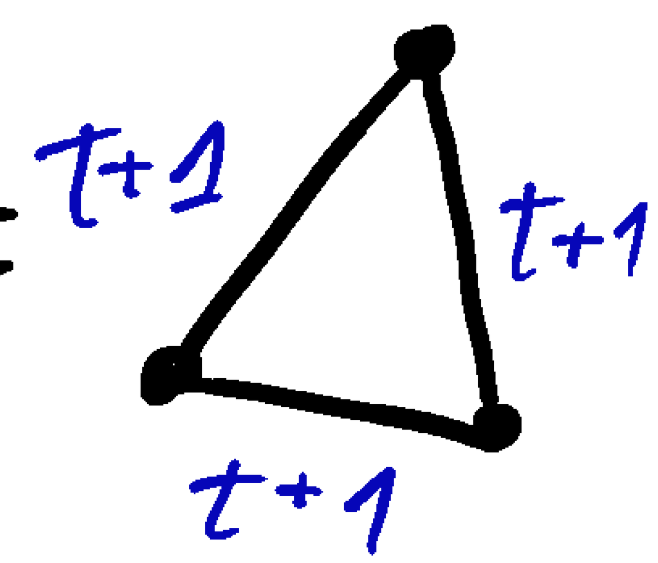

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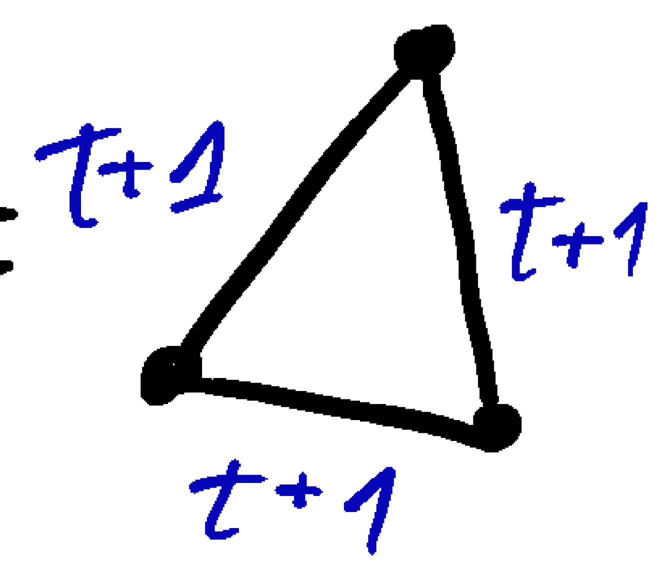
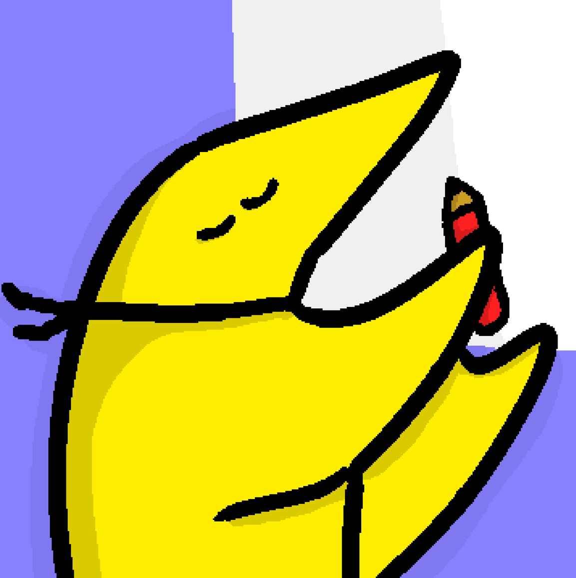


$N(C)$ = d-representable



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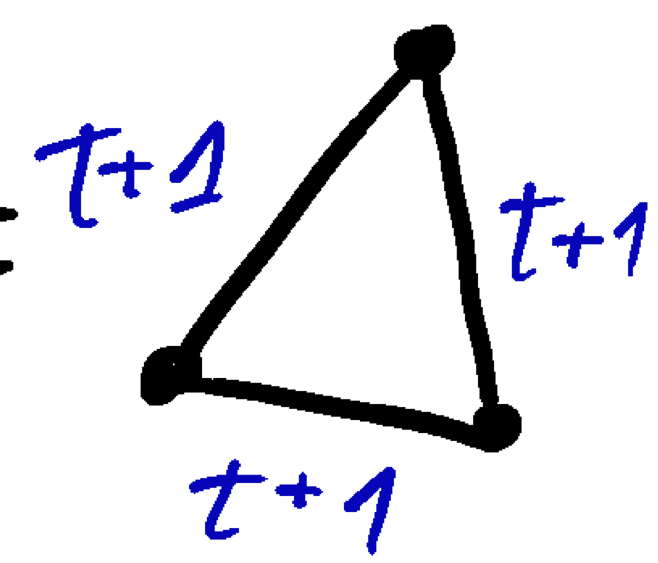
$C =$ 

$N(C) = d$-representable $\left(\begin{array}{c} d\text{-collapsible} \\ d\text{-Leray} \end{array}\right)$

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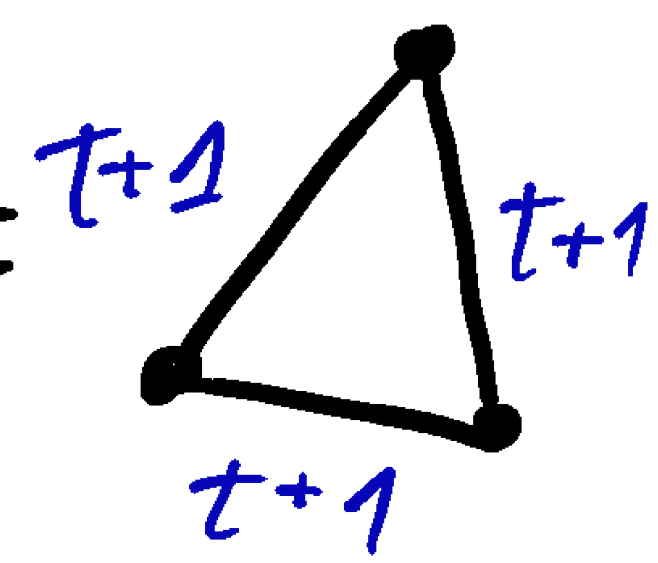
$$T_t(N(\mathcal{C})) = ?$$

$\mathcal{C}$ = 

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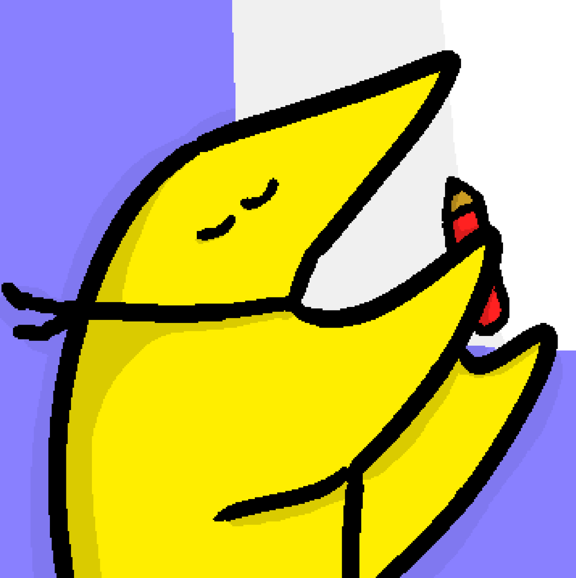

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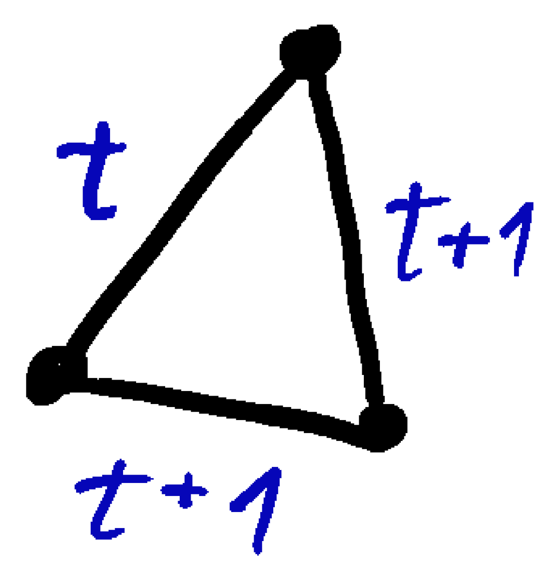
$C \setminus \{A\} =$ 

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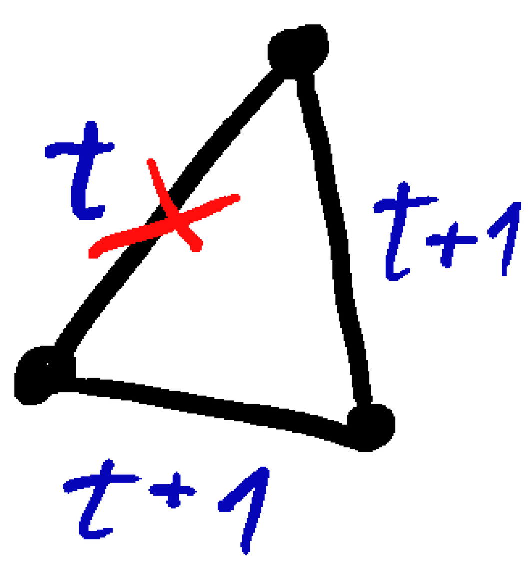
$\mathcal{C} \setminus \{A\} =$ 

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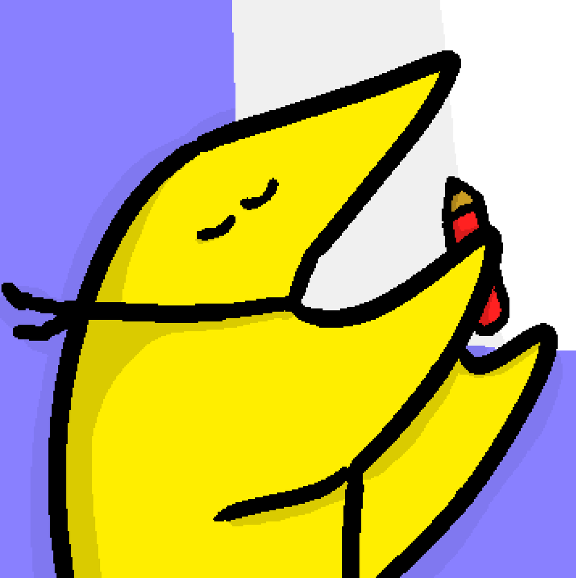

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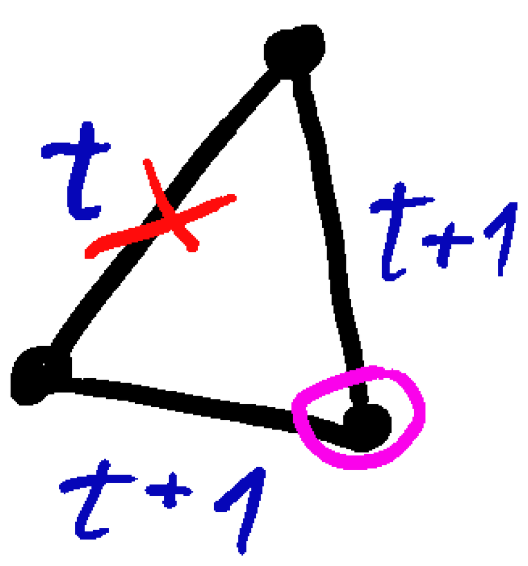
$\mathcal{C} \setminus \{A\} =$ 

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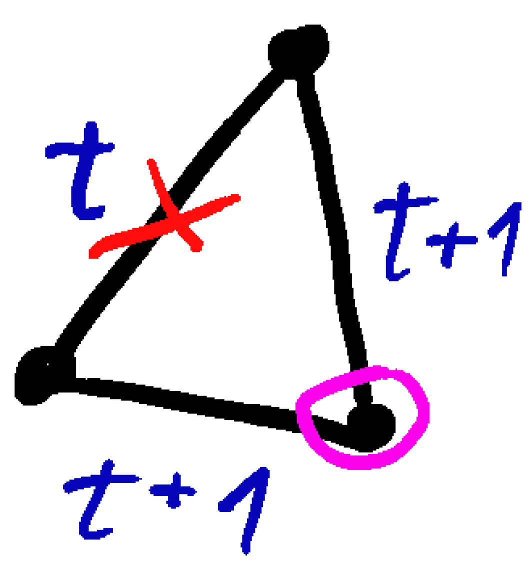
$\mathcal{C} \setminus \{A\} =$ 

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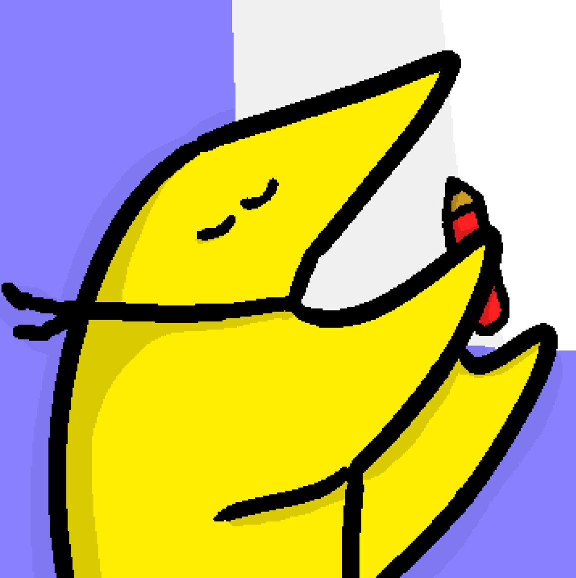
$T_t(N(\mathcal{C}))$

= boundary of

$((d+1)(t+1)-1)$-dim. simplex



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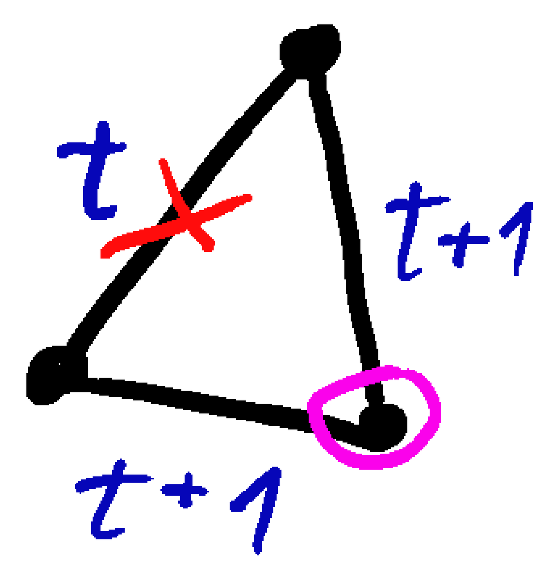
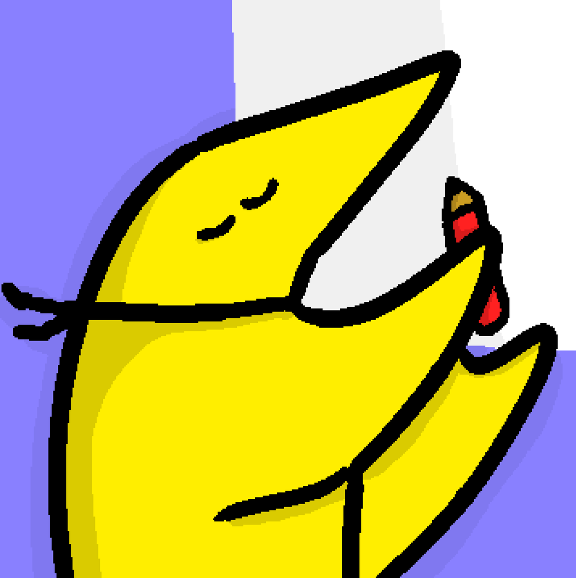


$N(\mathcal{C})$ = d-representable $\left(\begin{array}{l}\text{d-collapsible}\\ \text{d-Leray}\end{array}\right)$

$T_t(N(\mathcal{C}))$

$=$

$((d+1)(t+1)-2)$-dim. sphere



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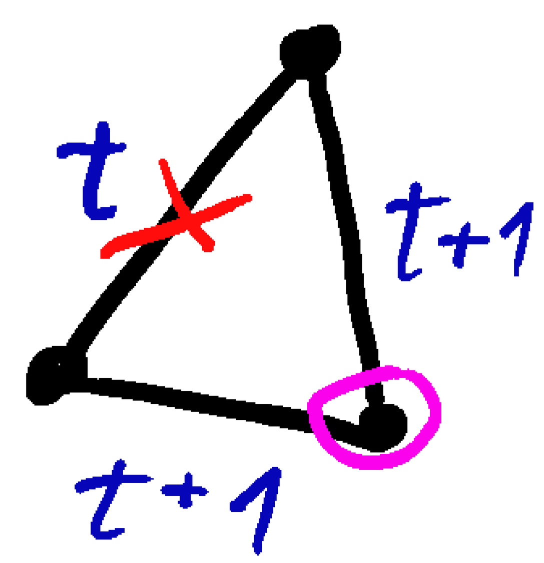

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$T_t(N(\mathcal{C}))$

=

$((d+1)(t+1)-2)$-dim. sphere

$\Downarrow$

$T_t(N(\mathcal{C}))$ is **NOT** $((d+1)(t+1)-2)$-Leray

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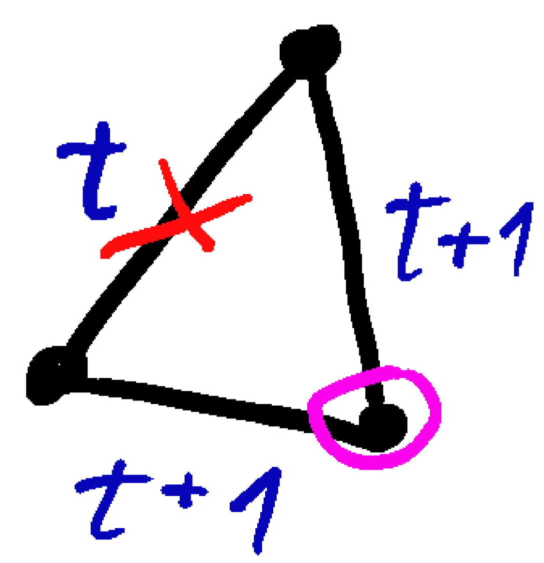
$\mathcal{C} \setminus \{A\} =$ 

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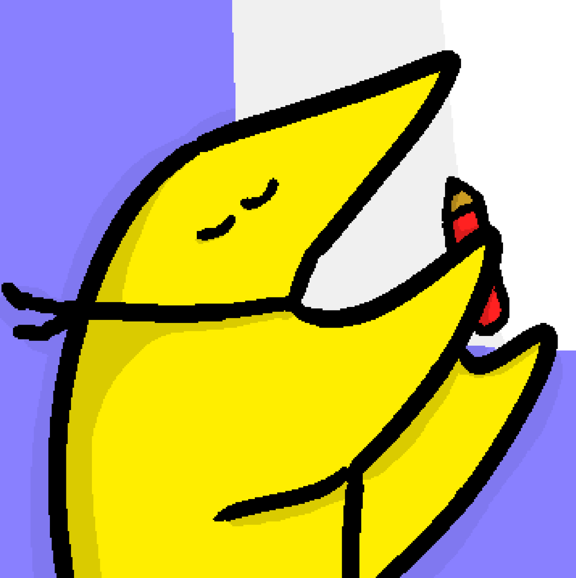
$T_t(N(\mathcal{C}))$ is **NOT** $((d+1)(t+1)-2)$-Leray

- For $d=1$, $(d+1)(t+1)-2 = 2t$



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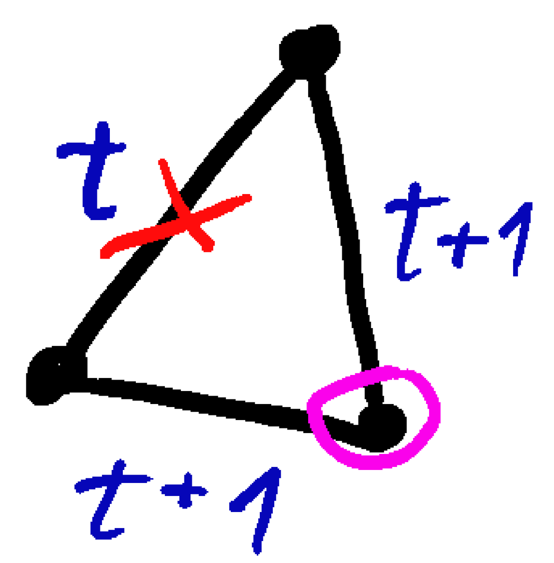

$N(\mathcal{C})$ = d-representable $\left(\begin{array}{l} d\text{-collapsible} \\ d\text{-Leray} \end{array}\right)$

$T_t(N(\mathcal{C}))$

$=$

$((d+1)(t+1)-2)$-dim. sphere

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$T_t(N(\mathcal{C}))$ is **NOT** $((d+1)(t+1)-2)$-Leray

- For $d=1$, $(d+1)(t+1)-2 = 2t = \eta(2, t+1)-2$

Extremal examples

Montejano-Oliveros ('10):

$\exists\ \mathcal{C}$ family of convex sets in $\mathbb{R}^d$

s.t. $T_1(N(\mathcal{C}))$ is **not** $\left(\left\lfloor\left(\frac{d+3}{2}\right)^2\right\rfloor-2\right)$-Leray

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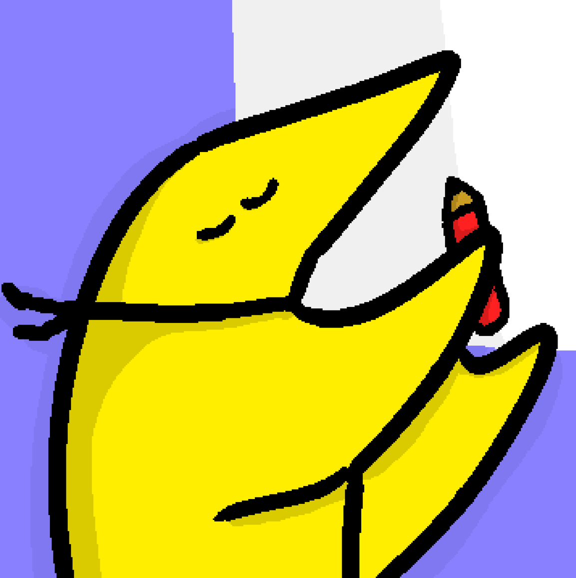
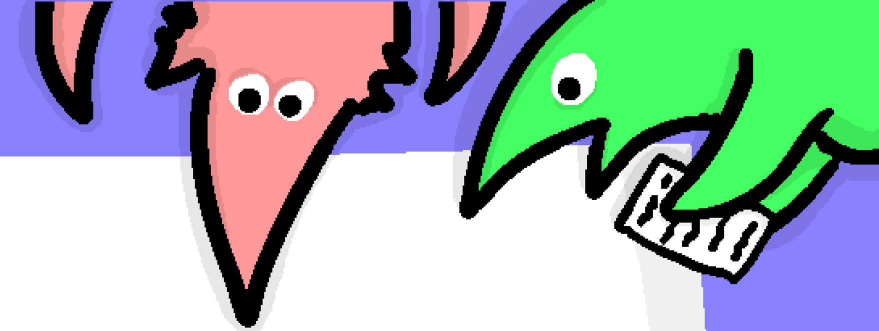
Montejano-Oliveros ('10):

$\exists\, \mathcal{C}$ family of convex sets in $\mathbb{R}^d$

s.t. $T_1(N(\mathcal{C}))$ is **not** $\left(\left\lfloor\left(\frac{d+3}{2}\right)^2\right\rfloor-2\right)$-Leray

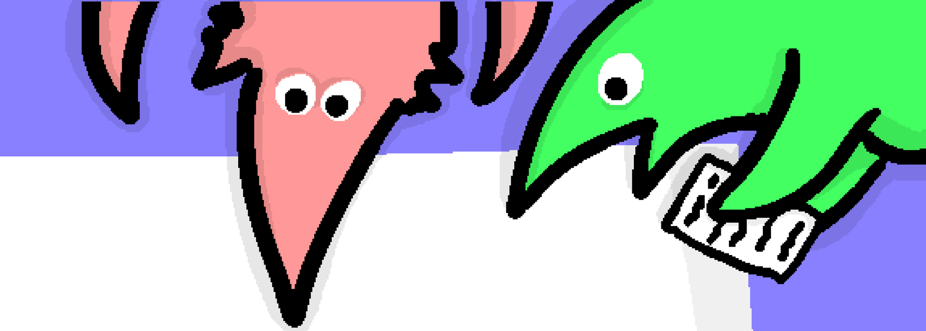
- For $d=1$, $(d+1)(t+1)-2 = 2t = \eta(2,t+1)-2$
- $t=1$: $\left\lfloor\left(\frac{d+3}{2}\right)^2\right\rfloor-2 = \eta(d+1,2)-2$





Main results

$$h(t,d) = \begin{cases} d & ; t=0 \\ \left[\sum_{s=1}^{\min\{t,d\}} \binom{d}{s}\left(h(t-s,d)+1\right)\right] + d & ; t>0 \end{cases}$$



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Thm (Kim-L. '21):

Let K be a d-collapsible complex. Then, $T_t(K)$ is $h(t,d)$-Leray.

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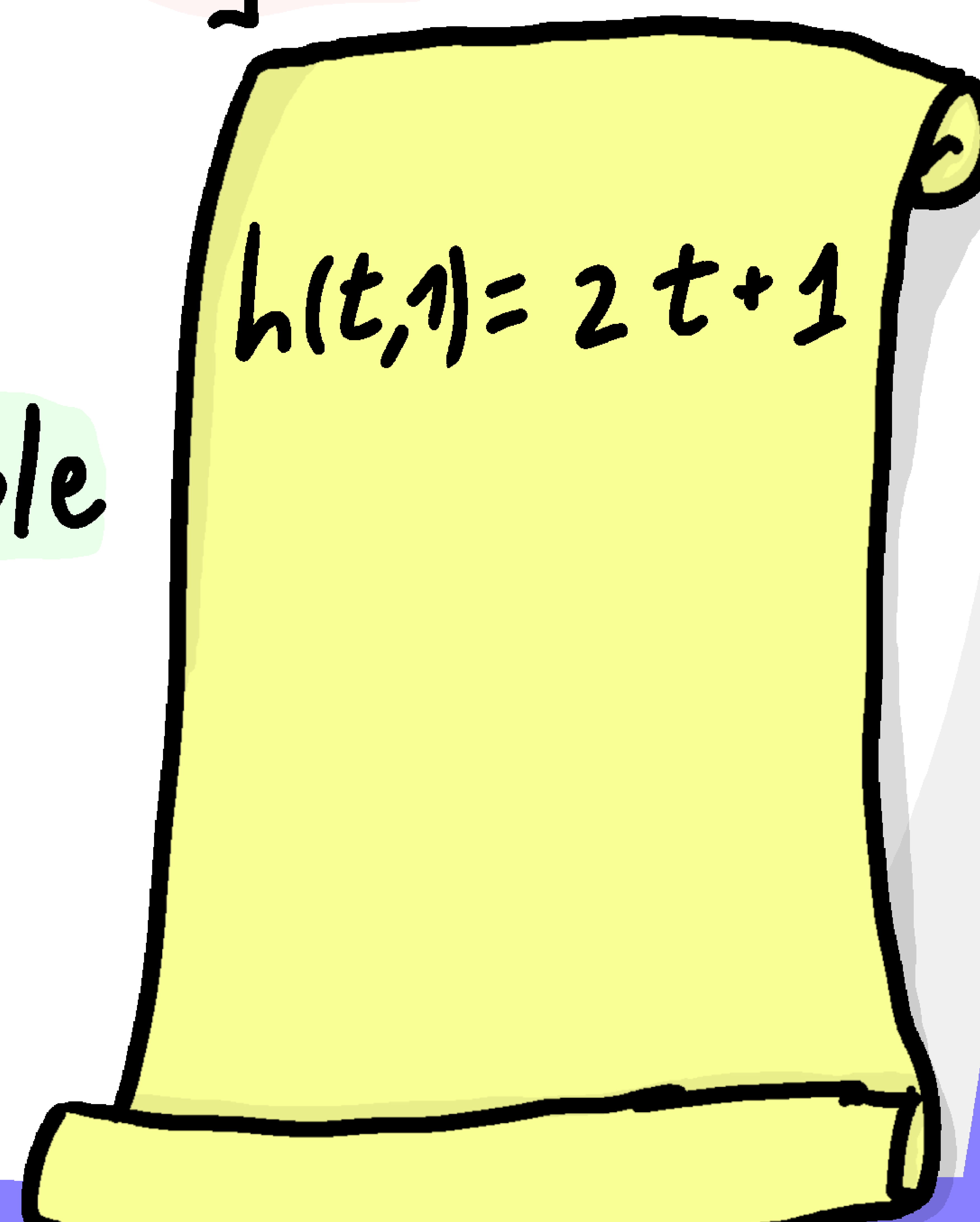

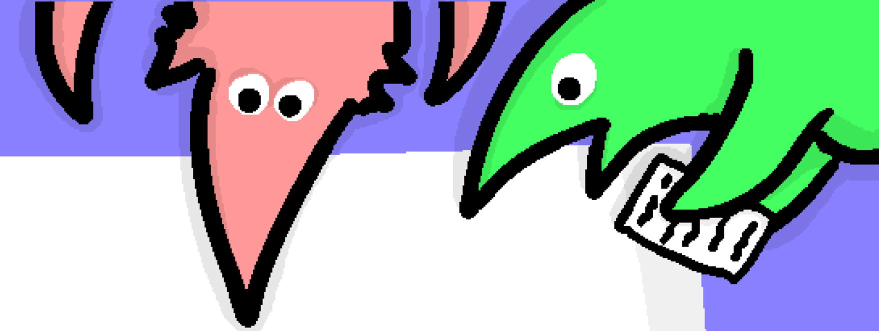

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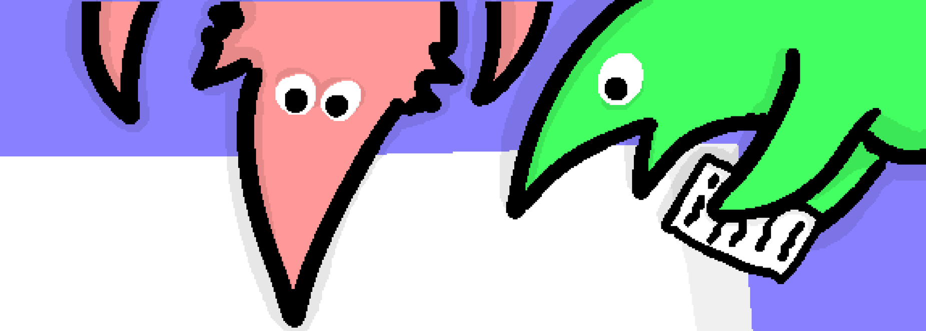

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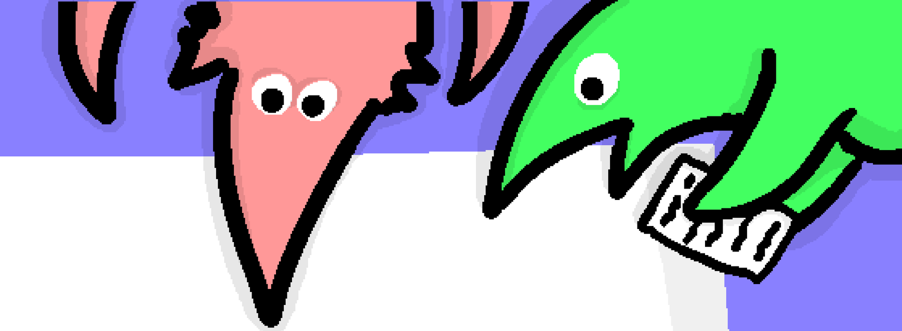

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$h(t,d) = O(d^{t+1})$

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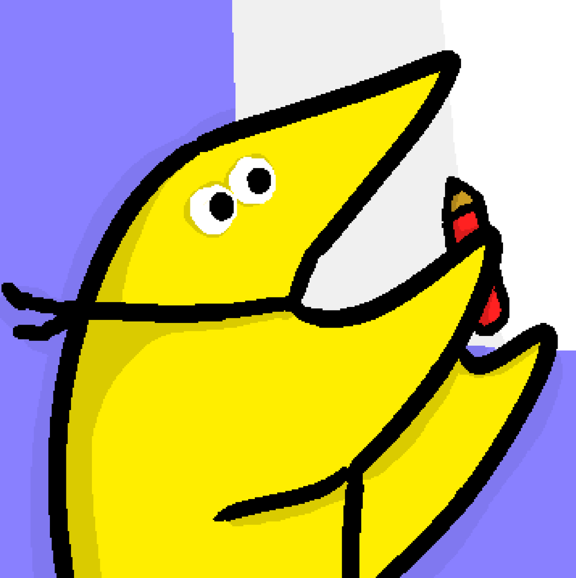
$$K \text{ } d\text{-collapsible}$$
$$\Downarrow$$
$$T_1(K) \text{ } (d^2+2d)\text{-Leray}$$



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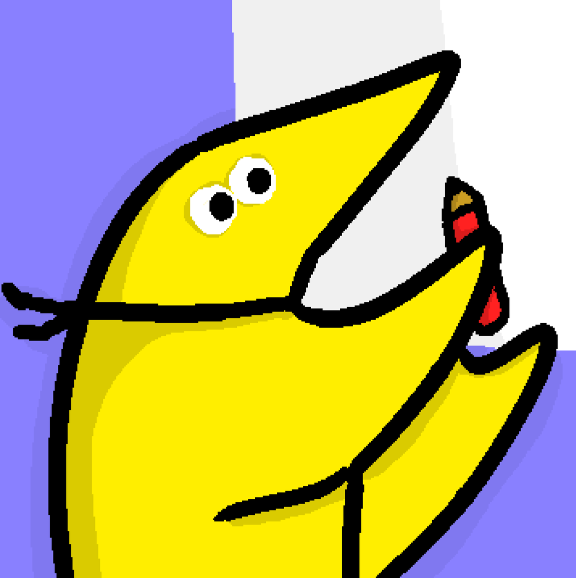

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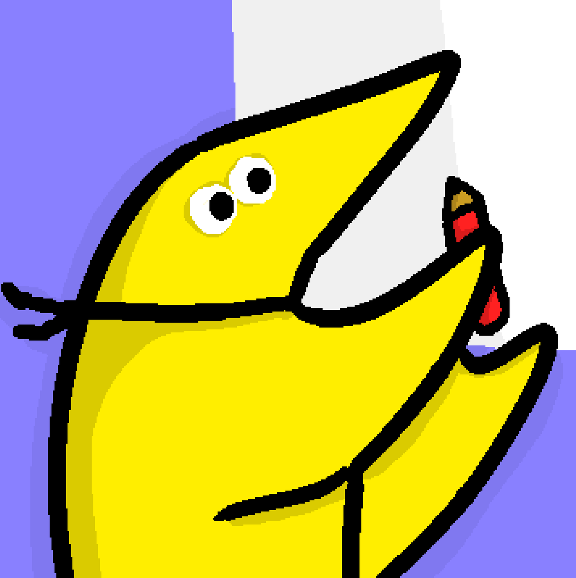

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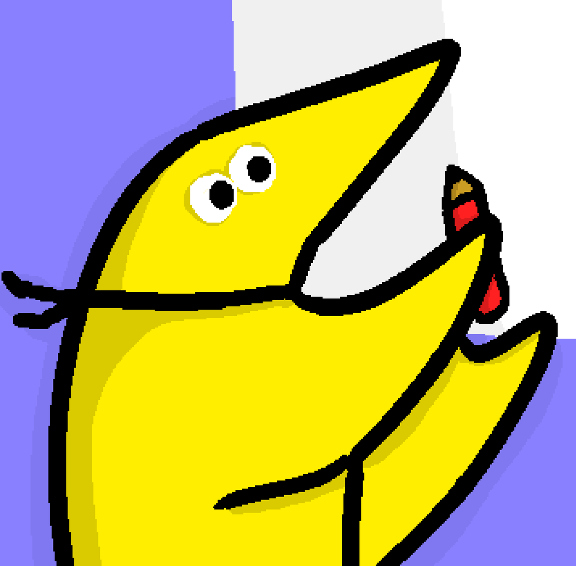
Thm (Kim–L. '21): Let K be 2-collapsible. Then, $T_1(K)$ is 5-Leray.



Some ideas from the proof

Some definitions:

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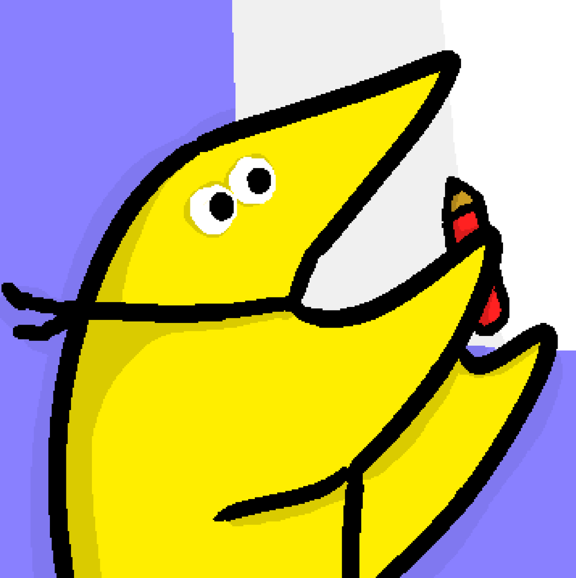
- $\mathrm{cost}(K,\sigma) = \{ \tau \in K : \sigma \notin \tau \}$



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Thm (Tancer '10): K is d-collapsible iff either $\dim(K) < d$ or

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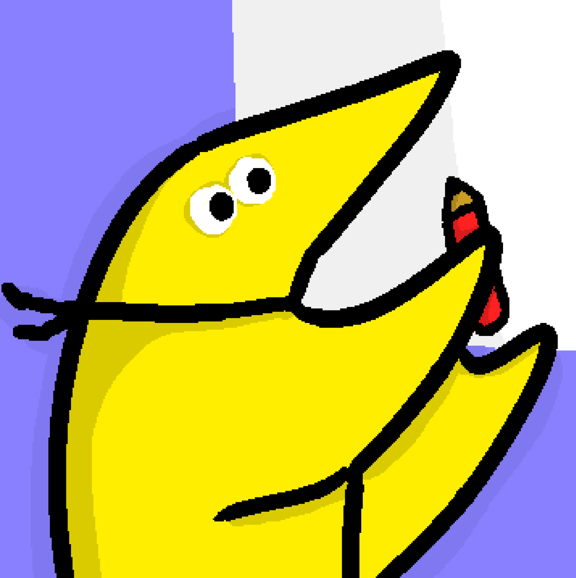
Thm (Tancer '10): K is d-collapsible iff either $\dim(K) < d$ or $\exists\, \sigma \in K$, $|\sigma| = d$, contained in unique max face $\tau \neq \sigma$ and

$$\mathrm{cost}(K, \sigma) \text{ is } d\text{-collapsible}$$



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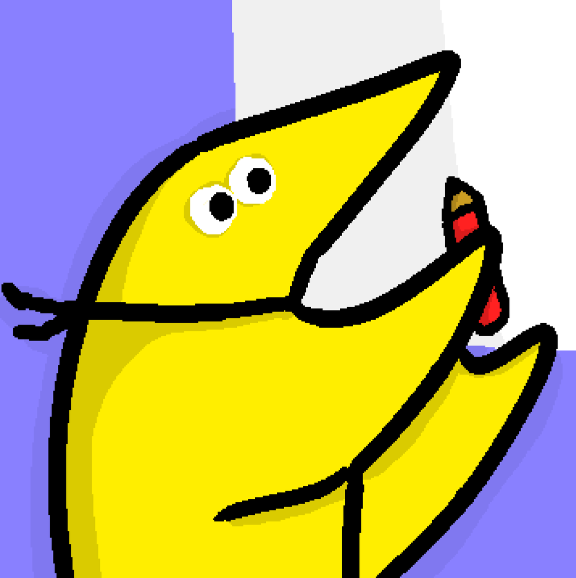

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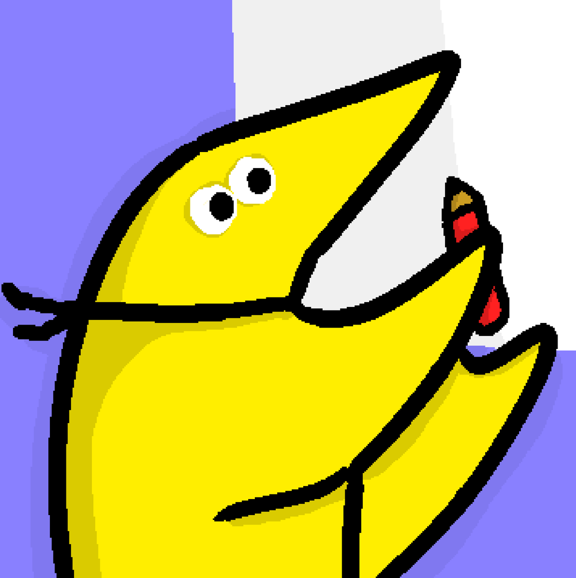

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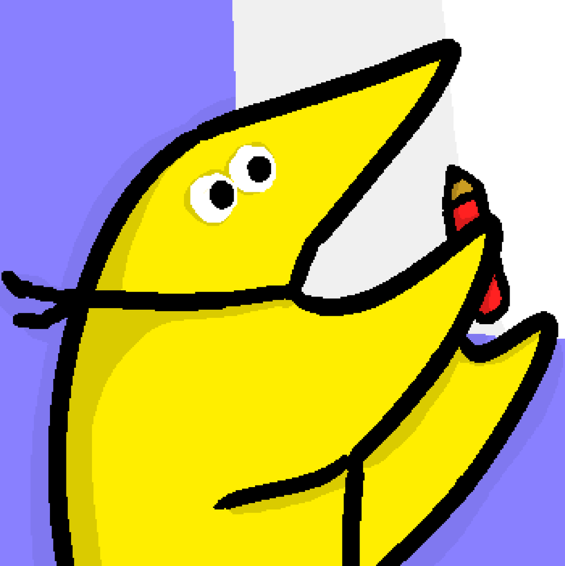
O.w., $\exists\, \sigma \in K$, $|\sigma| = d$, σ is contained in unique max. face

$\tau = \sigma \cup U$ $(U \neq \emptyset)$, $\mathrm{cost}(K, \sigma)$ d-col.



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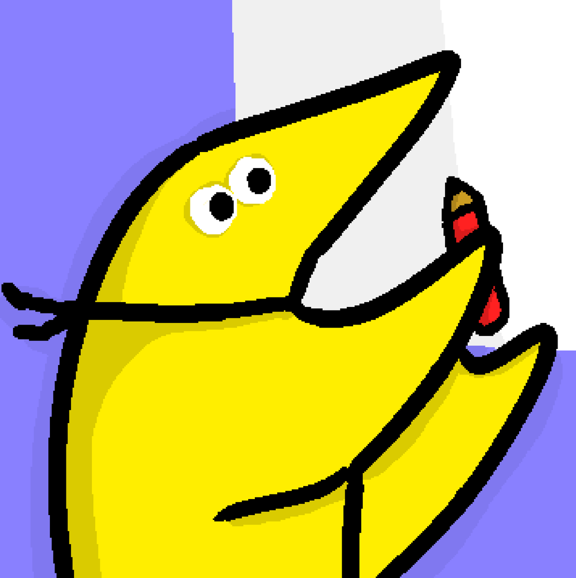

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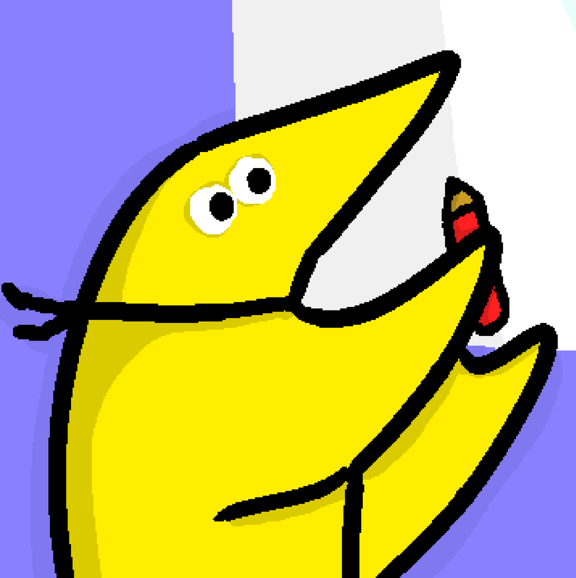
$$\to H_k(T_t(k), T_t(\mathrm{cost}(k,\sigma))) \to \cdots$$



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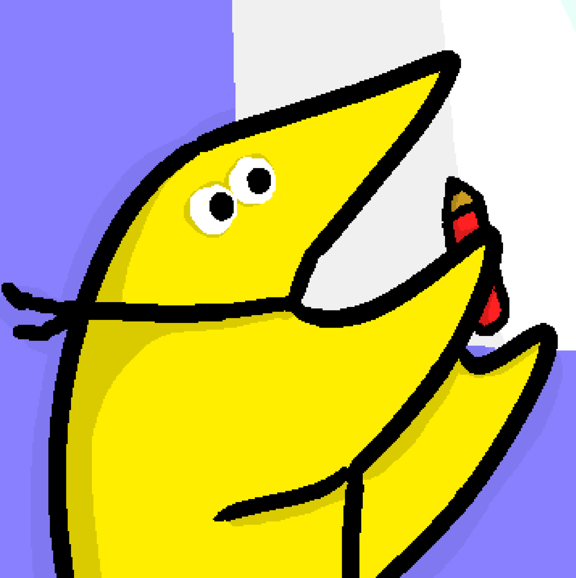
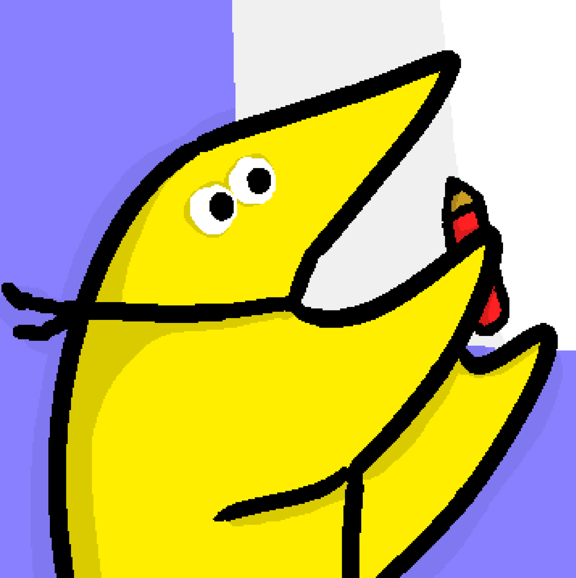


Some ideas from the proof

Prop.:

$$H_k\big(T_t(K), T_t(\mathrm{cost}(K,\sigma))\big) \cong \bigoplus_{\substack{W \subseteq V \setminus (\sigma \cup \tau) \\ |W| = t}} \tilde{H}_{k-d-1}\Bigg(\bigcup_{\substack{\sigma' \subsetneq \sigma, \\ 1 \le |\sigma'| \le t}} T_{t-|\sigma'|}\big(\mathrm{lk}(K, \sigma \setminus \sigma')[\tau \cup W]\big) \Bigg)$$



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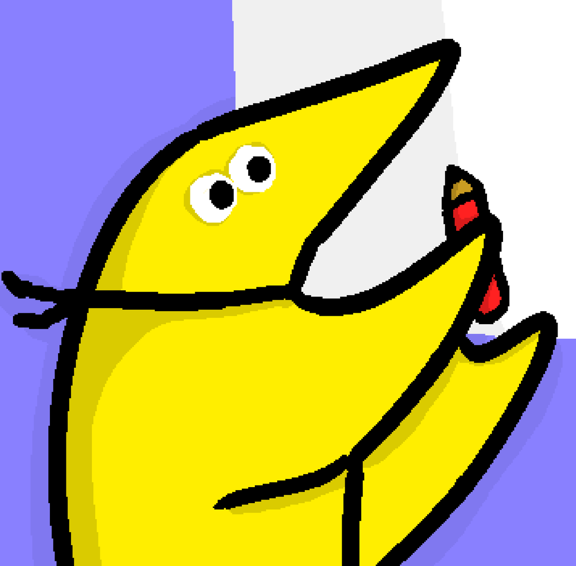
d-collapsible

(Khmelnitsky '18)



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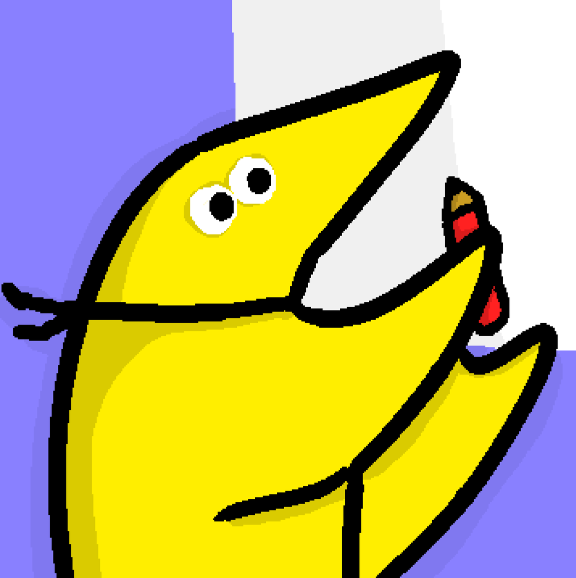
$\uparrow$

$h(t-|\sigma'|, d)$-Leray

(by induction on t)



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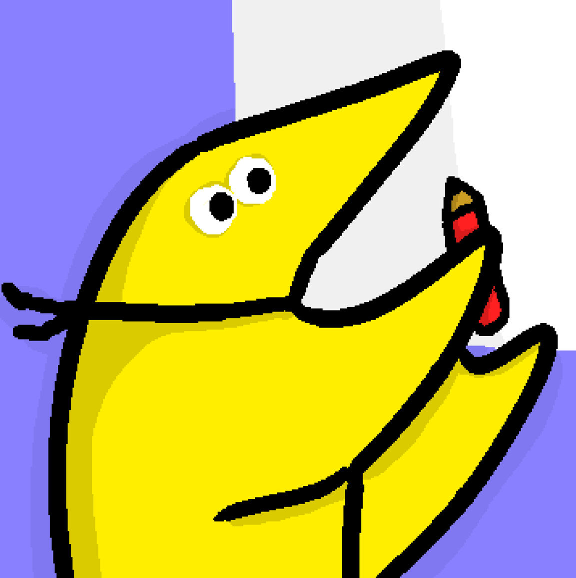
$$\Big[\Big(\sum_{\substack{\sigma' \subseteq \sigma \\ 1 \leq |\sigma'| \leq t}} h(t-|\sigma'|, d) + 1\Big) - 1\Big]\text{-Leray}$$

(Kalai-Meshulam '06)



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$\big[\, h(t,d) - d - 1 \,\big]$-Leray

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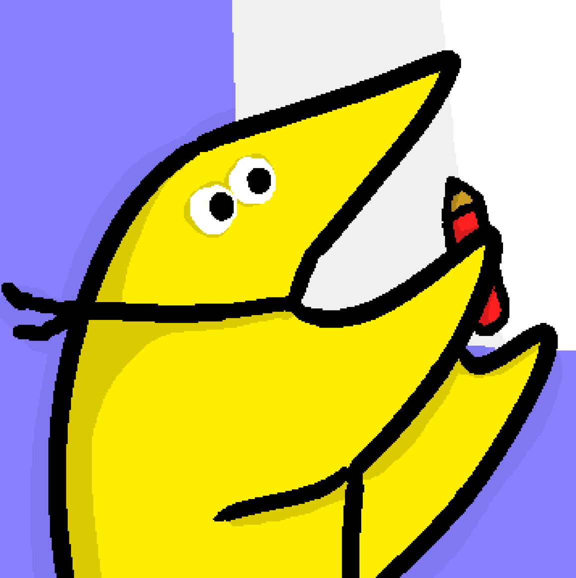

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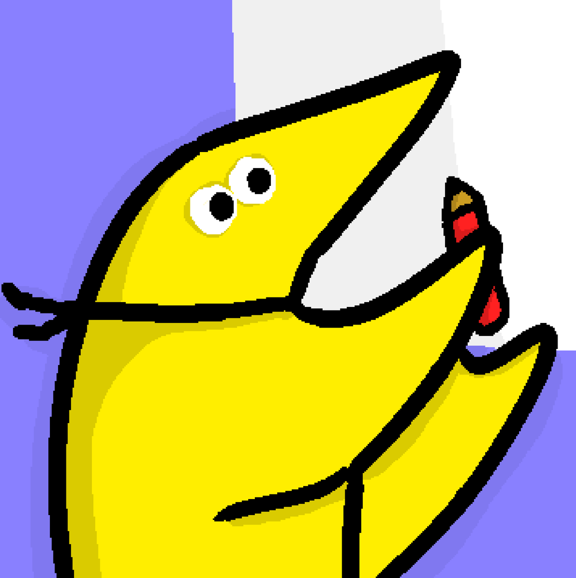

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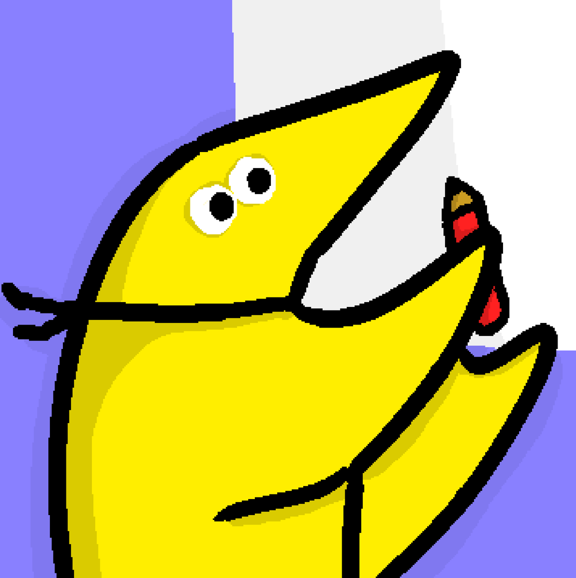
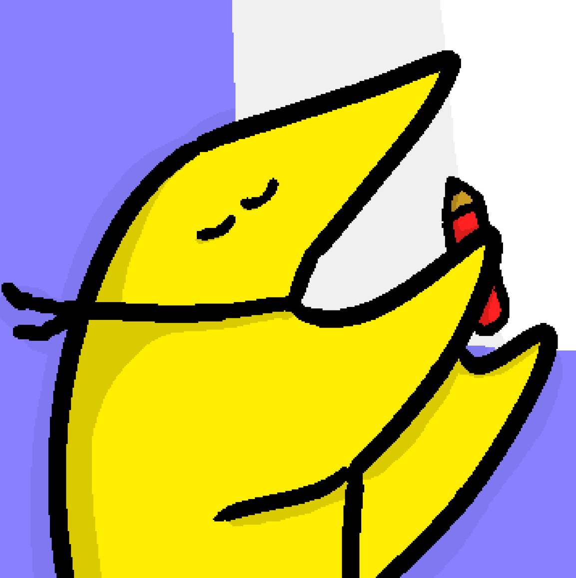


A geometric application

Thm (Kim, L. 21'): Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$ be families of convex sets in the plane.



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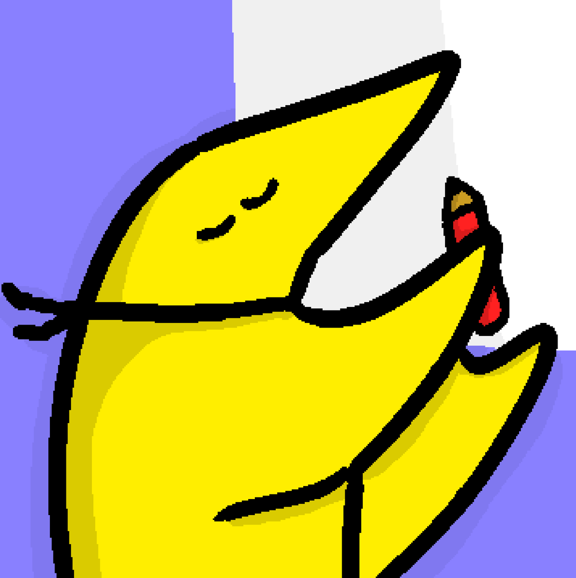

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Thm (Kim, L. 21'): Let

C_1, C_2, ...

Follows from application of Kalai and Meshulam's topological colorful Helly theorem ('05)

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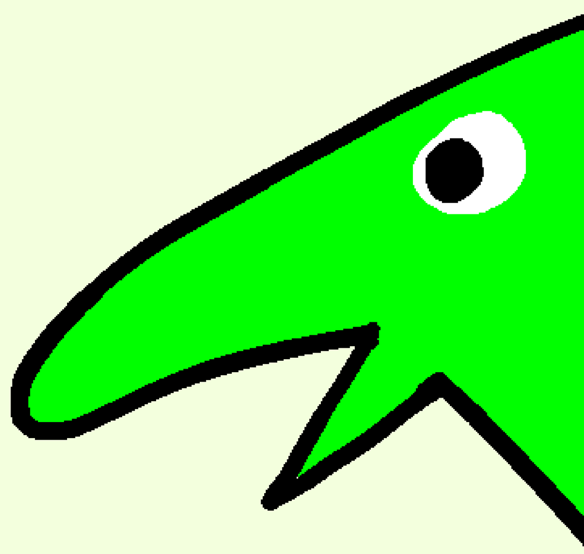


THANKS FOR LISTENING!